C. R. Mecanique 344 (2016) 510–520

Contents lists available at ScienceDirect

Comptes Rendus Mecanique

www.sciencedirect.com

New non-equilibrium matrix imbibition equation for double porosity model

Andrey Konyukhov [a,b], Leonid Pankratov [b,c,*]

[a] Joint Institute for High Temperatures of the Russian Academy of Sciences, Izborskaya 13 Bldg, 2, Moscow, 125412, Russian Federation
[b] Laboratory of Fluid Dynamics and Seismic, Moscow Institute of Physics and Technology, 9 Institutskiy per., Dolgoprudny, Moscow Region, 141700, Russian Federation
[c] Laboratoire de mathématiques et de leurs applications, CNRS – UMR 5142, Université de Pau et des pays de l'Adour, av. de l'Université, 64000 Pau, France

ARTICLE INFO

Article history:
Received 11 July 2015
Accepted 22 February 2016
Available online 30 March 2016

Keywords:
Porous media
Double porosity
Homogenization
Two-phase flow
Non-equilibrium model

ABSTRACT

The paper deals with the global Kondaurov double porosity model describing a non-equilibrium two-phase immiscible flow in fractured-porous reservoirs when non-equilibrium phenomena occur in the matrix blocks, only. In a mathematically rigorous way, we show that the homogenized model can be represented by usual equations of two-phase incompressible immiscible flow, except for the addition of two source terms calculated by a solution to a local problem being a boundary value problem for a non-equilibrium imbibition equation given in terms of the real saturation and a non-equilibrium parameter.

1. Introduction

The homogenized Kondaurov double-porosity-type model (see [1]) describing a non-equilibrium two-phase flow of immiscible incompressible fluids in fractured-porous reservoirs is considered. Two-phase flow in porous media is important to many practical problems, including those in petroleum reservoir engineering, soil science, etc. The modeling and numerical simulation of two-phase flow in porous media represents an important key in the design of cost-efficient, safe, and suitable clean-up tools. It can reduce the number of laboratory and field experiments, help to identify the significant mechanisms, optimize existing strategies and give indications about possible risks. In the existing physical and mathematical literature, the authors are dealing mainly with the equilibrium models. However, the experimental studies have invalidated this kind of models (see, e.g., [2]). The model considered in this paper corresponds physically to a non-equilibrium immiscible incompressible two-phase flow through fractured porous media. Notice that the crucial feature of a porous medium, saturated with immiscible fluids, is the fact that the process depends on the rate and direction of the change of state. The most well-known and often discussed phenomena of this type are the relaxation of capillary pressure, the "capillary pressure–saturation" hysteresis curve, and the dependence of the phase permeabilities and the value of the capillary sticking on the rate and direction of a change in the saturation. The generally accepted explanation of these phenomena is the non-equilibrium of the joint motion of the fluids (see, e.g., [3]).

* Corresponding author.
E-mail addresses: konyukhov_av@mail.ru (A. Konyukhov), leonid.pankratov@univ-pau.fr (L. Pankratov).

http://dx.doi.org/10.1016/j.crme.2016.02.011
1631-0721/

The homogenization of multiphase flow through heterogeneous porous media as well as the numerical simulation of this physical process has been a problem of interest for many years, and many methods have been developed. There is an extensive literature on this subject. Here we refer to the monographs [4,5] as well as to [6–8]. A recent review of the mathematical homogenization methods developed for two-phase flow in porous media can be viewed in [9,10]. It is important to notice that the microscopic models of the multiphase flow in porous media considered in all these works are equilibrium ones even if the homogenization process for single- and multiphase flows in double porosity media leads to the appearing of an additional source term which exhibits the global non-equilibrium behavior of the model (see, e.g., [5, 10–12]). In addition, it is shown in [13,14] that the homogenization procedure leads to the appearing of the non-equilibrium capillary pressure in the global model. However, there are few papers dealing with the homogenization of non-equilibrium two-phase flows in porous media. Here we refer to [8], where the authors deal with the upscaling of such flows in vertically fractured oil reservoirs. The homogenization process is carried out for Barenblatt's and Hassanizadeh's flow models (see, e.g., [15–17]). Concerning the rigorous mathematical studies in the domain of non-equilibrium two-phase flows, we also observe only few papers on the subject, dealing mainly with the existence and uniqueness problems (see, e.g., [18,19]).

In this paper, we study an immiscible non-equilibrium two-phase flow in double porosity media in the framework of the thermodynamically consistent Kondaurov model [20,21], which is, in fact, an integro-differential one due to the fact that mobility functions and capillary pressure depend on Kondaurov's non-equilibrium parameter, which satisfies a kinetic equation with respect to the real saturation (see Section 2 below). The detailed comparison of the Kondaurov model and Barenblatt's and Hassanizadeh's non-equilibrium flow models is done in [1,21]. Here we focus our attention on the homogenized non-equilibrium double-porosity-type model obtained recently in [1]. This model has a rather complicated form in view of numerical simulation. From the other hand, we know that the numerical methods are very sensitive to the choice of the form of the governing equations. Then the aim of the present paper is to find a more simple form for the local problem involved in the model. Namely, we will show that the homogenized problem can be represented as usual equations of two-phase incompressible, immiscible flow, with two source terms calculated by a solution to a local problem which is a boundary value problem for a *non-equilibrium imbibition equation*. The derivation of this equation is essentially based on the introduction of a *non-equilibrium global pressure*, which generalizes the notion of the well-known global pressure function (see, e.g., [22–24]) widely used in the mathematical analysis of multi-phase flows in porous media. To our knowledge, it is a first attempt to introduce the non-equilibrium imbibition equation in the homogenization process.

The rest of the paper is organized as follows. In Section 2, we present a mathematically rigorous adimensionalized non-equilibrium Kondaurov model focusing on the correct definitions of the capillary pressure and mobility functions. In Section 3, following the lines of [1], we introduce the global Kondaurov double-porosity model. Finally, in Section 4, we study the local problem involved in the homogenized model. Introducing the notion of non-equilibrium global pressure, we reduce the local problem formulated in terms of phase pressures to a unique non-equilibrium imbibition equation, which is an integro-differential one with respect to the real saturation. As it is shown in Remark 5 at the end of Section 4, the last one is a generalization of the well-known imbibition equation appearing in the homogenization of the two-phase double-porosity models (see, e.g., [25] and the references herein). The Note is completed by the section *Concluding remarks*.

2. Adimensionalized non-equilibrium Kondaurov model

In this section, we introduce the adimensionalized non-equilibrium Kondaurov flow model proposed in [20] and then developed in [21]. More recently, it was discussed in [1]. The equations of the model read:

$$\Phi\,\frac{\partial S_\kappa}{\partial t}+\operatorname{div}\mathbf{W}_\kappa=0,\quad \text{where the fluxes are defined by:}\quad \mathbf{W}_\kappa=-\frac{K f_\kappa(S_\kappa,\boldsymbol{\xi})}{\mu_\kappa}\nabla p_\kappa\ (\kappa=w,n) \tag{1}$$

Here the subscripts w,n denote the wetting and non-wetting fluids; S_κ is the saturation of the corresponding fluid; $0<\Phi<1$ is the porosity function; K is the absolute permeability tensor; p_κ is the pressure of the wetting (non-wetting) fluid; $f_\kappa=f_\kappa(S_\kappa,\boldsymbol{\xi})$ stands for the relative permeability of the wetting (non-wetting) fluid defined by:

$$f_w(S_w,\boldsymbol{\xi})=f_w^{\mathrm{e}}\,(2\,S_w+\beta\,\boldsymbol{\xi}/\alpha-1)\quad\text{and}\quad f_n(S_n,\boldsymbol{\xi})=f_n^{\mathrm{e}}\left(2\left[1-S_w\right]-\beta\,\boldsymbol{\xi}/\alpha\right) \tag{2}$$

where the superscript "e" denotes the equilibrium relative phase permeabilities in the Darcy–Muskat law (see, e.g., [23]) and $\alpha,\beta>0$ are the constitutive parameters of the model; μ_κ is the viscosity of the wetting (non-wetting) fluid; finally, $\boldsymbol{\xi}$ denotes the non-equilibrium Kondaurov parameter, which satisfies the following kinetic equation:

$$\frac{\partial\boldsymbol{\xi}}{\partial t}=\frac{1}{\tau}\,\Lambda(S_w,\boldsymbol{\xi})\quad\text{with}\ \Lambda(S_n,\boldsymbol{\xi})\overset{\text{def}}{=}\frac{\alpha}{\beta}\,[1-S_w]-\boldsymbol{\xi} \tag{3}$$

Here $\tau>0$ is the relaxation time. The model is completed as follows. By the definition of saturations, one has $S_w+S_n=1$ with $S_w,S_n\geqslant 0$. Then the curvature of the contact surface between the two fluids links the jump of pressure of two phases to the saturation by the capillary pressure law: $P_{\mathrm{c}}(S_w,\boldsymbol{\xi})=p_n-p_w$, where (see, e.g., [1]) the capillary pressure function has the form:

$$\Phi P_{\mathrm{c}}(S_w,\boldsymbol{\xi})\overset{\text{def}}{=}\gamma+M\,[1-S_w]-\alpha\,\boldsymbol{\xi} \tag{4}$$

Here $M, \gamma > 0$ are constitutive parameters of the model. Finally, we introduce the mobility functions λ_κ, which will be widely used below. They are defined as: $\lambda_\kappa(S_\kappa, \xi) \stackrel{\text{def}}{=} f_\kappa(S_\kappa, \xi)/\mu_\kappa$ $(\kappa = w, n)$.

