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An acoustical interpretation of the zeroes of ultraspherical polynomials

Une interprétation acoustique des zéros des polynômes ultrasphériques

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A B S T R A C T

In 1887, T.J. Stieltjes gave an electrostatical interpretation of the zeroes of Jacobi polynomials. This was extended later to Laguerre and Hermite polynomials by G. Szegö. An analogous interpretation is given here for ultraspherical polynomials in terms of piecewise cylindrical acoustical resonators.

R É S U M É

En 1887, T.J. Stieltjes a donné une interprétation des zéros des polynômes de Jacobi, en termes d'équilibre d'un système de charges électriques. L'extension aux polynômes de Laguerre et de Hermite en fut donnée ensuite par G. Szegö. Nous donnons ici une interprétation analogue des zéros des polynômes ultrasphériques en termes des fréquences de résonance de résonateurs acoustiques cylindriques par morceaux.

Version française abrégée

Outre le cas bien connu du cône et du cylindre, il a été démontré [1] qu'une classe particulière de résonateurs acoustiques possède des propriétés d'harmonicité des fréquences de résonance : les résonateurs cylindriques par morceaux. Les portions cylindriques constitutives possèdent toutes, dans ce cas, la même longueur et les aires a_n des sections droites successives forment une suite de terme général : $a_n = \frac{n(n+1)}{2} a_1, n = 2, \ldots, N$ où N est le nombre de portions cylindriques. Ces résonateurs ont été dénommés « cônes en escaliers » (*stepped cones*) en raison du fait qu'ils constituent une approximation discrète d'un cône. Dans un travail récent [2], on a montré que les fréquences de résonance des résonateurs cylindriques par morceaux formés d'un nombre quelconque de cylindres peuvent aussi être obtenues à partir des racines d'un certain polynôme p_N, construit au moyen d'une récurrence à trois termes. On a pu alors montrer que, pour $N \geq 3$, une infinité de tels résonateurs possède cette propriété d'harmonicité des fréquences de résonance. Néanmoins, dans cette généralité, la suite des aires des sections droites ainsi calculées n'est pas nécessairement monotone et peut admettre en théorie de

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grandes valeurs pour a_n [3], incompatibles avec l'hypothèse de propagation par ondes planes. Il y a donc un intérêt à rechercher une sous-classe de résonateurs possédant certaines propriétés comme la monotonie et une croissance contrôlée des a_n. Par ailleurs, dans le même travail [2], il avait été remarqué sur quelques valeurs faibles de N que la classe initiale des cônes en escaliers découverte dans [1] semblait donner lieu à la propriété remarquable d'orthogonalité de la famille des polynômes $p_n, n = 1, \ldots, N$. Il devenait alors naturel de s'interroger sur le degré de généralité de cette propriété. C'est ce qui est fait dans le présent travail, qui explicite les conditions sur les a_n pour que la suite des polynômes $p_n, n = 1, \ldots, N$ forme une famille orthogonale. Ceci permet en retour de donner une interprétation acoustique des racines des polynômes ultrasphériques, analogue à celle donnée en premier lieu par T.J. Stieltjes [4] pour les polynômes de Jacobi, puis par G. Szegö [5] pour les polynômes de Laguerre et Hermite, en termes d'équilibre d'un système de charges électriques. Cette interprétation fournit, par ailleurs, une méthode pour le calcul numérique des racines des polynômes ultrasphériques. La question de l'harmonicité des fréquences de résonance n'est pas abordée ici.

1. Introduction

Consider a piecewise cylindrical resonator (PCR), with N cylindrical pieces, in which the acoustical propagation is assumed to be one dimensional and lossless. Denote a_n the cross-section area of the nth cylindrical piece and assume in the sequel that all cylindrical pieces have the same length L. Denote also by ν the sound velocity inside the resonator and $k = \omega/\nu$ the wave number. It was shown in [1] that when choosing $a_n = \frac{n(n+1)}{2} a_1, n = 2, \ldots, N$, the natural resonance frequencies of the PCR closed at one end and open at the other one are all harmonic, i.e. that their ratio to some fundamental frequency is an integer. These resonators were then called "stepped cones", because they constitute a discrete approximation of a conical resonator. It was shown later [2] that, for a number $N \geq 3$ of pieces, the family of such PCR is infinite. Nevertheless, the general situation is that the sequence of cross-section areas obtained with this method is non-monotonic, with values of some a_n that can be very large [3] and are thus incompatible with the plane wave propagation assumption. On another hand, it was also observed [2] on the first few N that, for the special case of stepped cones, the polynomial whose roots give the PCR natural frequencies is an ultraspherical polynomial, which would lead to an orthogonal family on the interval $[-1, +1]$. This last observation is developed below and an acoustical interpretation of the zeroes of ultraspherical polynomials is provided, paralleling the results for classical orthogonal polynomials given by T.J. Stieltjes [4] and G. Szegö [5] in the domain of electrostatics. One by-product of this analogy is a numerical method to compute the roots of ultraspherical polynomials through simple linear algebra. Studying the harmonicity property of PCRs within this framework is currently under study.

