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Asymptotics of natural oscillations of a spatial junction of thin elastic rods ☆

Développement asymptotique des oscillations naturelles de la jonction spatiale de barres élastiques fines

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ABSTRACT

A one-dimensional model of an elastic junction that contains hard- and readily-movable thin rods is derived, and asymptotic formulas for eigenvalues with rigorous estimates for remainders are given. In addition to vector functions satisfying classical ordinary differential equations, the model involves algebraic unknowns and algebraic relations corresponding to the longitudinal rigid motion of readily-movable rods.

RÉSUMÉ

Nous obtenons un modèle 1D d'une jonction élastique qui contient des barres dures fines facilement déplaçables, et nous donnons un développement asymptotique des valeurs propres justifié par des estimations d'erreur rigoureuses. En plus de fonctions vectorielles satisfaisant des équations différentielles ordinaires, le modèle met en jeu des inconnues et des relations algébriques correspondant au mouvement rigide longitudinal des barres.

1. Statement of problem

Let $Q_j(h)$, $j = 1, 2, 3$ be three vertical thin ($h \ll 1$) rods

$$Q_j(h) = \{x = (y, z) \in \mathbb{R}^2 \times \mathbb{R} : |y - Y^j| < h, z = x_3 \in (0, L_j)\} \tag{1}$$

with circular cross-sections of radius $h > 0$ and length L^j. By rescaling, the smallest length is reduced to one so that the Cartesian coordinates $x = (x_1, x_2, x_3)$ and all geometrical parameters become dimensionless. The centers Y^j of the rod soles

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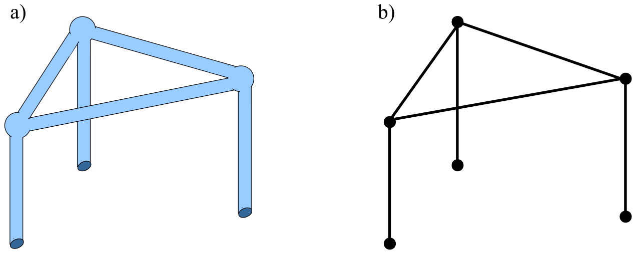

Fig. 1. The elastic junction (a) and its one-dimensional skeleton (b).

$\omega_j^0(h)$ are vertices of a nontrivial triangle, and the upper ends $\omega_j^{L_j}(h) = \{x : |y - Y^j| < h, z = L_j\}$ are embedded into the nodes $\theta_j(h) = \{x : h^{-1}(y - Y^j, z - L_j) \in \theta_j\}$ where $\theta_j \subset \mathbb{R}^3$ are bounded Lipschitz domains. Moreover, the nodes are connected with other three rods $\mathbf{Q}_k(h), k = 1, 2, 3$, also of radius h and with length $\mathbf{L}_k = \left(|Y^k - Y^{k+1}|^2 + |L_k - L_{k+1}|^2\right)^{1/2}$, where $Y^4 = Y^1$ and $L_4 = L_1$. All rods and nodes composing the spatial junction $\Xi(h)$, see Fig. 1a, are isotropic and homogeneous with the Lamé constants $\lambda \geq 0$ and $\mu > 0$. The spectral elasticity problem reads:

$$-\mu(\nabla \cdot \nabla)u^h(x) - (\lambda + \mu)\nabla\nabla \cdot u^h(x) = \Lambda^h u^h(x), \; x \in \Xi(h) \tag{2}$$

$$\sigma^n(u^h; x) = 0, \; x \in \partial\Xi(h) \setminus \left(\omega_1^0(h) \cup \omega_2^0(h) \cup \omega_3^0(h)\right) \tag{3}$$

$$u^h(x) = 0, \; x \in \omega_j^0(h), \; j = 1, 2, 3 \tag{4}$$

Here, $\nabla = \text{grad}$, $\nabla\cdot = \text{div}$, n is the outward normal defined almost everywhere on the Lipschitz (our assumption) boundary $\partial\Xi(h)$, $u^h = (u_1^h, u_2^h, u_3^h)$ is the displacement vector and $\sigma^{(n)}(u^h)$ the normal stress vector. In other words, the surface $\partial\Xi(h)$ is traction-free, see (3), except for the rod slopes $\omega_j^0(h)$ which are fixed, see (4).

The main goal of the paper is to construct asymptotics as $h \to +0$ of eigenvalues $\Lambda_m^h = \rho(\varkappa_m^h)^2$ of the problem (2)–(4) where $\rho > 0$ is the constant material density and $\varkappa_m^h > 0$ is a frequency of natural oscillations of the junction $\Xi(h)$.

