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On unique solvability of the full three-dimensional Ericksen–Leslie system ☆

Sur la solvabilité unique du système tridimensionnel complet d'Ericksen–Leslie

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A B S T R A C T

In this paper, we study the full three-dimensional Ericksen–Leslie system of equations for the nematodynamics of liquid crystals. We announce the short-time existence and uniqueness of strong solutions for the initial value problem in the periodic case and in a bounded domain with Dirichlet- and Neumann-type boundary conditions.

R É S U M É

Dans cet article, nous étudions le système tridimensionnel complet des équations d'Ericksen–Leslie decrivant la nématodynamique des cristaux liquides. Nous donnons la formulation des théorèmes d'existence en temps court et d'unicité des solutions fortes pour le problème de valeur initiale dans le cas périodique et dans un domaine borné avec conditions au bord de types Dirichlet et Neumann.
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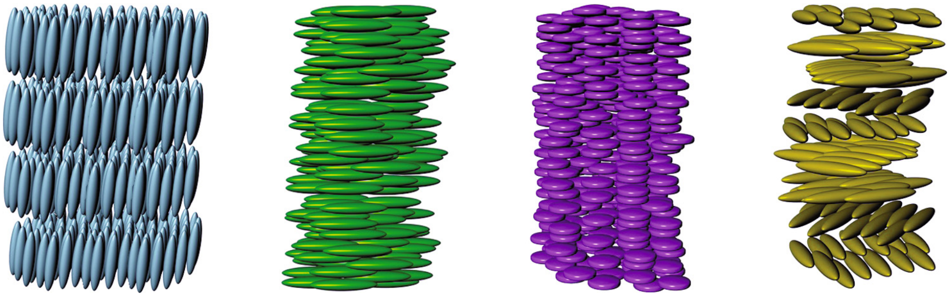

Fig. 1. The structure of horizontal flow of smectic (left), nematic calamitic and nematic discotic (center), cholesteric (right) liquid crystals.

1. Introduction

Mathematical models of the behavior of liquid crystals (see Fig. 1), attract much attention of scientists. The mathematical models of the hydrodynamics of incompressible, homogeneous nematic liquid crystals were firstly developed in the 1960s by J. Ericksen and F. Leslie (see, for instance, [1,2]).

In this paper, we consider the full Ericksen–Leslie system of equations (see, for instance, [3]). In our previous papers, we investigated plane periodic model [4,5] and plane problem in a bounded domain [6,5], homogenization of micro inhomogeneous nematic liquid crystals ([7] for periodic, [8] for random) in the case of a zero molecular moment of inertia, and two-dimensional nematodynamics in the case of a non-zero molecular moment of inertia [6]. We study the existence and uniqueness of solutions to the following Ericksen–Leslie system

$$\begin{cases} \dot{\mathbf{u}} - \mu\,\Delta\mathbf{u} = -\nabla p - \frac{\partial}{\partial x_j}\left(\frac{\partial\mathcal{F}}{\partial\mathbf{n}_{x_j}}\cdot\nabla\mathbf{n}\right) + \mathbf{F} \quad \operatorname{div}\mathbf{u} = 0 \\ J\ddot{\mathbf{n}} - 2q\,\mathbf{n} + \mathbf{h} = \mathbf{G}, \quad \|\mathbf{n}\| = 1 \end{cases} \tag{1}$$

where summation on repeated indices is understood and $\mathbf{n}_{x_j} := \frac{\partial}{\partial x_j}\mathbf{n}$. Here, $\mathbf{u}$ is the *Eulerian*, or *spatial velocity vector field*, $\mathbf{n}$ is the *director field*, the constant $\mu > 0$ is the *viscosity coefficient*, the constant $J > 0$ is the *moment of inertia of the molecule*, $\mathbf{F}(x,t)$ and $\mathbf{G}(x,t)$ are given *external forces*, and the overdot $^{\boldsymbol{\cdot}} := \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla$ is the *material derivative*. The function $\mathcal{F}(\mathbf{n},\nabla\mathbf{n})$ is the *Oseen–Zöcher–Frank free energy* and is defined by

$$\mathcal{F}(\mathbf{n},\nabla\mathbf{n}) := \frac{1}{2}\Big(K_1(\operatorname{div}\mathbf{n})^2 + K\big((\mathbf{n}\cdot\operatorname{curl}\mathbf{n})^2 + \|\mathbf{n}\times\operatorname{curl}\mathbf{n}\|^2\big)\Big)$$

where K, K_1 are real positive constants. The *molecular field* $\mathbf{h}$ is defined by

$$\mathbf{h} := \frac{\partial\mathcal{F}}{\partial\mathbf{n}} - \frac{\partial}{\partial x_j}\left(\frac{\partial\mathcal{F}}{\partial\mathbf{n}_{x_j}}\right)$$