Now we discuss in more details the definitions and the properties of the capillary pressure and of the mobility functions. We also formulate the conditions on the constitutive parameters of the model. We start the analysis by establishing the explicit dependence of the non-equilibrium parameter on the wetting saturation function. Namely, denoting $S \stackrel{\text{def}}{=} S_w$, one can easily show that

$$\xi = \xi^{\text{init}}(x)\, \mathrm{e}^{-t/\tau} + \frac{\alpha}{\tau\beta} \int_0^t \mathrm{e}^{(\varsigma - t)/\tau} \big(1 - S(x, \varsigma)\big)\, \mathrm{d}\varsigma \quad \text{with } \xi^{\text{init}}(x) \stackrel{\text{def}}{=} \xi(x, 0) > 0 \tag{5}$$

Consider the capillary pressure function. The initial boundary value problem for the two-phase filtration is well posed if and only if the capillary pressure function P_{c} is a decreasing function of the saturation S. In order to prove this fact, we often deal with the derivative of the parameter ξ with respect to S. This derivative involves the function $\xi'_S \stackrel{\text{def}}{=} \frac{\partial \xi}{\partial S}(x, 0)$. From now on, for the sake of definiteness, we assume that $\xi'_S = \xi'_S(x, 0) \geqslant 0$ in Ω, where Ω is our reservoir of interest. We have the following result.

Lemma 2.1. *Let the function $\xi'_S \geqslant 0$ satisfy the bound $\max_{x\in\Omega} \xi'_S(x, 0) < +\infty$ in Ω and let M, α, β be such that $M > 2\alpha^2/\beta$. Then the function P_{c} is a positive decreasing function of S.*

The proof of the lemma is based on the application of the kinetic equation (3).

Now, we turn to the mathematically rigorous definition and the properties of the mobility functions λ_w, λ_n in the non-equilibrium case. Let us recall that for an equilibrium two-phase flow in porous medium (see, e.g., [9,13], and the references therein) the standard assumptions on the mobility functions are: $0 \leqslant \lambda^{\mathrm{e}}_w(S), \lambda^{\mathrm{e}}_n(1 - S) \leqslant 1$ for $S \in [0, 1]$ and $\lambda^{\mathrm{e}}_w(S = 0) = 0$, $\lambda^{\mathrm{e}}_w(S = 1) = 1$ and $\lambda^{\mathrm{e}}_n(S = 0) = 1$, $\lambda^{\mathrm{e}}_n(S = 1) = 0$. Here S stands for the wetting phase saturation in the equilibrium case. Our goal now is to establish similar properties of the mobility functions, which depend, in the non-equilibrium case, both on the real saturation S and on the non-equilibrium parameter ξ. To this end, it is natural to introduce a new non-equilibrium parameter ϑ given by:

$$\vartheta \stackrel{\text{def}}{=} 2\, S + \beta\, \xi/\alpha - 1 \tag{6}$$

and to consider the properties of the mobility functions in terms of this parameter. As functions of ϑ, the mobilities become: $\lambda_w(S, \xi) = \lambda^{\mathrm{e}}_w(\vartheta)$ and $\lambda_n(S, \xi) = \lambda^{\mathrm{e}}_n(1 - \vartheta)$. We have:

Lemma 2.2. *Let ϑ be the parameter defined in (6). Assume that $0 < \max_\Omega \frac{\beta}{\alpha}\, \xi^{\text{init}}(x) < 1$ in Ω. Then we have: **(i)** There are the values of the saturation S, denoted by $S_{\vartheta=0}$ and $S_{\vartheta=1}$, such that*

$$\vartheta = 0 \quad \textit{for} \quad S_{\vartheta=0} \stackrel{\text{def}}{=} \frac{\mathrm{e}^{-t/(2\tau)}}{2} \left(1 - \frac{\beta}{\alpha}\, \xi^{\text{init}}\right) \qquad \textit{and} \qquad \vartheta = 1 \quad \textit{for} \quad S_{\vartheta=1} \stackrel{\text{def}}{=} 1 - \frac{\beta}{2\alpha} \xi^{\text{init}}(x)\, \mathrm{e}^{-t/(2\tau)} \tag{7}$$

***(ii)** The values $S_{\vartheta=0}, S_{\vartheta=1}$ are such that $0 < S_{\vartheta=0} < S_{\vartheta=1} < 1$ and $S_{\vartheta=0} \to 0$, $S_{\vartheta=1} \to 1$ as $t \to +\infty$.*

In order to prove the lemma, one have to solve a Volterra nonhomogeneous equation coming from the representation (5) of the non-equilibrium parameter ξ.

The behavior of the relative phase permeabilities in various processes is depicted in Figs. 1 and 2, where the relative phase permeabilities of the wetting and nonwetting fluids are given as the functions of the wetting fluids' saturation S_w and the parameter $\widehat{\xi} \stackrel{\text{def}}{=} \beta\xi/\alpha$ varying in the interval $[0, 1]$. In the case of slow quasi-static processes, the variation of the relative phase permeabilities occurs along the equilibrium curves shown in Figs. 1 and 2 by the bold line. This case corresponds to the Muscat–Leveret model. The non-equilibrium dependencies are determined by the history of the saturation variation. In the case of a weak non-equilibrium $\tau/t_{\text{ref}} \ll 1$ (t_{ref} is the reference time value), they lie in a small neighborhood of the equilibrium curves $f^{\mathrm{e}}_w(S_w)$, $f^{\mathrm{e}}_n(S_w)$. In these figures, the examples of non-equilibrium dependencies corresponding to the drainage with the model saturation $S(t) = 0.4 + 0.6\exp(-t)$ and imbibition with the saturation $S(t) = 0.6\,(1 - \exp(-t))$, respectively, are depicted for different relaxation times $\tau = 0.1, 0.2, 0.5, 1.5$ (the characteristic time for which the variation of the saturation is equal to one) by the curves with arrows, which show the direction of the corresponding process.

Now, let us study the dependance of the parameter ϑ on the saturation S. We have:

Lemma 2.3. *Let ϑ be the parameter defined by (6). Then ϑ is an increasing function of S.*

The proof of the lemma is based on the positiveness of the function ξ'_S.

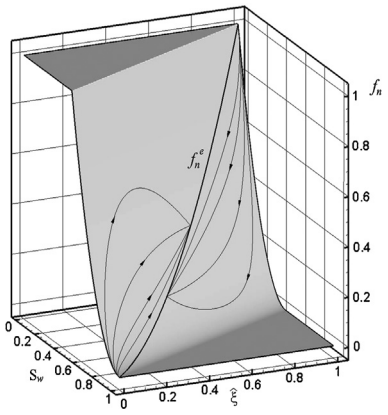

Fig. 1. Relative permeability of the nonwetting fluid as the function of $\widehat{\xi}$ and S_w.

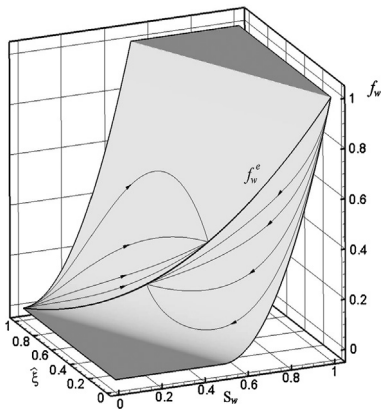

Fig. 2. Relative permeability of the wetting fluid as the function of $\widehat{\xi}$ and S_w.

Thus we conclude that, with the following assumptions on the constitutive parameters:

$$\boldsymbol{\xi}'_S(x,0) \geqslant 0, \quad M > 2\alpha^2/\beta, \quad \text{and} \quad 0 < \max_{\Omega}\left(\beta\,\boldsymbol{\xi}^{\text{init}}(x)/\alpha\right) < 1 \quad \text{in } \Omega \tag{8}$$

we have that: **(i)** the capillary pressure is a decreasing function of the saturation S; **(ii)** the parameter $\boldsymbol{\vartheta}$ equals 0 and 1 for $S_{\boldsymbol{\vartheta}=0}$ and $S_{\boldsymbol{\vartheta}=1}$ given by (7), and **(iii)** the parameter $\boldsymbol{\vartheta}$ is an increasing function of S.