2. Motivation

The common knowledge of different reactions of thin solids on longitudinal and bending deformations comes from everyday life, but brings quite big complications into the rigorous asymptotic analysis of elastic junctions. Although the limit one-dimensional models of beams and rods with various boundary conditions at their ends, see [1–4] and many others, results on their finite junctions are much less developed (we here do not touch the homogenization of lattices, which are rather different elastic objects). We mention the original paper [5] investigating an angular joint of two beams and an asymptotic procedure for arbitrary shaped finite junctions of anisotropic inhomogeneous beams [6] and rods [7]. In the latter papers also, the classification "readily/hard-movable" of elements, rods and nodes, was introduced while in the presence of readily-movable rods an adequate one-dimensional model of the junction is no longer differential, but gains algebraic unknowns and equations. These results are related to static elasticity problems, but in this note we adapt those asymptotic procedures to the formulated eigenvalue problem.

In Section 7 we will comment on possible generalizations, but here we restrict the geometry and elastic properties of the junction in order to make formulas shorter and more explicit.

3. Asymptotic expansions for hard-movable rods

On the vertical rods (1), we accept the standard asymptotic ansatz

$$u^h(x) = h^{-2}U^{-2}(z) + h^{-1}U^{-1}(\eta^j, z) + h^0 U^0(\eta^j, z) + h^1 U^1(\eta^j, z) + \ldots, \quad x \in Q^j(h) \tag{5}$$

where $\eta^j = h^{-1}(y - Y^j)$ are stretched coordinates,

$$U^{-2}(z) = \sum_{i=1}^{2} e_{(i)} w_i^j(z), \quad U^{-1}(\eta^j, z) = e_{(3)}\Big(w_3^j(z) - \sum_{i=1}^{2} \eta_i^j \partial_z w_i^j(z)\Big) + \theta(\eta^j) w_4^j(z) \tag{6}$$

$w_k^j(z)$ is the projection on the x_k-axis (with the unit vector $e_{(k)}, k = 1, 2, 3$) of the displacement vector averaged over the cross-section of $Q_j(h), \theta(\eta^j) = \eta_1^j e_{(2)} - \eta_2^j e_{(1)}$ and $w_4^j(z)$ is the averaged twist. Explicit expressions for U^0 and U^1 are available too, see, e.g., [1–4], but in the sequel we will only need asymptotic formulas for the averaged force F^j and torque T^j at the end $\omega_j^{L_j}(h)$ of $Q_j(h)$, namely

$$F_i^{jh} = h^2 \frac{E}{4} \partial_z^2 w_i^j(L_j) + \ldots, \quad F_3^{jh} = h^1 E \partial_z w_3^j(L_j) + \ldots, \quad E = \mu \frac{3\lambda + 2\mu}{\lambda + \mu} \tag{7}$$

$$T_i^{jh} = (-1)^i E \partial_z^2 w_{3-i}^j(L_j) + \ldots, \quad T_3^{jh} = \frac{\mu}{4} \partial_z w_4^j(L_j) + \ldots \tag{8}$$

where E is the Young modulus.

The vector functions $w^j = (w_1^j, w_2^j, w_3^j, w_4^j)$, columns in $\mathbb{R}^4$, satisfy the ordinary differential equations

$$D(-\partial_z) B D(\partial_z) w^j(z) = M K w^j(z), \quad z \in (0, L_j) \tag{9}$$

where the diagonal matrices are given by

$$D(\partial_z) = \mathrm{diag}\{\partial_z^2, \partial_z^2, \partial_z, \partial_z\}, \ K = \mathrm{diag}\{1, 1, 0, 0\}, \ B = \mathrm{diag}\left\{\frac{E}{4}, \frac{E}{4}, E, \frac{\mu}{4}\right\}$$

and the new spectral parameter M comes from the following asymptotic expansion in the low-frequency range of the spectrum of the problem (2)–(4):

$$\Lambda^h = hM + \widetilde{\Lambda}^h$$

According to [3, Ch. 7, §3], the spectral parameter appears only in the first two lines of the system (9). The very reason for this effect is nothing but the lower order h^{-1} of the longitudinal displacement u_3^h compared with h^{-2} of the transversal ones in the vertical rods $Q_j(h)$ which are to be regarded as hard-movable due to the Dirichlet condition (4). In view of formulas (6), the latter leads to

$$w_i^j(0) = 0, \quad \partial_z w_i^j(0) = 0, \quad w_{2+i}^j(0) = 0, \ i = 1, 2 \tag{10}$$

At the same time, a simple mechanical observation recognizes the cross-bars $\mathbf{Q}_k(h)$ as readily movable because the vertical thin stands $Q_j(h)$ cannot gravely resist against horizontal movements of the frame $\mathbf{Q}_1(h) \cup \mathbf{Q}_2(h) \cup \mathbf{Q}_3(h)$. Hence, we need to modify ansatz (5), (6), according to [6,7].