The *pressure* p and the Lagrange multiplier $2q$ are determined, respectively, by the conditions $\operatorname{div}\mathbf{u} = 0$ and $\|\mathbf{n}\| = 1$. In this case, the ith component of the molecular field has the expression

$$h^i = (K - K_1)n^k_{x_k x_i} - Kn^i_{x_k x_k} + q'n^i$$

where q' is a scalar function depending on $\mathbf{n}$ and its derivatives. Define the linear differential operator $\mathcal{L}$ by

$$\mathcal{L}\mathbf{v} := (K - K_1)\nabla(\operatorname{div}\mathbf{v}) - K\Delta\mathbf{v} \tag{2}$$

Given the Ericksen–Leslie system (1), define the new vector field (first introduced in [9])

$$\boldsymbol{\nu} := \mathbf{n}\times\dot{\mathbf{n}} \in \mathcal{F}(\mathcal{D},\mathbb{R}^3)$$

With all these hypotheses and notations, system (1) becomes

$$\dot{\mathbf{u}} - \mu\Delta\mathbf{u} = -\nabla p + (\mathcal{L}\mathbf{n}\cdot\nabla\mathbf{n}) + \mathbf{F}, \quad \operatorname{div}\mathbf{u} = 0 \tag{3}$$

$$J\dot{\boldsymbol{\nu}} = \mathcal{L}\mathbf{n}\times\mathbf{n} + \mathbf{n}\times\mathbf{G} \tag{4}$$

$$\dot{\mathbf{n}} = \boldsymbol{\nu}\times\mathbf{n} \tag{5}$$

with unknowns $\mathbf{u}$, $\boldsymbol{\nu}$, $\mathbf{n}$. Thus, the Ericksen–Leslie system (1) implies the new first-order system (3)–(5).

Conversely, if the initial conditions of the first order system (3)–(5) satisfy the identities

$$\|\mathbf{n}(x,0)\| = 1, \qquad \mathbf{n}(x,0)\perp\boldsymbol{\nu}(x,0)$$

at time $t = 0$, then for any $t > 0$ we have

$$\|\mathbf{n}\| \equiv 1, \qquad \boldsymbol{\nu} = \mathbf{n} \times \dot{\mathbf{n}}, \qquad 2q = \mathbf{n} \cdot \mathbf{h} - J\|\boldsymbol{\nu}\|^2$$

and (3)–(5) turns into (1). Thus, under these hypotheses on the initial conditions, the first-order system (3), (4), (5) is equivalent to the original Ericksen–Leslie system (1) (as was first noticed in [9]).

The preceding papers (see [10–18] and [19]) are mostly concerned with the case $J = 0$.

In this article, we focus on the system (3)–(5) with $J \neq 0$ and announce the existence and uniqueness of solutions for 3-dimensional periodic media (Theorem 2.2) as well as for the problem in a bounded domain with Dirichlet (Theorem 3.2) and Neumann-type (Theorem 4.2) boundary conditions, in appropriate natural spaces. The uniqueness theorem holds under weaker conditions than the existence theorem, i.e. the spaces in the uniqueness theorem are bigger. We also give a result on the finite speed of propagation of the director field disturbance in such media (Theorem 3.3, Corollary 3.4).

2. Solutions in a periodic domain

Let $Q_T := (0, T) \times \mathbb{T}$, where $\mathbb{T} = \mathbb{R}^3/\mathbb{Z}^3$ is the 3-dimensional flat torus. We study the system (3)–(5) in Q_T with initial conditions

$$\mathbf{u}(0, x) = \mathbf{u}_0, \quad \boldsymbol{\nu}(0, x) = \boldsymbol{\nu}_0, \quad \mathbf{n}(0, x) = \mathbf{n}_0 \tag{6}$$

Here $\mathbf{u}$, $\boldsymbol{\nu}$, $\mathbf{n}$ are unknown vector fields, p is an unknown scalar function, and $J > 0$, $K_i > 0$, $\mu > 0$ are given constants.