Now let us explain how do we understand the mobility functions λ_w, λ_n in our further analysis. We set:

$$\lambda_w(S,\boldsymbol{\xi}) := \begin{cases} 1, & \text{when } S > S_{\boldsymbol{\vartheta}=1}; \\ \lambda_w^{\text{e}}(\boldsymbol{\vartheta}) & \text{when } S \in I_{\boldsymbol{\vartheta}}; \\ 0, & \text{when } S < S_{\boldsymbol{\vartheta}=0} \end{cases} \quad \text{and} \quad \lambda_n(S,\boldsymbol{\xi}) := \begin{cases} 1, & \text{when } S < S_{\boldsymbol{\vartheta}=0}; \\ \lambda_n^{\text{e}}(1-\boldsymbol{\vartheta}) & \text{when } S \in I_{\boldsymbol{\vartheta}}; \\ 0, & \text{when } S > S_{\boldsymbol{\vartheta}=1} \end{cases} \tag{9}$$

where $I_{\boldsymbol{\vartheta}} \stackrel{\text{def}}{=} [S_{\boldsymbol{\vartheta}=0}, S_{\boldsymbol{\vartheta}=1}]$ stands for our interval of interest. This definition is depicted in Figs. 1, 2 above.

3. The global Kondaurov double-porosity model

In this section we formulate the mesoscopic flow equations of the Kondaurov model and then introduce the homogenized model obtained earlier in [1]. We consider a reservoir $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$, which is assumed to be a bounded, connected domain with a periodic structure (see Fig. 3). More precisely, we will scale this periodic structure by a parameter ε which represents the ratio of the cell size to the whole region Ω and we assume that $\varepsilon \downarrow 0$. Let $Y \stackrel{\text{def}}{=} (0,1)^d$ be a basic cell of a fractured porous medium. We assume that Y is made up of two homogeneous porous media Y_{m} and Y_{f} corresponding to the parties of the mesoscopic domain occupied by the matrix block and the fracture, respectively. Thus $Y = Y_{\text{m}} \cup Y_{\text{f}} \cup \Gamma_{\text{fm}}$, where Γ_{fm} denotes the interface between the two media. Let Ω_ℓ^ε with $\ell =$ "f" or "m" denotes the open set corresponding to the porous medium with index ℓ. Then $\Omega = \Omega_{\text{m}}^\varepsilon \cup \Gamma_{\text{fm}}^\varepsilon \cup \Omega_{\text{f}}^\varepsilon$, where $\Gamma_{\text{fm}}^\varepsilon \stackrel{\text{def}}{=} \partial\Omega_{\text{f}}^\varepsilon \cap \partial\Omega_{\text{m}}^\varepsilon \cap \Omega$ and the subscripts "m", "f" refer to the matrix and fracture, respectively.

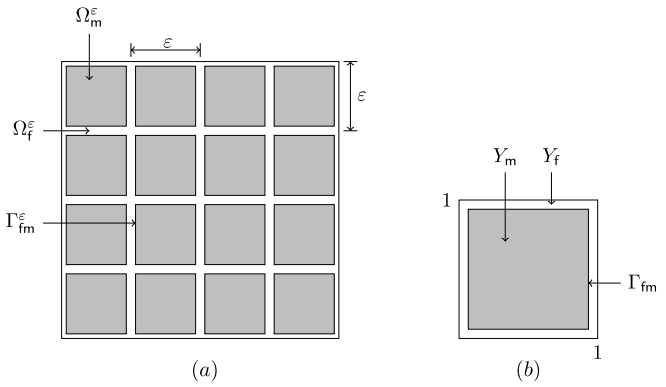

Fig. 3. (a) The domain Ω with the mesostructure. (b) The reference cell Y.

Before describing the equations of the model (1) for the nonhomogeneous porous medium Ω with the periodic structure, we give the corresponding notation and also define the porosity function and the global permeability tensor adopted to the double porosity medium Ω. We have that $\Phi^\varepsilon(x) = \Phi(\frac{x}{\varepsilon})$ is the porosity of the reservoir Ω. The function Φ^ε is a Y-periodic defined by: $\Phi^\varepsilon(x) \stackrel{\text{def}}{=} \Phi_{\mathsf{f}}\mathbf{1}^\varepsilon_{\mathsf{f}}(x) + \Phi_{\mathsf{m}}\mathbf{1}^\varepsilon_{\mathsf{m}}(x)$, where $\mathbf{1}^\varepsilon_{\mathsf{f}}, \mathbf{1}^\varepsilon_{\mathsf{m}}$ are the characteristic functions of the media $\Omega^\varepsilon_{\mathsf{f}}, \Omega^\varepsilon_{\mathsf{m}}$, respectively, and where the constants $0 < \Phi_{\mathsf{f}}, \Phi_{\mathsf{m}} < 1$ do not depend on ε; $K^\varepsilon(x) = K(\frac{x}{\varepsilon})$ is the absolute permeability tensor of Ω it is defined by: $K^\varepsilon(x) \stackrel{\text{def}}{=} K_{\mathsf{f}}\mathbf{1}^\varepsilon_{\mathsf{f}}(x) + \check{\mathsf{e}}^2 K_{\mathsf{m}}\mathbf{1}^\varepsilon_{\mathsf{m}}$, where $0 < K_{\mathsf{f}}, K_{\mathsf{m}} < +\infty$ are positive constants that do not depend on ε; $S^\varepsilon_{\ell,w} = S^\varepsilon_{\ell,w}(x,t)$, $S^\varepsilon_{\ell,n} = S^\varepsilon_{\ell,n}(x,t)$ are the saturations of wetting and nonwetting fluids in Ω^ε_ℓ, respectively; $p^\varepsilon_{\ell,w} = p^\varepsilon_{\ell,w}(x,t)$, $p^\varepsilon_{\ell,n} = p^\varepsilon_{\ell,n}(x,t)$ are the pressures of wetting and nonwetting fluids in Ω^ε_ℓ, respectively; $\boldsymbol{\xi}^\varepsilon_\ell = \boldsymbol{\xi}^\varepsilon_\ell(x,t)$ is the non-equilibrium parameter in the medium Ω^ε_ℓ; $\lambda_{\ell,w}, \lambda_{\ell,n}$ are the mobilities of wetting and nonwetting fluids in Ω^ε_ℓ, respectively; τ_ℓ is the relaxation time in Ω^ε_ℓ; $\alpha_\ell, \beta_\ell, \gamma_\ell, M_\ell > 0$ denote the constitutive parameters in Ω^ε_ℓ which do not depend on ε. Denoting $S^\varepsilon_\ell \stackrel{\text{def}}{=} S^\varepsilon_{\ell,w}$, we obtain the following flow equations:

$$\begin{cases} \Phi^\varepsilon(x)\dfrac{\partial \mathsf{S}^\varepsilon}{\partial t} - \operatorname{div}\left\{K^\varepsilon(x)\lambda_w\left(\dfrac{x}{\varepsilon}, \mathsf{S}^\varepsilon, \boldsymbol{\xi}^\varepsilon\right)\nabla \mathsf{p}^\varepsilon_w\right\} = 0 \quad \text{in } \Omega_T; \\ -\Phi^\varepsilon(x)\dfrac{\partial \mathsf{S}^\varepsilon}{\partial t} - \operatorname{div}\left\{K^\varepsilon(x)\lambda_n\left(\dfrac{x}{\varepsilon}, \mathsf{S}^\varepsilon, \boldsymbol{\xi}^\varepsilon\right)\nabla \mathsf{p}_n\right\} = 0 \quad \text{in } \Omega_T; \\ P^\varepsilon_c\left(\dfrac{x}{\varepsilon}, \mathsf{S}^\varepsilon, \boldsymbol{\xi}^\varepsilon\right) = \mathsf{p}^\varepsilon_n - \mathsf{p}^\varepsilon_w \text{ with } \Phi^\varepsilon(x)P^\varepsilon_c\left(\dfrac{x}{\varepsilon}, \mathsf{S}^\varepsilon, \boldsymbol{\xi}^\varepsilon\right) \stackrel{\text{def}}{=} \gamma^\varepsilon(x) + M^\varepsilon(x)\left[1 - \mathsf{S}^\varepsilon\right] - \alpha^\varepsilon(x)\boldsymbol{\xi}^\varepsilon \end{cases} \tag{10}$$