2. Computing the natural frequencies of an acoustical PCR

It has been shown [2] that the natural resonance frequencies of a PCR, closed at the left end and open at the right one can be found from the roots $r_{N,i}$ of the polynomial p_N, $\deg(p_N) = N$, defined through the recursion:

$$\begin{cases} p_{-1}(x) = 0 \\ p_0(x) = 1 \\ p_n(x) = -(a_{n-1} + a_n) x p_{n-1}(x) - a_{n-1}^2 p_{n-2}(x) \ , \ n = 1, \ldots, N \end{cases} \tag{1}$$

with $x = \cos(kL)$ and initial values $a_0 = 0$, a_1 given. The natural frequencies are then: $f_{N,i} = \frac{\nu}{2\pi L} \arccos(r_{N,i})$. In the same way, following [2], it can be shown that when the PCR is open at both ends, its natural resonance frequencies can be obtained from the roots $s_{N-1,i}$ of the polynomial q_{N-1}, $\deg(q_{N-1}) = N - 1$, defined by the recursion:

$$\begin{cases} q_{-1}(x) = 0 \\ q_0(x) = 1 \\ q_n(x) = -(a_n + a_{n+1}) x q_{n-1}(x) - a_n^2 q_{n-2}(x) \ , \ n = 1, \ldots, N-1 \end{cases} \tag{2}$$

with a_1, a_2 a priori given parameters. Both recursions are three terms ones and are thus reminiscent of those that have been known for a long time for classical orthogonal polynomials [5,6]. Notice that when setting $a_1 = 0$, (2) reduces to (1), up to a renumbering ($a_{n+1} \to a_n$).

3. Search for orthogonal polynomial families

The polynomial functions $p_n(x), q_n(x)$ are clearly defined on the interval $[-1, 1]$. Consequently, when wondering about orthogonality, it is natural to focus on the most general such family: the Jacobi polynomials [5,6]. To be more precise, what are the consequences of assuming that p_n or q_n is a Jacobi polynomial? Firstly, comparing the recursions given for Jacobi polynomials under usual normalization [5,6] to recursions (1), (2), it is apparent that p_n and q_n can only be ultraspherical, because the coefficients in front of p_{n-1} in recursion (1) and in front of q_{n-1} in recursion (2) are merely linear in x and not strictly affine. In the sequel, one thus considers the recurrence for ultraspherical polynomials P_n^α under the normalization given in [5,6]:

$$xP_{n-1}^\alpha(x) = \frac{n(n+2\alpha)}{(2n+2\alpha-1)(n+\alpha)} P_n^\alpha(x) + \frac{n+\alpha-1}{2n+2\alpha-1} P_{n-2}^\alpha(x) \tag{3}$$

Remind that they are orthogonal on the interval $[-1,+1]$ with respect to the weight $(1-x^2)^\alpha$, $\alpha > -1$.

3.1. Closed–open resonators

In order for p_n to be an ultraspherical polynomial, set $p_n = c_n P_n$ where c_n is a normalizing coefficient. A simple substitution into recursion (3) gives:

$$p_n = \frac{c_n}{c_{n-1}} \frac{(n+\alpha)(2n+2\alpha-1)}{n(n+2\alpha)} x p_{n-1} - \frac{c_n}{c_{n-2}} \frac{(n+\alpha-1)(n+\alpha)}{n(n+2\alpha)} p_{n-2} \tag{4}$$

Identifying both recursions (1) and (4) leads to:

$$\begin{cases} \dfrac{c_n}{c_{n-1}} \dfrac{(n+\alpha)(2n+2\alpha-1)}{n(n+2\alpha)} = -(a_{n-1}+a_n) \\ \dfrac{c_n}{c_{n-2}} \dfrac{(n+\alpha-1)(n+\alpha)}{n(n+2\alpha)} = a_{n-1}^2 \end{cases} \tag{5}$$