Throughout the paper we use the following notations:

- $\dot{f} := \frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla f = f_t + u^j f_{x_j}$ is the material time derivative of f;
- a bold letter $\mathbf{b}$ denotes a 3-dimensional vector $\mathbf{b} = (b^1, b^2, b^3)$, or a vector field with values in $\mathbb{R}^3$;
- a standard summation convention is used on repeated indices, independent of their position, e.g., $a_i b_i := \sum_i a_i b_i$;
- $L_2(\mathbb{T}) := \left\{\mathbf{v} : \mathbb{T} \to \mathbb{R}^3 \mid \|\mathbf{v}\|_2^2 := \int_{\mathbb{T}} \|\mathbf{v}\|^2 \mathrm{d}\mathbf{x} < \infty\right\}$;
- $(\mathbf{u}, \mathbf{v}) := \int_{\mathbb{T}} \mathbf{u} \cdot \mathbf{v}\, \mathrm{d}\mathbf{x}$ is the inner product in $L_2(\mathbb{T})$;
- $W_2^m(\mathbb{T})$ is the Sobolev space of functions on $\mathbb{T}$ having m distributional derivatives in $L_2(\mathbb{T})$;
- for any $\mathbf{v} \in W_2^m(\mathbb{T})$, $m \in \mathbb{N}$, define

$$\|D^m \mathbf{v}\|_2^2 := \sum_{i_1+i_2+i_3=m} \left\| \frac{\partial^m \mathbf{v}}{\partial x_1^{i_1} \partial x_2^{i_2} \partial x_3^{i_3}} \right\|_2^2$$

- $Sol(\mathbb{T}) := \{\mathbf{v} : \mathbb{T} \to \mathbb{R}^3 \mid \mathbf{v} \in C^\infty(\mathbb{T}),\ \operatorname{div} \mathbf{v} = 0\}$;
- $Sol(Q_T) := \{\mathbf{v} \in C^\infty(Q_T) \mid \mathbf{v}(t, \cdot) \in Sol(\mathbb{T}),\ \forall t \in (0, T)\}$;
- $Sol_2(\mathbb{T})$ is the closure of $Sol(\mathbb{T})$ in the norm $L_2(\mathbb{T})$;
- $Sol_2^m(\mathbb{T})$ is the closure of $Sol(\mathbb{T})$ in the norm $W_2^m(\mathbb{T})$.

Definition 2.1. A quadruple $(\mathbf{u}, \boldsymbol{\nu}, \mathbf{n}, \nabla p)$ is a *strong solution* to problem (3)–(6) in the domain Q_T if

(i) $\mathbf{u}$ is a time-dependent vector field in $L_2((0, T); Sol_2^3(\mathbb{T}))$, $\mathbf{u}_t \in L_2(Q_T)$;
(ii) $\boldsymbol{\nu}$ is a vector field in $L_\infty((0, T); W_2^2(\mathbb{T}))$, $\boldsymbol{\nu}_t \in L_\infty((0, T); L_2(\mathbb{T}))$;
(iii) $\mathbf{n}$ is a vector field in $L_\infty((0, T); W_2^3(\mathbb{T}))$, $\mathbf{n}_t \in L_\infty((0, T); W_2^1(\mathbb{T}))$;
(iv) $\nabla p \in L_2(Q_T)$;
(v) $\mathbf{u}$, $\mathbf{n}$, $\boldsymbol{\nu}$ satisfy the initial conditions (6), i.e. $(\mathbf{u}, \mathbf{n}, \boldsymbol{\nu}) \rightharpoonup (\mathbf{u}_0, \mathbf{n}_0, \boldsymbol{\nu}_0)$ in $L_2(\mathbb{T})$ as $t \to 0$;
(vi) Eqs. (3)–(5) hold almost everywhere.

The following assertion is valid.