where $\Omega_T \stackrel{\text{def}}{=} \Omega \times (0,T)$ ($T > 0$ is fixed); the mobilities $\lambda_{\ell,w}, \lambda_{\ell,n}$ are defined (in accordance with (9)) by:

$$\lambda_{\ell,w}(S^\varepsilon_\ell, \boldsymbol{\xi}^\varepsilon_\ell) = \lambda_{\ell,w}\left(2\,S^\varepsilon_\ell - 1 + \beta_\ell\,\boldsymbol{\xi}^\varepsilon_\ell/\alpha_\ell\right) \quad \text{and} \quad \lambda_{\ell,n}(S^\varepsilon_\ell, \boldsymbol{\xi}^\varepsilon_\ell) = \lambda_{\ell,n}\left(2\left[1 - S^\varepsilon_\ell\right] - \beta_\ell\,\boldsymbol{\xi}^\varepsilon_\ell/\alpha_\ell\right) \tag{11}$$

and each function $u^\varepsilon := \mathsf{S}^\varepsilon, \mathsf{p}^\varepsilon_w, \mathsf{p}^\varepsilon_n, \boldsymbol{\xi}^\varepsilon$ as well as the piece-wise constant functions $\Phi^\varepsilon, K^\varepsilon, \gamma^\varepsilon, M^\varepsilon, \alpha^\varepsilon$ are defined as: $u^\varepsilon \stackrel{\text{def}}{=} u^\varepsilon_{\mathsf{f}}\mathbf{1}^\varepsilon_{\mathsf{f}}(x) + u^\varepsilon_{\mathsf{m}}\mathbf{1}^\varepsilon_{\mathsf{m}}(x)$. The system (10) is completed by the corresponding interface and initial conditions, which are omitted here for the sake of brevity (for more details, see [1]).

Now we introduce the global non-equilibrium Kondaurov flow model obtained by the method of two-scale asymptotic expansions (see, e.g., [14,26–28]) in Section 4.2 of [1]. Here we also restrict ourselves to a special case of the homogenized model. Namely, as in [8], we consider the non-equilibrium effects for the matrix part only and not for the fracture system, which is related to the fact that the non-equilibrium effects for fractures, due to their high permeabilities and, consequently, low capillary forces, are negligible. First, we introduce the notation: S, P_w, P_n denote the homogenized wetting liquid saturation, the wetting and nonwetting liquid pressures, respectively; $\Phi^\star$ denotes the effective porosity and is given by: $\Phi^\star \stackrel{\text{def}}{=} \Phi_{\mathsf{f}}|Y_{\mathsf{f}}|/|Y_{\mathsf{m}}|$, where $|Y_\ell|$ is the measure of the set Y_ℓ ($\ell = \mathsf{f}, \mathsf{m}$); $\mathbb{K}^\star$ is the homogenized tensor with the entries

$$\mathbb{K}^\star_{ij} \stackrel{\text{def}}{=} \frac{K_{\mathsf{f}}}{|Y_{\mathsf{m}}|}\int\limits_{Y_{\mathsf{f}}}\left[\nabla_y\zeta_i + \boldsymbol{e}_i\right]\left[\nabla_y\zeta_j + \boldsymbol{e}_j\right]\mathrm{d}y, \text{ where } \zeta_j \text{ satisfies: } \begin{cases} -\Delta_y\,\zeta_j = 0 \text{ in } Y_{\mathsf{f}}; \\ \nabla_y\zeta_j \cdot \boldsymbol{\nu}_y = -\boldsymbol{e}_j \cdot \boldsymbol{\nu}_y \text{ on } \Gamma_{\mathsf{fm}} \\ y \mapsto \zeta_j(y) \quad Y\text{-periodic} \end{cases} \tag{12}$$

Then the homogenized system has the form:

$$\begin{cases} \Phi^\star \dfrac{\partial S}{\partial t} - \mathrm{div}_x \left\{ \mathbb{K}^\star \lambda_{\mathrm{f},w}(S) \nabla P_w \right\} = \mathcal{Q}_w \quad \text{in } \Omega_T; \\ -\Phi^\star \dfrac{\partial S}{\partial t} - \mathrm{div}_x \left\{ \mathbb{K}^\star \lambda_{\mathrm{f},n}(1-S) \nabla P_n \right\} = \mathcal{Q}_n \quad \text{in } \Omega_T; \\ P_c(S) = P_n - P_w \text{ with } \Phi_\mathrm{f} P_c(S) \overset{\mathrm{def}}{=} \mathsf{a}_{\mathrm{f},1} S + \mathsf{a}_{\mathrm{f},3} \quad \text{in } \Omega_T \end{cases} \tag{13}$$

where the constants $\mathsf{a}_{\mathrm{f},j}$ $(j = 1, 2, 3)$ in view of condition (8) are defined as:

$$\mathsf{a}_{\ell,1} \overset{\mathrm{def}}{=} -\left(M_\ell - \alpha_\ell^2/\beta_\ell \right) < 0, \quad \mathsf{a}_{\ell,2} \overset{\mathrm{def}}{=} \tau_\ell \left(M_\ell - 2\alpha_\ell^2/\beta_\ell \right) > 0, \ \mathsf{a}_{\ell,3} \overset{\mathrm{def}}{=} \gamma_\ell + M_\ell - \alpha_\ell^2/\beta_\ell > 0 \ (\ell = \mathrm{f}, \mathrm{m}) \tag{14}$$

Remark 1. Notice that the functions S, P_w, P_n appearing in (13) are, in fact, zero-order terms in the asymptotic expansions for the saturations S_f^ε, and phase pressures $p_{\mathrm{f},w}^\varepsilon$, $p_{\mathrm{f},n}^\varepsilon$ in the fracture domain $\Omega_\mathrm{f}^\varepsilon$ (for more details, see formulae (3.3)–(3.4) and the beginning of Section 3.2 in [1]). In a similar way, we introduce below the functions s, p_w, p_n in (15) for the matrix block.

For almost all point $x \in \Omega$, the equations for flow in a matrix block are given by:

$$\begin{cases} \Phi_\mathrm{m} \dfrac{\partial s}{\partial t} - \mathrm{div}_y \left\{ K_\mathrm{m} \lambda_{\mathrm{m},w}(\boldsymbol{\vartheta}_\mathrm{m}) \nabla_y p_w \right\} = 0 \quad \text{in } Y_\mathrm{m} \times \Omega_T; \\ -\Phi_\mathrm{m} \dfrac{\partial s}{\partial t} - \mathrm{div}_y \left\{ K_\mathrm{m} \lambda_{\mathrm{m},n}(1-\boldsymbol{\vartheta}_\mathrm{m}) \nabla_y p_n \right\} = 0 \quad \text{in } Y_\mathrm{m} \times \Omega_T; \\ p_c\left(\boldsymbol{\vartheta}_\mathrm{m}, \dfrac{\partial \boldsymbol{\vartheta}_\mathrm{m}}{\partial t} \right) = p_n - p_w \text{ with } \Phi_\mathrm{m} p_c\left(\boldsymbol{\vartheta}_\mathrm{m}, \dfrac{\partial \boldsymbol{\vartheta}_\mathrm{m}}{\partial t} \right) \overset{\mathrm{def}}{=} \mathsf{a}_{\mathrm{m},1} \boldsymbol{\vartheta}_\mathrm{m} + \mathsf{a}_{\mathrm{m},2} \dfrac{\partial \boldsymbol{\vartheta}_\mathrm{m}}{\partial t} + \mathsf{a}_{\mathrm{m},3}; \\ p_w(x,y,t) = P_w(x,t) \quad \text{and} \quad p_n(x,y,t) = P_n(x,t) \quad \text{on } \Gamma_{\mathrm{fm}} \times \Omega_T \end{cases} \tag{15}$$

Here we make use of the following notation: s, p_w, p_n denote the wetting liquid saturation, the wetting and nonwetting liquid pressures in the matrix block Y_m, respectively; $\boldsymbol{\xi}_\mathrm{m}$ denotes the local non-equilibrium parameter in the matrix block Y_m, it is defined as the solution to the following equation:

$$\frac{\partial \boldsymbol{\xi}_\mathrm{m}}{\partial t} = \frac{1}{\tau_\mathrm{m}} \Lambda(s, \boldsymbol{\xi}_\mathrm{m}) \quad \text{with } \Lambda(s, \boldsymbol{\xi}_\mathrm{m}) \overset{\mathrm{def}}{=} \frac{\alpha_\mathrm{m}}{\beta_\mathrm{m}} \left[1 - s \right] - \boldsymbol{\xi}_\mathrm{m} \tag{16}$$

the parameter $\boldsymbol{\vartheta}_\mathrm{m}$ is defined by: $\boldsymbol{\vartheta}_\mathrm{m} \overset{\mathrm{def}}{=} 2s + \beta_\mathrm{m} \boldsymbol{\xi}_\mathrm{m}/\alpha_\mathrm{m} - 1$. For any $x \in \Omega$ and $t > 0$, the matrix-fracture sources are given by:

$$\mathcal{Q}_w \overset{\mathrm{def}}{=} -\frac{\Phi_\mathrm{m}}{|Y_\mathrm{m}|} \int\limits_{Y_\mathrm{m}} \frac{\partial s}{\partial t}(x, y, t) \, \mathrm{d}y = -\mathcal{Q}_n \tag{17}$$

Remark 2. Notice that in the case of the equilibrium flow, from (16), we have that $\boldsymbol{\xi}_\mathrm{m} = \frac{\alpha_\mathrm{m}}{\beta_\mathrm{m}} \left[1 - s \right]$ for $\tau_\mathrm{m} = 0$. Then the macroscopic model (13)–(17) is exactly (evidently, with a specified capillary pressure) the well-known homogenized double-porosity model for the immiscible incompressible two-phase flow in porous media considered by many authors (see, e.g., [12,14,25] and the references therein).