4. Asymptotic expansions for readily-movable rods

The rod $\mathbf{Q}_k(h) = \{x : |\mathbf{y}^k| < h, \ \mathbf{z}^k \in (0, \mathbf{L}_k)\}$ is supplied with the Cartesian coordinate system $x^k = (x_1^k, x_2^k, x_3^k) = (\mathbf{y}^k, \mathbf{z}^k)$ with the unit vector $\mathbf{e}_{(j)}^k$ of the x_j^k-axis. The asymptotic expansion in these rods

$$u^h(x) = h^{-2}\mathbf{U}^{-2}(\mathbf{z}^k) + h^{-1}\mathbf{U}^{-1}(\boldsymbol{\eta}^k, \mathbf{z}^k) + h^0 \mathbf{U}^0(\boldsymbol{\eta}^k, \mathbf{z}^k) + h^1 \mathbf{U}^1(\boldsymbol{\eta}^k, \mathbf{z}^k) + \ldots, \quad x \in \mathbf{Q}_k(h) \tag{11}$$

involves the stretched transversal coordinates $\boldsymbol{\eta}^k = h^{-1}\mathbf{y}^k$ and the main term

$$\mathbf{U}^{-2}(\mathbf{z}^k) = \sum_{i=1}^{2} \mathbf{e}_{(i)}^k \mathbf{w}_i^k(\mathbf{z}^k) + \mathbf{e}_{(3)}^k \mathbf{a}^k \tag{12}$$

with the new, cf. (6), unknown constant longitudinal displacement $\mathbf{e}_{(3)}^k \mathbf{a}^k$. Other terms in (11), written in local coordinates, depend on the unknown vector function $\mathbf{w}^k = (\mathbf{w}_1^k, \mathbf{w}_2^k, \mathbf{w}_3^k, \mathbf{w}_4^k)$ but, owing to differentiation in $\mathbf{z}^k$, are independent of $\mathbf{a}^k$ and therefore look quite similar to the asymptotic terms in (5). In this way, the last term in (12) does not affect the procedure of derivation of the differential equations

$$D(-\partial_z) B D(\partial_z) \mathbf{w}^k(z) = M K \mathbf{w}^k(z), \quad z \in (0, \mathbf{L}_k) \tag{13}$$

The longitudinal inertia force $hM\mathbf{e}_{(3)}^k h^{-1} \mathbf{a}^k$ is too small and does not enter the limit system, but it will appear in the transmission conditions and make them nonlocal.

As usual in multistage asymptotic procedures, cf. [2], [3], [8, Ch. 4], [9] introduction of the free parameter $\mathbf{a}^k$ in (12) requires for the orthogonality condition

$$\int_0^{\mathbf{L}_k} \mathbf{w}_3^k(\mathbf{z}) \, d\mathbf{z} = 0 \tag{14}$$

imposed on the lower-order asymptotic term in the same component of the displacement vector.

5. Transmission conditions

To close the system (13) and (9) which had been supplied with the boundary conditions (10) only, we must attribute transmission conditions at the points $z = L_j$ relating w^j with $\mathbf{w}^j, \mathbf{w}^{j+1}$, while $\mathbf{w}^4$ coincides with $\mathbf{w}^1$. In papers [6] and [7], this has been done in stationary problems by regarding boundary layers in the vicinity of the nodes $\theta_j(h)$, cf. [8]. Here, we will conclude the conditions by comparing the asymptotic expansions (5), (6) and (11), (12) while taking into account the equilibrium of certain fragments of the elastic junction. A rigorous justification of such an approach is supported by the final theorem on asymptotics.

Formulas (7) and (8), although in the local coordinates and on the base of (10), (11), apply to the rotations $\mathbf{R}^{kh}(0), \mathbf{R}^{kh}(\mathbf{L}_j)$ and the torques $\mathbf{T}^{kh}(0), \mathbf{T}^{kh}(\mathbf{L}_j)$ at the ends of the rod $\mathbf{Q}_k(h)$. Since all components of these vectors gain the same order in h, we just state the traditional transmission conditions of Kirchhoff's type:

$$R^{j1} = \mathbf{R}^{j1}(\mathbf{L}_j) = \mathbf{R}^{j+1\,1}(0), \quad j = 1,2,3 \tag{15}$$

$$-T^{j\,1} - \mathbf{T}^{j\,1}(\mathbf{L}_j) + \mathbf{T}^{j+1\,1}(0) = 0, \quad j = 1,2,3 \tag{16}$$

In [10], the conditions (15) and (16) are classified as stable and intrinsic, respectively.