Theorem 2.2. *Suppose $\mathbf{u}_0 \in Sol_2^2(\mathbb{T})$, $\boldsymbol{\nu}_0 \in W_2^2(\mathbb{T})$, $\mathbf{n}_0 \in W_2^3(\mathbb{T})$, and $\mathbf{F} \in L_2((0, T); W_2^1(\mathbb{T}))$, $\mathbf{G} \in L_1((0, T); W_2^2(\mathbb{T}))$. Then there is a $T > 0$ such that the solution to problem (3)–(5), (6) (as given in Definition 2.1) does exist.*

Let $(\mathbf{u}_1, \boldsymbol{\nu}_1, \mathbf{n}_1, p_1)$ and $(\mathbf{u}_2, \boldsymbol{\nu}_2, \mathbf{n}_2, p_2)$ be solutions to problem (3)–(6) in the domain Q_T. Then, for some $0 < T_0 \le T$

$$(\mathbf{u}_2, \boldsymbol{\nu}_2, \mathbf{n}_2, \nabla p_2) = (\mathbf{u}_1, \boldsymbol{\nu}_1, \mathbf{n}_1, \nabla p_1)$$

almost everywhere in Q_{T_0}.

The proof is based on Galerkin-type approximations.

3. Solutions with Dirichlet-type boundary conditions

Let Ω be a bounded domain in $\mathbb{R}^3$ and consider nematic liquid crystal flow in the cylinder $\Omega \times \mathbb{R}$.

We study Eqs. (3)–(5) in the domain $(0, T) \times \Omega$ with initial conditions (6) and additional boundary conditions

$$\mathbf{u}\big|_{\partial\Omega} = 0, \quad \mathbf{n} - \boldsymbol{n}_1\big|_{\partial\Omega} = 0, \quad \boldsymbol{\nu}|_{\partial\Omega} = 0 \quad \text{for any} \quad t > 0 \tag{7}$$

where $\boldsymbol{n}_1$ is a given vector field on Ω.

Condition $\mathbf{u}\big|_{\partial\Omega} = 0$ means that the domain has impenetrable boundary and that the fluid moves without slipping; $\mathbf{n} - \boldsymbol{n}_1\big|_{\partial\Omega} = 0$ describes the director position at the boundary. The third condition comes from the original Ericksen–Leslie system and means that $\dot{\mathbf{n}} = 0$ at the boundary.

In this section, we let $Q_T := (0, T) \times \Omega$ and introduce the function spaces

$$\overset{\circ}{Sol}(\Omega) := \{\mathbf{v} : \Omega \to \mathbb{R}^3 \mid \mathbf{v} \text{ has compact support}, \ \operatorname{div} \mathbf{v} = 0\}$$

$$\overset{\circ}{Sol}(Q_T) := \{\mathbf{v} \in C^\infty(Q_T) \mid \mathbf{v}(t, \cdot) \in \overset{\circ}{Sol}(\Omega), \forall t\}$$

$$\overset{\circ}{Sol}{}_2^m(\Omega) \text{ is the closure of } \overset{\circ}{Sol}(\Omega) \text{ in the norm } W_2^m(\Omega)$$

The definition of a solution to the Ericksen–Leslie equations is the natural modification for the case of a bounded domain with boundary of the one given in Definition 2.1.

Definition 3.1. The quadruple $(\mathbf{u}, \boldsymbol{\nu}, \mathbf{n}, \nabla p)$ is a *strong solution* to problem (3)–(6), (7) in the domain Q_T if

- $\mathbf{u}$ is a vector field in $L_2((0, T); \overset{\circ}{Sol}{}_2^1(\Omega)) \cap L_2((0, T); W_2^3(\Omega))$, $\mathbf{u}_t \in L_2(Q_T)$;
- $\boldsymbol{\nu}$ is a vector field in $L_\infty((0, T); \overset{\circ}{W}{}_2^1(\Omega)) \cap L_\infty((0, T); W_2^2(\Omega))$, $\boldsymbol{\nu}_t \in L_\infty((0, T); L_2(\Omega))$;
- $\mathbf{n} - \boldsymbol{n}_1$ is a vector field in $L_\infty((0, T); \overset{\circ}{W}{}_2^1(\Omega)) \cap L_\infty((0, T); W_2^3(\Omega))$, where $\boldsymbol{n}_1$ is a given constant vector field, and $\mathbf{n}_t \in L_\infty((0, T); W_2^1(\Omega))$;
- $\nabla p \in L_2(Q_T)$;
- $\mathbf{u}$, $\mathbf{n}$, $\boldsymbol{\nu}$ satisfy initial conditions (6), i.e. $(\mathbf{u}, \mathbf{n}, \boldsymbol{\nu}) \rightharpoonup (\mathbf{u}_0, \mathbf{n}_0, \boldsymbol{\nu}_0)$ in $L_2(\Omega)$;
- Eqs. (3)–(5) hold almost everywhere.