4. Non-equilibrium matrix imbibition equation

Let us recall that when a porous medium filled with some fluid is brought into contact with another fluid that preferentially wets the medium, there is a spontaneous flow of the wetting fluid into the medium and a counterflow of the resident fluid from the medium. This phenomenon is called imbibition and arises in physical situations involving multiphase flow systems (see, e.g., [29]).

The goal of this section is to reduce the local problem (15) formulated in terms of the phase pressures to a new problem, for a *non-equilibrium imbibition equation* given in terms of the real saturation s and the parameter $\boldsymbol{\vartheta}_\mathrm{m}$ which is, in fact, the functional of s. To this end, let us rewrite the capillary pressure function given in $(15)_3$ as follows:

$$p_c\left(\boldsymbol{\vartheta}_\mathrm{m}, \frac{\partial \boldsymbol{\vartheta}_\mathrm{m}}{\partial t} \right) = \pi_c(\boldsymbol{\vartheta}_\mathrm{m}) + \widehat{\mathsf{a}}_{\mathrm{m},2} \frac{\partial \boldsymbol{\vartheta}_\mathrm{m}}{\partial t} \quad \text{with } \pi_c(\boldsymbol{\vartheta}_\mathrm{m}) \overset{\mathrm{def}}{=} \widehat{\mathsf{a}}_{\mathrm{m},1} \boldsymbol{\vartheta}_\mathrm{m} + \widehat{\mathsf{a}}_{\mathrm{m},3} \quad \text{and} \quad \widehat{\mathsf{a}}_{\mathrm{m},j} \overset{\mathrm{def}}{=} \mathsf{a}_{\mathrm{m},j}/\Phi_\mathrm{m} \tag{18}$$

Inspired by [9], we introduce the notion of *non-equilibrium global pressure* P, which is a generalization of the global pressure function defined earlier (see, e.g., [22–24]) in the equilibrium case:

$$p_w \overset{\text{def}}{=} \mathsf{P} + \mathsf{G}_w(\vartheta_\mathsf{m}) + \widehat{\mathsf{a}}_{\mathsf{m},2}\,\mathcal{F}_w\left(\vartheta_\mathsf{m}, \frac{\partial \vartheta_\mathsf{m}}{\partial t}\right) \quad \text{and} \quad p_n \overset{\text{def}}{=} \mathsf{P} + \mathsf{G}_n(\vartheta_\mathsf{m}) + \widehat{\mathsf{a}}_{\mathsf{m},2}\,\mathcal{F}_n\left(\vartheta_\mathsf{m}, \frac{\partial \vartheta_\mathsf{m}}{\partial t}\right) \tag{19}$$

where the functions $\mathsf{G}_w, \mathsf{G}_n, \mathcal{F}_w, \mathcal{F}_n$ will be specified later using several conditions. Roughly speaking, these conditions are a consequence of the definition of capillary pressure function $(15)_3$. First, we define the functions $\mathsf{G}_w, \mathsf{G}_n$. Namely, the function $\mathsf{G}_n(\vartheta_\mathsf{m})$ we choose in the following way:

$$\mathsf{G}_n(\vartheta_\mathsf{m}) \overset{\text{def}}{=} \int_0^{\vartheta_\mathsf{m}} \frac{\lambda_{\mathsf{m},w}(\varsigma)}{\lambda_\mathsf{m}(\varsigma)}\,\pi_\mathsf{c}'(\varsigma)\,\mathrm{d}\varsigma \quad \text{with } \lambda_\mathsf{m}(\vartheta_\mathsf{m}) \overset{\text{def}}{=} \lambda_{\mathsf{m},w}(\vartheta_\mathsf{m}) + \lambda_{\mathsf{m},n}(\vartheta_\mathsf{m}) \tag{20}$$

From now on, $\lambda_{\mathsf{m},n}(\vartheta_\mathsf{m}) := \lambda_{\mathsf{m},n}(1-\vartheta_\mathsf{m})$ and π_c' denotes the derivative of the function π with respect to its argument. Notice that the standard assumption on the function λ_m (see, e.g., [9] and the references herein) is that there exists a strictly positive constant L_0 such that $\lambda_\mathsf{m}(\varsigma) \geqslant L_0 > 0$ in $\varsigma \in [0,1]$. Now, taking into account that $\pi_\mathsf{c}'(\varsigma) = \widehat{\mathsf{a}}_{\mathsf{m},1}$, where $\widehat{\mathsf{a}}_{\mathsf{m},1} < 0$ (see (14)), from (20) we get:

$$\mathsf{G}_n(\vartheta_\mathsf{m}) = \widehat{\mathsf{a}}_{\mathsf{m},1} \int_0^{\vartheta_\mathsf{m}} \frac{\lambda_{\mathsf{m},w}(\varsigma)}{\lambda_\mathsf{m}(\varsigma)}\,\mathrm{d}\varsigma \quad \text{with } \nabla_y \mathsf{G}_n(\vartheta_\mathsf{m}) = \widehat{\mathsf{a}}_{\mathsf{m},1}\,\frac{\lambda_{\mathsf{m},w}(\vartheta_\mathsf{m})}{\lambda_\mathsf{m}(\vartheta_\mathsf{m})}\,\nabla_y \vartheta_\mathsf{m} \tag{21}$$

The function G_w is then defined by $\mathsf{G}_w(\vartheta_\mathsf{m}) \overset{\text{def}}{=} \mathsf{G}_n(\vartheta_\mathsf{m}) - \pi_\mathsf{c}(\vartheta_\mathsf{m})$. This implies the following formula for the gradient of the function G_w:

$$\nabla_y \mathsf{G}_w(\vartheta_\mathsf{m}) = -\frac{\lambda_{\mathsf{m},n}(\vartheta_\mathsf{m})}{\lambda_\mathsf{m}(\vartheta_\mathsf{m})}\pi_\mathsf{c}'(\vartheta_\mathsf{m})\,\nabla_y \vartheta_\mathsf{m} = -\widehat{\mathsf{a}}_{\mathsf{m},1}\,\frac{\lambda_{\mathsf{m},n}(\vartheta_\mathsf{m})}{\lambda_\mathsf{m}(\vartheta_\mathsf{m})}\nabla_y \vartheta_\mathsf{m} \tag{22}$$

We notice that $\lambda_{\mathsf{m},w}(\vartheta_\mathsf{m})\nabla_y \mathsf{G}_w(\vartheta_\mathsf{m}) = \mathfrak{a}(\vartheta_\mathsf{m})\nabla_y \vartheta_\mathsf{m}$ and $\lambda_{\mathsf{m},n}(\vartheta_\mathsf{m})\nabla_y \mathsf{G}_n(\vartheta_\mathsf{m}) = -\mathfrak{a}(\vartheta_\mathsf{m})\nabla_y \vartheta_\mathsf{m}$, where

$$\mathfrak{a}(\vartheta_\mathsf{m}) \overset{\text{def}}{=} |\widehat{\mathsf{a}}_{\mathsf{m},1}|\,\frac{\lambda_{\mathsf{m},n}(\vartheta_\mathsf{m})\,\lambda_{\mathsf{m},w}(\vartheta_\mathsf{m})}{\lambda_\mathsf{m}(\vartheta_\mathsf{m})} \tag{23}$$

Let us introduce the following function:

$$\mathfrak{b}(\vartheta_\mathsf{m}) \overset{\text{def}}{=} \int_0^{\vartheta_\mathsf{m}} \mathfrak{a}(\varsigma)\,\mathrm{d}\varsigma = |\widehat{\mathsf{a}}_{\mathsf{m},1}| \int_0^{\vartheta_\mathsf{m}} \frac{\lambda_{\mathsf{m},n}(\varsigma)\,\lambda_{\mathsf{m},w}(\varsigma)}{\lambda_\mathsf{m}(\varsigma)}\,\mathrm{d}\varsigma \tag{24}$$