Observing that the coefficient h^{-1} of $e^j_{(3)} w^j_3$ and $\mathbf{e}^k_{(3)}\mathbf{w}^k_3$ is much less than the coefficient h^{-2} of $e^j_{(i)} w^j_i$ and $\mathbf{e}^k_{(i)}\mathbf{w}^k_i + \mathbf{e}^k_{(3)}\mathbf{a}^k$, there is no need to endow w^k_3 and $\mathbf{w}^k_3$ with any stable transmission condition, but only with the intrinsic ones

$$F^{j1}_3 = \mathbf{F}^{j1}_3(\mathbf{L}_j) = \mathbf{F}^{j+1\,1}_3(0) = 0, \quad j = 1,2,3 \tag{17}$$

However, other components enjoy conditions of both types. First of all, the continuity of displacements requires that

$$\sum_{i=1}^{2} e^j_{(i)} w^j_i(L_j) = \sum_{i=1}^{2} \mathbf{e}^j_{(i)}\mathbf{w}^j_i(\mathbf{L}_j) + \mathbf{e}^j_{(3)}\mathbf{a}^j = \sum_{i=1}^{2} \mathbf{e}^{j+1}_{(i)}\mathbf{w}^{j+1}_i(0) + \mathbf{e}^{j+1}_{(3)}\mathbf{a}^{j+1} \tag{18}$$

The differential equation for $\mathbf{w}^k_3$ in (13) and the relations (7), (14) and (17) show that $\mathbf{w}^k_3 = 0$ in $(0, \mathbf{L}_k)$. Thus, formulas (17) are valid automatically and we have to deal with the averaged longitudinal forces $\mathbf{f}^k_3(0)$ and $\mathbf{f}^k_3(\mathbf{L}_k)$ at the ends of the rod $\mathbf{Q}_k(h)$ with the next order in h. We write the equilibrium equation of this rod

$$\mathbf{f}^k_{(3)}(0) - \mathbf{f}^k_{(3)}(\mathbf{L}_k) + \pi\Lambda\mathbf{L}_k\mathbf{a}^k = 0, \quad k = 1,2,3 \tag{19}$$

the latter term being the averaged inertia force mentioned above. Also the equilibrium equation of the node $\theta_j(h)$

$$\sum_{i=1}^{3}\Big(-\mathbf{e}^j_{(i)}\mathbf{F}^{j1}_i - \mathbf{e}^j_{(i)}\mathbf{F}^{j1}_i(\mathbf{L}_j) + \mathbf{e}^{j+1}_{(i)}\mathbf{F}^{j+1\,1}_i(0)\Big) + \mathbf{e}^j_{(3)}\mathbf{f}^j_3(\mathbf{L}_j) - \mathbf{e}^{j+1}_{(3)}\mathbf{f}^{j+1}_3(0) = 0, \quad j = 1,2,3 \tag{20}$$

must hold. The differential-algebraic formulation of the one-dimensional model is completed.

6. The variational formulation

Let $\mathfrak{H} \ni \mathfrak{M} = \{w^j, \mathbf{w}^k, \mathbf{a}^k\}$ be a space of vector functions $w^j \in H^2(0, L_j)^2 \times H^1(0, L_j)$, $\mathbf{w}^k \in H^2(0, \mathbf{L}_j)^2 \times H^1(0, \mathbf{L}_k)^2$ and constants $\mathbf{a}^k$ with $j, k = 1,2,3$, which are subject to the stable boundary and transmission conditions (10) and (15), (18) as well as the orthogonality conditions (14). As usual, see [10], we multiply the systems (9) and (13) scalarly with test vector functions v^j and $\mathbf{v}^k$, respectively, and integrate by parts using the intrinsic transmission conditions (16), (17), (18) together with the algebraic equations (19) to process addends outside integrals. As a result of algebraic transformations, we obtain:

$$\begin{aligned}&\sum_{p=1}^{3}\Big(\big(BD(\partial_z)w^p, D(\partial_z)v^p\big)_{(0,L_p)} + \big(BD(\partial_{\mathbf{z}})\mathbf{w}^p, D(\partial_{\mathbf{z}})\mathbf{v}^p\big)_{(0,\mathbf{L}_p)}\Big)\\ &= \Lambda\sum_{p=1}^{3}\Big(\big(Kw^p, v^p\big)_{(0,L_p)} + \big(K\mathbf{w}^p, \mathbf{v}^p\big)_{(0,\mathbf{L}_p)} + \mathbf{L}_p\mathbf{a}^p\mathbf{b}^p\Big), \quad \forall\, \mathfrak{V} = (v^j, \mathbf{v}^k, \mathbf{b}^k) \in \mathfrak{H}\end{aligned} \tag{21}$$