The following result is proved in the same way as Theorem 2.2, with natural modifications.

Theorem 3.2. *Assume that for all $\mathbf{x} \in \partial\Omega$ the boundary is the graph of a C^2-function in some neighborhood of $\mathbf{x}$. Let $\boldsymbol{n}_1 = const$, $\mathbf{n}_0 \in \overset{\circ}{W}{}_2^3(\Omega)$, $\boldsymbol{\nu}_0 \in W_2^2(\Omega)$, $\mathbf{u_0} \in \overset{\circ}{Sol}{}_2^1(\Omega) \cap W_2^2(\Omega)$, $\Delta \mathbf{u}_0\big|_{\partial\Omega} = 0$, and assume that, for some $d > 0$, we have*

$$\mathbf{n}_0(x) = \text{const}, \quad \boldsymbol{\nu}_0(x) = 0 \quad \textit{if} \quad \operatorname{dist}(x, \partial\Omega) < d$$

Assume also that $\mathbf{F} \in L_2((0, T); W_2^1(\Omega))$, $\mathbf{G} \in L_1((0, T); W_2^2(\Omega))$, G equal to zero in a neighborhood of $\partial\Omega$. Then problem (3)–(6), (7) has a unique solution in Q_T for some $T > 0$.

Instead of (3)–(5), we consider the system

$$\dot{\mathbf{u}} - \mu \Delta \mathbf{u} = -\nabla p + (\mathcal{L}\mathbf{n} \cdot \nabla \mathbf{n})\Psi, \qquad \operatorname{div} \mathbf{u} = 0 \tag{8}$$

$$J(\boldsymbol{\nu}_t + \Psi u^i \boldsymbol{\nu}_{x_i}) = (\mathcal{L}\mathbf{n} \times \mathbf{n})\Psi \tag{9}$$

$$\mathbf{n}_t + \Psi u^i \mathbf{n}_{x_i} = (\boldsymbol{\nu} \times \mathbf{n})\Psi \tag{10}$$

where $\Psi(\mathbf{x}) \in C^\infty(\Omega)$ is a given smooth non-negative function with compact support.

Theorem 3.3. *Fix $\mathbf{u} \in L_2((0, T); \overset{\circ}{Sol}{}_2^1(\Omega)) \cap L_2((0, T); W_2^2(\Omega))$ and $\Psi \in C^\infty(\Omega)$ with compact support, $0 \le \Psi \le 1$. Consider Eqs. (9), (10) for this given vector field $\mathbf{u}$.*

Suppose, in addition, that for some $1 < \alpha \le \infty$ and for all i, j, there are constants $m > 0$, $M > 0$ such that the vector field $\mathbf{u}$ satisfies

$$\| \operatorname*{esssup}_x |u^i_{x_j}(x, t)| \|_{L_\alpha(0,T)} \le M \quad \textit{and} \quad \|\mathbf{u}(x, t)\| \le m, \quad \forall (x, t) \in Q_T$$

Assume also that the initial conditions $\mathbf{n}_0$ and $\boldsymbol{\nu}_0$ of (9), (10), with this given vector field $\mathbf{u}$, are such that $\nabla \mathbf{n}_0$ and $\boldsymbol{\nu}_0$ vanish for $\|x - x_0\| < r$. Then there exist constants m', $t_0 > 0$ such that $\nabla \mathbf{n}$ and $\boldsymbol{\nu}$ are equal to zero for all (x, t) satisfying

$$\|x - x_0\| < r - m't, \quad t < t_0$$

Remark 1. A similar result, with identical proof, holds in a periodic domain. In this case, we assume $\mathbf{u} \in L_2((0,T); Sol_2^1(\mathbb{T})) \cap L_2((0,T); W_2^2(\mathbb{T}))$ and take $\Psi \equiv 1$.