Then taking into account the definition of the function $\mathfrak{b}$ we have:

$$\lambda_{\mathsf{m},w}(\vartheta_\mathsf{m})\nabla_y p_w = \lambda_{\mathsf{m},w}(\vartheta_\mathsf{m})\nabla_y \mathsf{P} + \nabla_y \mathfrak{b}(\vartheta_\mathsf{m}) + \lambda_{\mathsf{m},w}(\vartheta_\mathsf{m})\,\widehat{\mathsf{a}}_{\mathsf{m},2}\,\nabla_y \mathcal{F}_w\left(\vartheta_\mathsf{m}, \frac{\partial \vartheta_\mathsf{m}}{\partial t}\right) \tag{25}$$

$$\lambda_{\mathsf{m},n}(\vartheta_\mathsf{m})\nabla_y p_n = \lambda_{\mathsf{m},n}(\vartheta_\mathsf{m})\nabla_y \mathsf{P} - \nabla_y \mathfrak{b}(\vartheta_\mathsf{m}) + \lambda_{\mathsf{m},n}(\vartheta_\mathsf{m})\,\widehat{\mathsf{a}}_{\mathsf{m},2}\,\nabla_y \mathcal{F}_n\left(\vartheta_\mathsf{m}, \frac{\partial \vartheta_\mathsf{m}}{\partial t}\right) \tag{26}$$

Now, we turn to the functions $\mathcal{F}_w, \mathcal{F}_n$. The relation $(15)_3$ along with the previous assumptions on the functions $\mathsf{G}_w, \mathsf{G}_n$ leads to the following condition:

$$\mathcal{F}_n\left(\vartheta_\mathsf{m}, \frac{\partial \vartheta_\mathsf{m}}{\partial t}\right) - \mathcal{F}_w\left(\vartheta_\mathsf{m}, \frac{\partial \vartheta_\mathsf{m}}{\partial t}\right) = \frac{\partial \vartheta_\mathsf{m}}{\partial t} \tag{27}$$

Let us rewrite (15) in terms of the non-equilibrium global pressure P, saturation s, and the non-equilibrium parameter ϑ_m. From (25), (26), we get:

$$\Phi_\mathsf{m}\,\frac{\partial s}{\partial t} - K_\mathsf{m}\,\mathrm{div}_y\left\{\lambda_{\mathsf{m},w}(\vartheta_\mathsf{m})\nabla_y \mathsf{P} + \nabla_y \mathfrak{b}(\vartheta_\mathsf{m}) + \widehat{\mathsf{a}}_{\mathsf{m},2}\,\lambda_{\mathsf{m},w}(\vartheta_\mathsf{m})\,\nabla_y \mathcal{F}_w\left(\vartheta_\mathsf{m}, \frac{\partial \vartheta_\mathsf{m}}{\partial t}\right)\right\} = 0 \tag{28}$$

$$-\Phi_\mathsf{m}\,\frac{\partial s}{\partial t} - K_\mathsf{m}\,\mathrm{div}_y\left\{\lambda_{\mathsf{m},n}(\vartheta_\mathsf{m})\nabla_y \mathsf{P} - \nabla_y \mathfrak{b}(\vartheta_\mathsf{m}) + \widehat{\mathsf{a}}_{\mathsf{m},2}\,\lambda_{\mathsf{m},n}(\vartheta_\mathsf{m})\,\nabla_y \mathcal{F}_n\left(\vartheta_\mathsf{m}, \frac{\partial \vartheta_\mathsf{m}}{\partial t}\right)\right\} = 0 \tag{29}$$

We add the equations (28) and (29), to have:

$$-\mathrm{div}_y\left\{\lambda_\mathsf{m}(\vartheta_\mathsf{m})\nabla_y \mathsf{P} + \widehat{\mathsf{a}}_{\mathsf{m},2}\left[\lambda_{\mathsf{m},w}(\vartheta_\mathsf{m})\,\nabla_y \mathcal{F}_w\left(\vartheta_\mathsf{m}, \frac{\partial \vartheta_\mathsf{m}}{\partial t}\right) + \lambda_{\mathsf{m},n}(\vartheta_\mathsf{m})\,\nabla_y \mathcal{F}_n\left(\vartheta_\mathsf{m}, \frac{\partial \vartheta_\mathsf{m}}{\partial t}\right)\right]\right\} = 0 \tag{30}$$

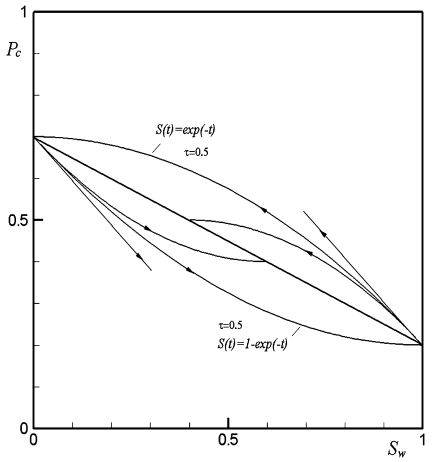

Fig. 4. Kondaurov's capillary pressure as the function of the wetting fluid saturation.

Then we can impose the second condition on the functions $\mathcal{F}_w, \mathcal{F}_n$. Namely, we set:

$$\lambda_{\mathsf{m},w}(\vartheta_{\mathsf{m}})\,\nabla_y \mathcal{F}_w\left(\vartheta_{\mathsf{m}}, \frac{\partial \vartheta_{\mathsf{m}}}{\partial t}\right) + \lambda_{\mathsf{m},n}(\vartheta_{\mathsf{m}})\,\nabla_y \mathcal{F}_n\left(\vartheta_{\mathsf{m}}, \frac{\partial \vartheta_{\mathsf{m}}}{\partial t}\right) = 0 \tag{31}$$

Now the simple calculations lead to the following result:

Lemma 4.1. *Let the functions $\mathcal{F}_w, \mathcal{F}_n$ satisfy the conditions (27) and (31). Then*

$$\nabla_y \mathcal{F}_w\left(\vartheta_{\mathsf{m}}, \frac{\partial \vartheta_{\mathsf{m}}}{\partial t}\right) = -\frac{\lambda_{\mathsf{m},n}(\vartheta_{\mathsf{m}})}{\lambda_{\mathsf{m}}(\vartheta_{\mathsf{m}})}\nabla_y \frac{\partial \vartheta_{\mathsf{m}}}{\partial t} \quad \textit{and} \quad \nabla_y \mathcal{F}_n\left(\vartheta_{\mathsf{m}}, \frac{\partial \vartheta_{\mathsf{m}}}{\partial t}\right) = \frac{\lambda_{\mathsf{m},w}(\vartheta_{\mathsf{m}})}{\lambda_{\mathsf{m}}(\vartheta_{\mathsf{m}})}\nabla_y \frac{\partial \vartheta_{\mathsf{m}}}{\partial t} \tag{32}$$

Lemma 4.1 implies that (30) becomes: $-\mathrm{div}_y\left\{\lambda_{\mathsf{m}}(\vartheta_{\mathsf{m}})\nabla_y \mathsf{P}\right\} = 0$. Then it follows from $(15)_4$ that the boundary conditions for the real saturation s as well as for the function P on the interface Γ_{fm} (see the beginning of Section 3 for the definition of Γ_{fm}) do not depend on the variable y. This fact allows us to prove the following result (see Lemma 1 in [25] for similar arguments).

Lemma 4.2. *The function P does not depend on the variable y, i.e. $\nabla_y \mathsf{P} = 0$ in $Y_{\mathsf{m}} \times \Omega_T$.*

Now, taking into account the results of Lemma 4.2, from equation (28) and (32), we obtain the following non-equilibrium imbibition equation. It reads:

$$\Phi_{\mathsf{m}}\frac{\partial s}{\partial t} - K_{\mathsf{m}}\,\mathrm{div}_y\left\{|\widehat{\mathsf{a}}_{\mathsf{m},1}|\,F(\vartheta_{\mathsf{m}})\,\nabla_y \vartheta_{\mathsf{m}} - \widehat{\mathsf{a}}_{\mathsf{m},2}\,F(\vartheta_{\mathsf{m}})\,\nabla_y \frac{\partial \vartheta_{\mathsf{m}}}{\partial t}\right\} = 0 \quad \text{in } Y_{\mathsf{m}} \times \Omega_T \tag{33}$$

where

$$F(\vartheta_{\mathsf{m}}) \stackrel{\text{def}}{=} \frac{\lambda_{\mathsf{m},n}(\vartheta_{\mathsf{m}})\,\lambda_{\mathsf{m},w}(\vartheta_{\mathsf{m}})}{\lambda_{\mathsf{m}}(\vartheta_{\mathsf{m}})} \tag{34}$$

Finally, due to $(15)_3$, equation (33) can be rewritten as:

$$\Phi_{\mathsf{m}}\frac{\partial s}{\partial t} + K_{\mathsf{m}}\,\mathrm{div}_y\left\{F(\vartheta_{\mathsf{m}})\,\nabla_y p_c\left(\vartheta_{\mathsf{m}}, \frac{\partial \vartheta_{\mathsf{m}}}{\partial t}\right)\right\} = 0 \tag{35}$$

The dependence of the non-equilibrium Kondaurov capillary pressure on the water saturation is given below in Fig. 4.