Here, $(\cdot,\cdot)_{(0,l)}$ is the inner product in the Lebesgue space $L^2(0,l)$. The left-hand side of the integral identity (21) with $\mathfrak{V} = \mathfrak{W}$ is the double kinetic energy of harmonic in time oscillations of the junction, while the last terms reflect the longitudinal rigid motion $h^{-2}\mathbf{a}^k\mathbf{e}^k_{(3)}$ of the rods $\mathbf{Q}_k(h)$.

Lemma. *The spectral variational problem (21) (or (9), (10), (13)–(20) in the differential-algebraic form) has the positive monotone unbounded sequence of eigenvalues*

$$0 < M_1 \le M_2 \le \ldots \le M_n \le \ldots \to +\infty \tag{22}$$

listed according to their multiplicity. The corresponding eigenvectors $\mathfrak{W}^1, \mathfrak{W}^2, \ldots, \mathfrak{W}^n, \ldots \in \mathfrak{H}$ *can be subject to the normalization and orthogonality conditions* $\mathfrak{A}(\mathfrak{W}^n, \mathfrak{W}^m) = \delta_{n,m}$, *where* $\delta_{n,m}$ *is the Kronecker symbol.*

7. Theorem on asymptotics

In [11], an anisotropic Korn inequality is proved that distinguishes between hard and readily-movable rods. Since, for brevity, we do not present here estimates of remainders in the asymptotics of the elastic fields, we write the inequality in a very condensed form

$$\sum_{p=1}^{3} \Big(\|u_3^h; L^2(Q_p(h))\|^2 + h^2 \sum_{i=1}^{2} \|u_i^h; L^2(Q_p(h))\|^2 + h^2 \|u^h; L^2(\mathbf{Q}_p(h))\|^2 \Big) \le C \sum_{\alpha,\beta=1}^{3} \Big\| \frac{\partial u_\alpha^h}{\partial x_\beta} + \frac{\partial u_\beta^h}{\partial x_\alpha}; L^2(\Xi(h)) \Big\|^2$$

The difference between the displacements of stands and crossbars in Fig. 1 is easily recognized in (23): the norm $\|\mathbf{u}_3^h; L^2(\mathbf{Q}_k(h))\|$ gets the factor h, but $\|u_3^h; L^2(Q_j(h))\|$ does not. This distinction explains why the classical asymptotic ansatz (5), (6) ought to be modified in (11), (12).

A routine but combersome scheme, cf., [12] and [6], [7], based on the intact form [11] of the Korn inequality, leads to the final assertion for our asymptotic analysis.

Theorem. *The eigenvalue sequences* (22) *and*

$$0 < \Lambda_1^h \le \Lambda_2^h \le \ldots \le \Lambda_n^h \le \ldots \to +\infty \tag{23}$$

of the model problem and the original elasticity problem (2)–(4) *are in the relationship*

$$|\Lambda_n^h - hM_n| \le c_n h^{3/2} \quad \forall h \in (0, h_n]$$

where c_n *and* h_n *are some positive numbers depending on the number n of eigenvalues.*

We deliberately missed an opportunity to apply an extremely complicated and involved scheme of the direct and inverse reduction [12,13], which describes an explicit dependence of the numbers c_n and h_n on certain characteristics of the limit spectrum (22).

Many specifications of the elastic junction $\Xi(h)$ were introduced to simplify the demonstration only. For example, rods can have varying cross-sections and are made of inhomogeneous anisotropic materials. These modifications do not influence the model at all, but the constant matrix B in (9) and (13) is no longer diagonal and becomes dependent on z and $\mathbf{z}^k$, cf. [1–4]. Changes in the configurations of rods are available as well, but require an application of the classification procedure [7]. For example, if the sole $\omega_1^0(h)$ is released, then the term $U^{-2}(z)$ in formula (5) on the rod $Q_1(h)$ must be replaced by the sum (12), and a new algebraic unknown must be introduced as well.

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