Corollary 3.4. *Consider a solution* $(\mathbf{u}, \boldsymbol{\nu}, \mathbf{n}, p)$ *of the problem* (3)–(5), (6) *in the periodic domain, as given in Definition* 2.1.

Assume also that the initial conditions $\mathbf{n}_0$ *and* $\boldsymbol{\nu}_0$ *are such that* $\nabla \mathbf{n}_0$ *and* $\boldsymbol{\nu}_0$ *vanish for* $\|x - x_0\| < r$. *Then there exist constants* $m', t_0 > 0$ *such that* $\nabla \mathbf{n}$ *and* $\boldsymbol{\nu}$ *are equal to zero for all* (x,t) *satisfying*

$$\|x - x_0\| < r - m't, \quad t < t_0$$

If $(\mathbf{u}, \boldsymbol{\nu}, \mathbf{n}, p)$ *is the solution to problem* (3)–(6), (7) *in a bounded domain, we need to assume, in addition, that* $\boldsymbol{\nu}$, $\nabla \mathbf{n}$ *vanish in some neighborhood of the boundary* $\partial\Omega$.

4. Solutions with Neumann-type boundary conditions

The problem considered in Section 3 has an important drawback: the director vector field is assumed to be constant near the boundary. As shown below, this condition can be neglected if we change the boundary conditions.

Suppose that $K_1 = K = K$ and consider the domain Ω as being a cuboid, i.e. $\Omega = \Pi_{i=1}^3 (a_i, b_i)$, where $-\infty < a_i < b_i < \infty$.

The equations of motion are (3)–(5) with initial conditions (6), but instead of the boundary conditions (7), we require

$$\mathbf{u} \cdot \mathbf{N}\Big|_{(0,T)\times\partial\Omega} = 0 \quad \text{and} \quad u^i_{x_j} \tau^i N^j \Big|_{(0,T)\times\partial\Omega} = 0 \ \forall \boldsymbol{\tau} \perp \mathbf{N} = 0 \tag{11}$$

The boundary condition on the director field $\mathbf{n}$ and the variable $\boldsymbol{\nu}$ are

$$\mathbf{n}_{x_i} N^i \Big|_{(0,T)\times\partial\Omega} = 0 \quad \text{and} \quad \boldsymbol{\nu}_{x_i} N^i \big|_{\Omega} = 0 \tag{12}$$

Definition 4.1. The quadruple $(\mathbf{u}, \boldsymbol{\nu}, \mathbf{n}, \nabla p)$ is a *strong solution* to problem (3)–(6), (11), (12) in the domain Q_T if

- $\mathbf{u}$ is a vector field in $L_2((0,T); Sol_2^1(\Omega)) \cap L_2((0,T); W_2^3(\Omega))$, $\mathbf{u}_t \in L_2(Q_T)$;
- $\boldsymbol{\nu}$ is a vector field in $L_\infty((0,T); W_2^2(\Omega))$, $\boldsymbol{\nu}_t \in L_\infty((0,T); L_2(\Omega))$;
- $\mathbf{n}$ is a vector field in $L_\infty((0,T); W_2^3(\Omega))$, where $\boldsymbol{n}_1$ is a given constant vector field, and $\mathbf{n}_t \in L_\infty((0,T); W_2^1(\Omega))$;
- $\nabla p \in L_2(Q_T)$;
- $\mathbf{u}$, $\mathbf{n}$, $\boldsymbol{\nu}$ satisfy the initial conditions (6), i.e. $(\mathbf{u}, \mathbf{n}, \boldsymbol{\nu}) \rightharpoonup (\mathbf{u}_0, \mathbf{n}_0, \boldsymbol{\nu}_0)$ in $L_2(\Omega)$;
- Eqs. (3)–(5) and boundary conditions (11), (12) hold almost everywhere.

Theorem 4.2. *Assume that* Ω *is the cuboid* $\Pi_{i=1}^3 (a_i, b_i)$ *with* $-\infty < a_i < b_i < \infty$. *Let* $\mathbf{n}_0 \in W_2^3(\Omega)$, $\frac{\partial \mathbf{n}}{\partial N}\big|_{\partial\Omega} = 0$, $\boldsymbol{\nu}_0 \in W_2^2(\Omega)$, $\mathbf{u_0} \in Sol_2^1(\Omega) \cap W_2^2(\Omega)$, $u_0^3|_{\partial\Omega} = 0$. *Suppose also that* $\mathbf{F} \in L_2((0,T); W_2^1(\Omega))$, $\mathbf{G} \in L_1((0,T); W_2^2(\Omega))$, *and* $G^i N^i|_{\partial\Omega} = 0$.