It is clear that: $\frac{c_n}{c_{n-1}} < 0$ and $\frac{c_n}{c_{n-2}} > 0$. As $P_0^\alpha(x) = 1, P_1^\alpha(x) = (\alpha+1)x$ [5] and $p_0 = 1, p_1 = -a_1 x$, thanks to (1), the initial terms are given by: $c_0 = 1, c_1 = -\frac{a_1}{\alpha+1}$, with $a_0 = 0$ and a_1 arbitrary. From (5), a mere elimination shows that the cross-section areas can be computed through the recursion:

$$\begin{cases} a_n = a_{n-1} \left(\dfrac{(2n+2\alpha-1)(2n+2\alpha-3)}{(n-1)(n+2\alpha-1)} \dfrac{a_{n-1}}{a_{n-2}+a_{n-1}} - 1 \right), \; n = 2, \ldots, N \\ a_0 = 0, \; a_1 > 0 \text{ given} \end{cases} \tag{6}$$

Setting, for $n \ge 2$, $\gamma_{n,\alpha} = (2n+2\alpha-1)(2n+2\alpha-3)/(n-1)(n+2\alpha-1)$ and $\delta_n = \frac{a_n}{a_{n-1}}$, recursion (6) rewrites:

$$\begin{cases} \delta_{n+1} = \gamma_{n+1,\alpha} \dfrac{\delta_n}{1+\delta_n} - 1, \; n = 2, \ldots, N-1 \\ \delta_2 = 2\alpha + 2 \end{cases} \tag{7}$$

that allows to set $a_1 = 1$ without loss of generality. By induction, a solution to this recursion is found to be:

$$\delta_n = \frac{2\alpha+n}{n-1} \Longrightarrow a_n = \Pi_{i=2}^n \frac{2\alpha+i}{i-1} a_1, \; n \ge 2 \tag{8}$$

For a fixed $\alpha > -1$, all a_ns are strictly positive: one gets a unique sequence of cross-sectional areas and a unique feasible PCR. As α uniquely determines a family of ultraspherical polynomials, it also uniquely determines their roots and the natural frequencies of the underlying PCR.

3.1.1. Special cases

In musical acoustics, the initial motivation for the present study, it is generally accepted that the two useful families of acoustical resonators are the cylinder and the cone. In the present setting of PCRs, they can be recovered from (8). Observing that: $\forall n > 2, \gamma_{n,-1/2} = 4$ and $\forall n \ge 2, \gamma_{n,1/2} = 4$:

- for $\alpha = -1/2$, one gets $\delta_n = 1 \Rightarrow a_n = a_{n-1}$ thus $a_n = a_1$, $\forall n = 2, \ldots, N$. The acoustical resonator is a mere cylinder with length NL. But ultraspherical polynomials for $\alpha = -1/2$ are nothing else than Tchebychev polynomials of the first kind;
- for $\alpha = 1/2$, one gets $\delta_n = (n+1)/(n-1) \Rightarrow a_n = \frac{n+1}{n-1} a_{n-1} \Rightarrow a_n = \frac{n(n+1)}{2} a_1$ i.e. one recovers the stepped cones. Here, ultraspherical polynomials are Tchebychev polynomials of second kind.

The two musically useful resonators thus correspond to Tchebychev polynomials of first and second kinds, respectively, i.e. to the extreme values for α on the interval $[-1/2, 1/2]$. Another well-known family can be recovered: for $\alpha = 0$, one gets by the same method the resonator defined by $\delta_n = \frac{n}{n-1}$, which is equivalent to $a_n = n a_1$. In that case, ultraspherical polynomials specialize to Legendre (spherical) polynomials.

3.1.2. Geometry of closed–open resonators

Recursion (8) gives interesting results about the global shape of a resonator. For mathematical [5] as well as for physical reasons (when $\alpha \leq -1$, (8) leads to vanishing or negative a_ns), one must have $\alpha > -1$. From (8) and for $-1 < \alpha < -1/2$, $\delta_n < 1$ and the sequence of a_n is decreasing, resulting in convergent resonators. In the opposite case, $\alpha > -1/2$, the sequence of a_n is increasing and resonators profiles are divergent. More precisely, for $-1/2 < \alpha < 1/2$, the resonator profile is underlinear and when $\alpha > 1/2$ it is superlinear, with the limit cases given by the cylinder, $\alpha = -1/2$, and the stepped cone, $\alpha = 1/2$, as seen above. But, due to the plane wave assumption, α cannot grow indefinitely. One answer is thus given to the above-mentioned question of designing acoustical PCRs with some regularity property such as monotonicity while controlling the growth of a_n: one parameter, α, is sufficient to rule it in a flexible way. Conversely, with each ultraspherical polynomial is associated a unique closed–open resonator.