Thus the homogenized double porosity Kondaurov model contains the global equations (13) coupled with the boundary value problem for the non-equilibrium imbibition equation (35).

Remark 3. We arrive at the imbibition equation due to the fact that the total velocity in the block is zero. The only force, which plays the role here, is the capillary pressure and this enables to simplify total formulation (15), taking into account the countercurrent character of the imbibition process. In the mathematical literature, the advantages of the imbibition formulation for the equilibrium two-phase flow in double-porosity media was discussed for the first time in [30]. The computational time for the imbibition equation is much smaller, because it is a scalar parabolic equation and the calculation

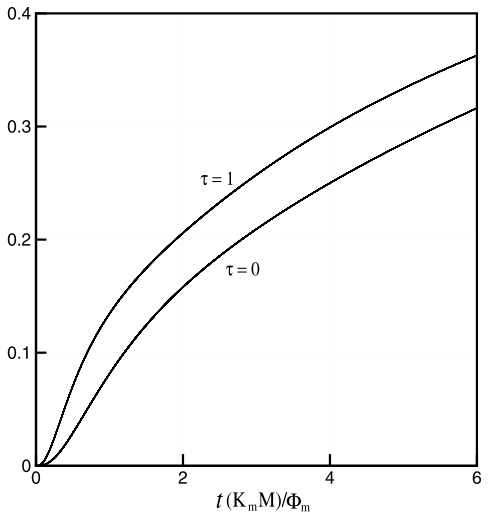

Fig. 5. The exchange term for equilibrium $\tau = 0$ and non-equilibrium $\tau = 1$ imbibition processes.

of the non-equilibrium parameter does not take much time (approximately 25% of total computational time in our case). In the full-phase formulation (15), in the framework of the widely used IMPES approach, an elliptic equation for the pressure should be solved at each time step of integration of the full system of equations. This elliptic equation is aimed to provide zero-flux condition at each time step. Depending on the computational grid and on numerical schemes, the phase formulation takes much more computational time (dozens or even more) in comparison with imbibition one. Our numerical calculations in the framework of IMPES method agree with this fact.

Remark 4. Notice that in contrast to the classical case (see, e.g., [25] and the references therein) or the case of the global Barenblatt model [31], equation (35) is integro-differential. This fact shows explicitly the impact of the capillary non-equilibrium on the mass exchange between the fracture system and blocks via the source terms $\mathcal{Q}_w, \mathcal{Q}_n$ in (13).

Remark 5. Notice that if $\tau_{\mathrm{m}} = 0$ (equilibrium state) then, as it was shown in Remark 2, $\vartheta_{\mathrm{m}} = s$ and, in addition, due to (14), $\widehat{\mathsf{a}}_{\mathrm{m},2} = 0$. Thus, equation (35) becomes:

$$\Phi_{\mathrm{m}} \frac{\partial s}{\partial t} - K_{\mathrm{m}} \Delta_y \mathfrak{b}(s) = 0 \quad \text{in } Y_{\mathrm{m}} \times \Omega_T$$

where the function $\mathfrak{b}$ is defined in (24). This is exactly (with evident modifications due to a special form of π_{c}) the classical imbibition equation in the equilibrium case (see, e.g., formula (24) in [25]).

The impact of the non-equilibrium phenomena is shown numerically in Figs. 5, 6, 7. In Fig. 5 for the test boundary condition $s = 1 - \exp(-t)$, the initial condition $s = 0$, and $\alpha/\beta = 1$, $\alpha/M = 0.8$, $\gamma = 1$, we calculate the exchange term, i.e. the quantity of the fluid phase which is displaced, as a function of the time variable. In Figs. 6 and 7, we present the solution to the imbibition equation for the equilibrium and non-equilibrium processes in the case of the test boundary condition given by $s(t) = 0.2 + 0.6\left[1 - \exp(-2t)\right]$.

5. Concluding remarks

In the framework of Kondaurov's formalism [20], a non-equilibrium porous medium saturated by two fluids is described by a dependence of the thermodynamical potential on a number of constitutive parameters. Using the relations that guarantee a thermodynamical model consistency, it is possible to calculate the capillary pressure function and the right-hand side of the kinetic equation. The first one determines the capillary-driven fluxes, and the second one describes the capillary relaxation processes. This approach has a number of advantages in contrast to Barenblatt's model (see, e.g., [15]) whose application is restricted to weakly non-equilibrium flows. Turning to the model considered in this Note, we observe that, in practice, the fractured porous medium is usually modeled by two-superimposed continua, a connected fracture system and a system of topologically disconnected matrix blocks (see, e.g., [1] and the reference therein). Therefore, we are facing a problem of description of a highly heterogeneous medium, where each block is described by Kondaurov's model. A distinctive feature of this model is as follows. It enables to take into account the impact of the capillary non-equilibrium on the mass exchange between the fissure system and the blocks. The analysis of the homogenized system carried out in this Note shows some new aspects, which are briefly discussed below. We focus our attention on two main points.

(i) Numerical aspects of Kondaurov's model. From the mathematical point of view, the double-porosity models like (13)–(15) are rather complex systems of PDE involving $(2d + 1)$ variables (x, y, t) instead of $(d + 1)$ for the initial meso-

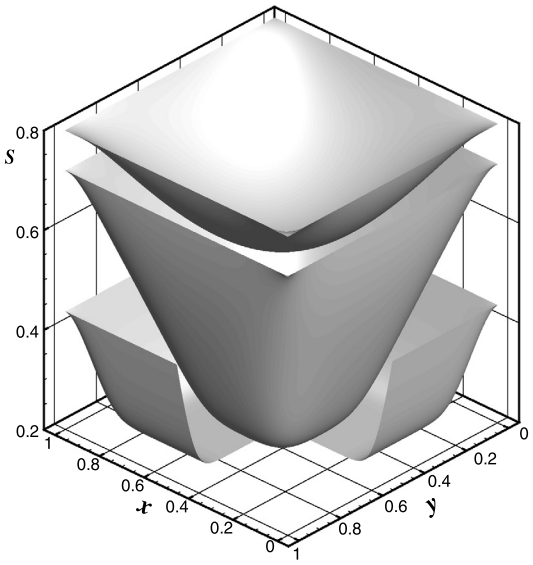

Fig. 6. The saturation function in the equilibrium case for $t = 1/16; 1/4; 1$.

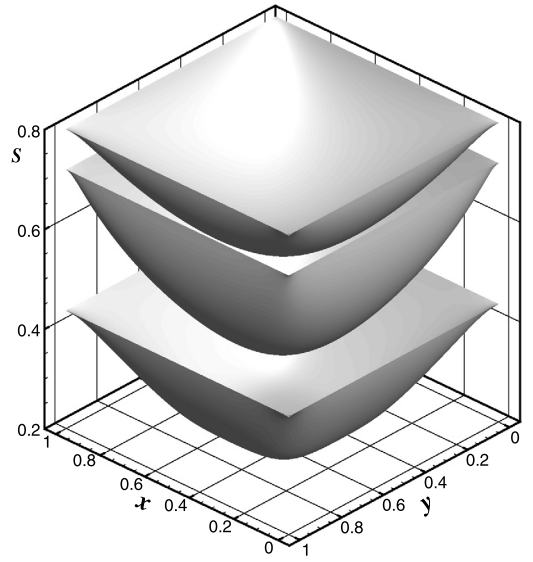

Fig. 7. The saturation function in the non-equilibrium case for $t = 1/16; 1/4; 1$.

scopic system. However, we know (see, e.g., Ch. 10 in [4]) that the double-porosity model in contrast to the mesoscopic one does not require the length scale of the block to be grid resolved. This enables us to solve macroscopic problems numerically and justifies the importance of the homogenization process in the study of non-equilibrium flows, like Kondaurov's flow model, in double porosity media. Our next step in this Note is to pass from the matrix problem formulated in terms of phase pressures to the non-equilibrium imbibition equation (35). Evidently, the new formulation of the homogenized problem is much easier for the numerical simulation because the number of the unknown functions and, consequently, the standing equations is lower than for problem (13)–(15). Some comments on the subject are given above in Remark 3. Notice that for the case of equilibrium two-phase flow in double porosity media (see, e.g., [4,11,12]), the numerical analysis of the global model can be done in two main ways. The first one is to deal directly with the global model involving an equilibrium imbibition equation, using the numerical resolution of this equation by the approach proposed in [32]. The second one is the linearization of the non-linear equilibrium imbibition equation in the sense of [13,30] or like in [25] for the case of double-porosity media with thin fissures. In this case, the homogenized system becomes fully homogenized (i.e. does not involve any coupling with a matrix problem) and the numerical simulation is much easier without great loss of accuracy. Thus, our further work is to generalize these approaches to the analysis of the global Kondaurov model (13)–(35).