Then problem (3)–(6), (11), (12) *has a unique solution in* Q_T *for some* $T > 0$.

References

[1] J. Ericksen, Conservation laws for liquid crystals, Trans. Soc. Rheol. 5 (1961) 22–34.
[2] F. Leslie, Some constitutive equations for anisotropic fluids, Q. J. Mech. Appl. Math. 19 (1966) 357–370.
[3] P.G. De Gennes, J. Prost, The Physics of Liquid Crystals, Clarendon Press, Oxford, 1993.
[4] G.A. Chechkin, T.S. Ratiu, M.S. Romanov, V.N. Samokhin, Nematic liquid crystals. Existence and uniqueness of periodic solutions to Ericksen–Leslie equations, Bull. Ivan Fedorov Mosc. State Univ. Print. Arts 12 (2012) 139–151.
[5] G.A. Chechkin, T.S. Ratiu, M.S. Romanov, V.N. Samokhin, Existence and uniqueness theorems for two-dimensional Ericksen–Leslie system, J. Math. Fluid Mech. (2016), http://dx.doi.org/10.1007/s00021-016-0250-0.
[6] G.A. Chechkin, T.S. Ratiu, M.S. Romanov, V.N. Samokhin, Existence and uniqueness theorems in two-dimensional nematodynamics. Finite speed of propagation, Russ. Acad. Sci. Dokl. Math. 91 (3) (2015) 354–358, Translated from: Dokl. Akad. Nauk 462 (5) (2015) 519–523.
[7] T.S. Ratiu, M.S. Romanov, G.A. Chechkin, Homogenization of the equations of the dynamics of nematic liquid crystals with inhomogeneous density, J. Math. Sci. 186 (2) (2012) 322–329, Translated from: Probl. Mat. Anal. 66 (2012) 167–173.
[8] G.A. Chechkin, T.P. Chechkina, T.S. Ratiu, M.S. Romanov, Nematodynamics and Random Homogenization, Appl. Anal. (2015), http://dx.doi.org/10.1080/00036811.2015.1036241, Online first.
[9] F. Gay-Balmaz, T.S. Ratiu, The geometric structure of complex fluids, Adv. Appl. Math. 42 (2009) 176–275.
[10] F.H. Lin, Nonlinear theory of defects in nematic liquid crystal: phase transition and flow phenomena, Commun. Pure Appl. Math. 42 (1989) 789–814.
[11] F.H. Lin, C. Liu, Nonparabolic dissipative system modeling the flow of liquid crystals, Commun. Pure Appl. Math. XLVIII (1995) 501–537.
[12] S. Shkoller, Well-posedness and global attractors for liquid crystals on Riemannian manifolds, Commun. Partial Differ. Equ. 27 (2002) 1103–1137.
[13] M.C. Hong, Global existence of solutions of the simplified Ericksen–Leslie system in dimension two, Calc. Var. Partial Differ. Equ. 40 (2011) 15–36.
[14] F.H. Lin, J.Y. Liu, C.Y. Wang, Liquid crystal flows in two dimensions, Arch. Ration. Mech. Anal. 197 (2010) 297–336.
[15] J. Huang, F. Lin, C. Wang, Regularity and existence of global solutions to the Ericksen–Leslie system in $\mathbb{R}^2$, arXiv:1305.5988.
[16] Y. Wu, X. Xu, C. Liu, On the general Ericksen–Leslie system: Parodi's relation, well-posedness and stability, Arch. Ration. Mech. Anal. 208 (2013) 59–107.
[17] M. Dai, Existence of regular solutions to the full liquid crystal system, arXiv:1309.0476v1, 2013.
[18] T. Huang, C. Wang, H. Wen, Strong solutions of the compressible nematic liquid crystal flow, J. Differ. Equ. 252 (2012) 2222–2265.
[19] C.Y. Wang, Well-posedness for the heat flow of harmonic maps and the liquid crystal flow with rough initial data, Arch. Ration. Mech. Anal. 200 (2011) 1–19.