3.2. Open–open resonators

Proceeding the same way as for closed–open resonators, in order for q_n (2) to be an ultraspherical polynomial, set $q_n = d_n P_n$, where d_n is a normalizing coefficient. A mere substitution into recursion (3) leads to:

$$q_n = \frac{d_n}{d_{n-1}} \frac{(n+\alpha)(2n+2\alpha-1)}{n(n+2\alpha)} x q_{n-1} - \frac{d_n}{d_{n-2}} \frac{(n+\alpha-1)(n+\alpha)}{n(n+2\alpha)} q_{n-2} \tag{9}$$

noticing that $q_0(x) = 1$ and $q_1(x) = -(a_1 + a_2)x$. Identifying recursions (2) and (9) leads to a system of two equations involving, d_n, d_{n-1}, d_{n-2} and a_{n+1}, a_n, a_{n-1}, analogous to (5). Omitting the details that are quite similar, the resulting recursion for the coefficients a_n is in that case:

$$\begin{cases} a_{n+1} = a_n \left(\dfrac{(2n+2\alpha-1)(2n+2\alpha-3)}{(n-1)(n+2\alpha-1)} \dfrac{a_n}{a_{n-1}+a_n} - 1 \right), \; n = 2, \dots, N-1 \\ a_1 > 0, \; a_2 > 0 \text{ given} \end{cases} \tag{10}$$

resembling (6) obtained for closed–open resonators, with different initial values. Using the notations of 3.1, (10) rewrites:

$$\begin{cases} \delta_{n+1} = \gamma_{n,\alpha} \dfrac{\delta_n}{1+\delta_n} - 1, \; n = 2, \dots, N-1 \\ \delta_2 > 0 \text{ given} \end{cases} \tag{11}$$

As (10) is a three terms recursion, a_1 and a_2 are a priori arbitrary parameters, being initial terms of the recursion. But here again the parenthesized term in (10) must be positive, hence:

$$\delta_n > \frac{1}{\gamma_{n,\alpha} - 1}, \forall n = 2, \dots, N-1 \tag{12}$$

so that a_1 and a_2 actually must satisfy the above inequality for $n = 2$: $a_2/a_1 = \delta_2 > 1/(2\alpha + 2)$. Due to the additional freedom given through δ_2, open–open resonators associated with one ultraspherical polynomial are far from being unique: for a given α, one gets a whole family, parameterized by δ_2. The possible solutions to recursion (11) can behave very differently from recursion (7). Observe however that both cylinder and stepped cones are found this time from the sole value $\alpha = 1/2$, with initial values $\delta_2 = 1$ and $\delta_2 = 3$, respectively.

4. Acoustical interpretation of ultraspherical polynomials

Let a piecewise cylindrical acoustical resonator with N cylindrical pieces with same length L and cross-section areas $a_n, n = 1, \dots, N$. Let $x^\alpha_{N,i} = \cos(\theta^\alpha_{N,i}),\ i = 1, \dots, N$, be the zeroes of the Nth ultraspherical polynomial $P^\alpha_N(x)$, $\alpha > -1$.

Proposition 4.1. *The zeroes of $P^\alpha_N(x)$, are given by setting: $\theta^\alpha_{N,i} = \frac{2\pi f^\alpha_{N,i} L}{\nu}$ where $f^\alpha_{N,i}$ are the resonance frequencies of:*

– *either the closed–open resonator whose cross-section areas are given by:*

$$\begin{cases} a_n = \dfrac{2\alpha+n}{n-1} a_{n-1}, \; n = 2, \dots, N \\ a_1 > 0, \text{ given} \end{cases}$$

– *either the open–open resonator whose cross-section areas are given by:*

$$\begin{cases} a_{n+1} = a_n \left(\dfrac{(2n+2\alpha-1)(2n+2\alpha-3)}{(n-1)(n+2\alpha-1)} \dfrac{a_n}{a_{n-1}+a_n} - 1 \right), \; n = 2, \dots, N-1 \\ a_1, \; a_2 > 0 \text{ given} \end{cases}$$

5. Application

Except for $\alpha = \pm\frac{1}{2}$, for which explicit formulae hold, only lower and upper estimates are known for the roots of ultraspherical polynomials ([5], chap. VI). But, adopting a reversed point of view from the one above, the previous results furnish a numerical method to compute at once, with good accuracy, all the roots of P_n^α for any $\alpha > -1$, using simple linear algebra. To this end, define firstly the cross-section areas $a_n, n = 1, 2, \ldots, N$ thanks to Proposition 4.1. Outside any acoustical framework, recursions (1) and (2) define ultraspherical polynomials p_N and q_{N-1}, respectively, whatever $\alpha > -1$. Then, the method in [2] allows us to compute their roots by solving a mere algebraic eigenvalue problem.

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