(ii) Mathematical analysis of Kondaurov's model. As it was underlined in [1], we carry out our work with an eye to a rigorous mathematical analysis of Kondaurov's model. To this end, in Section 2, we define rigorously the capillary pressure and mobility functions. The main results of the Note are given in Section 4. The key point here is the definition of non-equilibrium global pressure. The global pressure function for degenerate (i.e. when the mobility functions vanish for the wetting phase saturation taking the values zero or one) equilibrium multiphase flows in porous media plays a crucial role, in particular for the proof of compactness results. This is also the case for the non-equilibrium two-phase flows like Kondaurov's flow model or the Hassanizadeh model (see, e.g., [16]). It enables us to apply the ideas of [33] in the proof of the existence result, including the proof of the maximum principle for the real saturation. The notion of the non-equilibrium global pressure along with the non-equilibrium matrix imbibition equation will play an important role in the rigorous justification of the homogenization result obtained by formal asymptotic expansions in [1].

Thus the main novelty of the paper with respect to the existing literature is the introduction of the non-equilibrium global pressure and derivation of the non-equilibrium imbibition equation. These results enable us to continue the development of the theory of non-equilibrium multiphase flows in porous media.

Acknowledgements

This work was funded by the RSCF research project N 15-11-00015. The work was done in the Laboratory of Fluid Dynamics and Seismic of Moscow Institute of Physics and Technology (RAEP *5top100*). The authors are grateful to Prof. M. Jurak for fruitful discussions.

References

[1] A. Konyukhov, L. Pankratov, Upscaling of an immiscible non-equilibrium two-phase flow in double porosity media, Appl. Anal. (2015), http://dx.doi.org/10.1080/00036811.2015.1064524.
[2] S. Bottero, S.M. Hassanizadeh, P.J. Kleingeld, T. Heimovaara, Nonequilibrium capillarity effects in two-phase flow through porous media at different scales, Water Resour. Res. 47 (2011) 1–11.
[3] O. Coussy, Poromechanics, Wiley, New York, 2004.
[4] U. Hornung, Homogenization and Porous Media, Springer-Verlag, New York, 1997.
[5] M. Panfilov, Macroscale Models of Flow Through Highly Heterogeneous Porous Media, Kluwer Academic Publishers, London, 2000.
[6] H. Salimi, J. Bruining, Improved prediction of oil recovery from waterflooded fractured reservoirs using homogenization, SPE Reserv. Eval. Eng. 13 (2010) 44–55.
[7] H. Salimi, J. Bruining, Upscaling in vertically fractured oil reservoirs using homogenization, Transp. Porous Media 84 (2010) 21–53.
[8] H. Salimi, J. Bruining, Upscaling of fractured oil reservoirs using homogenization including non-equilibrium capillary pressure and relative permeability, Comput. Geosci. 16 (2012) 367–389.
[9] B. Amaziane, S. Antontsev, L. Pankratov, A. Piatnitski, Homogenization of immiscible compressible two-phase flow in porous media: application to gas migration in a nuclear waste repository, Multiscale Model. Simul. 8 (2010) 2023–2047.
[10] B. Amaziane, L. Pankratov, Homogenization of a model for water-gas ow through double-porosity media, Math. Methods Appl. Sci. 39 (2016) 425–451.
[11] A. Bourgeat, S. Luckhaus, A. Mikelić, Convergence of the homogenization process for a double-porosity model of immicible two-phase flow, SIAM J. Math. Anal. 27 (1996) 1520–1543.
[12] L.M. Yeh, Homogenization of two-phase flow in fractured media, Math. Methods Appl. Sci. 16 (2006) 1627–1651.
[13] B. Amaziane, J.P. Milišić, M. Panfilov, L. Pankratov, Generalized nonequilibrium capillary relations for two-phase flow through heterogeneous media, Phys. Rev. E 85 (2012) 016304.
[14] A. Bourgeat, M. Panfilov, Effective two-phase flow through highly heterogeneous porous media: capillary nonequilibrium effects, Comput. Geosci. 2 (1998) 191–215.
[15] G.I. Barenblatt, T.W. Patzek, D.B. Silin, The mathematical model of non-equilibrium effects in water-oil displacement, SPE J. 8 (2003) 409–416.
[16] S.M. Hassanizadeh, W.G. Gray, Thermodynamic basis of capillary pressure in porous media, Water Resour. Res. 29 (1993) 3389–3405.
[17] D.B. Das, S.M. Hassanizadeh (Eds.), Upscaling Multiphase Flow in Porous Media, Springer, Dordrecht, The Netherlands, 2005.
[18] X. Cao, I.S. Pop, Two-phase porous media flows with dynamic capillary effects and hysteresis: uniqueness of weak solutions, Comput. Math. Appl. 69 (2015) 688–695.
[19] J. Koch, A. Rätz, B. Schweizer, Two-phase flow equations with a dynamic capillary pressure, Eur. J. Appl. Math. 24 (2013) 49–75.
[20] V.I. Kondaurov, A non-equilibrium model of a porous medium saturated with immiscible fluids, J. Appl. Math. Mech. 73 (2009) 88–102.
[21] A. Konyukhov, A. Tarakanov, On two approaches in investigation of non-equilibrium effects of filtration in a porous medium, in: Proceedings of the Fifth Biot Conference on Poromechanics, ASCE 2013, 2013, pp. 2307–2316.
[22] S.N. Antontsev, A.V. Kazhikhov, V.N. Monakhov, Boundary Value Problems in Mechanics of Nonhomogeneous Fluids, North-Holland, Amsterdam, 1990.
[23] G. Chavent, J. Jaffré, Mathematical Models and Finite Elements for Reservoir Simulation, North-Holland, Amsterdam, 1986.
[24] C. Galusinski, M. Saad, Weak solutions for immiscible compressible multifluid flows in porous media, C. R. Acad. Sci. Paris, Sér. I 347 (2009) 249–254.
[25] M. Jurak, L. Pankratov, A. Vrbaški, A fully homogenized model for incompressible two-phase flow in double porosity media, Appl. Anal. (2015), http://dx.doi.org/10.1080/00036811.2015.1031221.
[26] N. Bakhvalov, G. Panasenko, Homogenisation: Averaging Processes in Periodic Media, Kluwer Academic Publishers, Dordrecht, 1989.
[27] A. Bensoussan, J.-L. Lions, G. Papanicolaou, Asymptotic Analysis for Periodic Structures, North-Holland, Amsterdam, 1978.
[28] E. Sanchez-Palencia, Non-homogeneous Media and Vibration Theory, Springer-Verlag, Berlin, 1980.
[29] J.G. Richardson, Flow through porous media, in: V.L. Streeter (Ed.), Handbook of Fluid Dynamics, McGraw Hill, 1961, pp. 16–65.
[30] T. Arbogast, A simplified dual-porosity model for two-phase flow, in: T.F. Russell, R.E. Ewing, C.A. Brebbia, W.G. Gray, G.F. Pindar (Eds.), Computational Methods in Water Resources IX, Denver, CO, 1992, in: Mathematical Modeling in Water Resources, vol. 2, Comput. Mech, Southampton, UK, 1992, pp. 419–426.
[31] B. Amaziane, M. Panfilov, L. Pankratov, Homogenized model of two-phase ow with local non-equilibrium in double porosity media, J. Math. Fluid Mech. (2015), submitted for publication.
[32] A.E. Berger, H. Brezis, J.C.W. Rogers, A numerical method for solving the problem $u_t - \Delta f(u) = 0$, RAIRO. Anal. Numér. 13 (1979) 297–312.
[33] B. Amaziane, L. Pankratov, A. Piatnitski, The existence of weak solutions to immiscible compressible two-phase flow in porous media: the case of fields with different rock-types, Discrete Contin. Dyn. Syst., Ser. B 18 (2013) 1217–1251.