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Transient response of elastic bodies connected by a thin stiff viscoelastic layer with evanescent mass

Réponse transitoire de corps élastiques liés par une mince et raide bande viscoélastique de faible masse

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A B S T R A C T

We extend the study [1] devoted to the dynamic response of a structure made up of two linearly elastic bodies connected by a thin soft adhesive layer made of a Kelvin–Voigt-type nonlinear viscoelastic material to the cases of stiff and very stiff adhesives whose mass vanishes. We use a nonlinear extension of Trotter's theory of convergence of semi-groups of operators acting on variable spaces to identify the asymptotic behavior of the mechanical state of the system, when some geometrical and mechanical parameters tend to their natural limits. The models we obtain describe the behavior of a structure consisting of two linearly elastic adherents perfectly bonded to a material deformable flat surface whose behavior is of the same kind as that of the genuine adhesive.

R É S U M É

Nous étendons aux adhésifs durs ou très durs, dont la masse est évanescente, l'étude menée en [1] consacrée au comportement dynamique d'un assemblage de deux corps linéairement élastiques liés par une couche adhésive mince et molle constituée d'un matériau viscoélastique non linéaire de type Kelvin–Voigt. Afin d'identifier le comportement asymptotique de l'état mécanique du système lorsque des paramètres mécanique et géométriques tendent vers leurs limites naturelles, nous utilisons une extension non linéaire de la théorie de Trotter de convergence de semi-groupes d'opérateurs agissant sur des espaces variables. Les modèles obtenus décrivent le comportement d'une structure constituée de

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deux adhérents élastiques parfaitement collés à une surface matérielle plate et déformable, dont le comportement est identique à celui de l'adhésif.

1. Setting the problem

We extend to the situations of high and very high stiffness the results obtained in [1] concerning the dynamics of elastic bodies connected by a thin soft viscoelastic layer. Let $\{e_1, e_2, e_3\}$ be an orthonormal basis of $\mathbb{R}^3$ assimilated to the Euclidean space. For all $\xi = (\xi_1, \xi_2, \xi_3)$ in $\mathbb{R}^3$, $\widehat{\xi}$ stands for (ξ_1, ξ_2). The space of all $(n \times n)$ symmetric matrices is denoted by $\mathbb{S}^n$ and equipped with the usual inner product and norm denoted by $\cdot$ and $|\ |$ (as in $\mathbb{R}^3$). For all η in $\mathbb{S}^3$, $\widehat{\eta}$ stands for the matrix $(\eta_{\alpha\beta})_{1\le\alpha,\beta\le 2}$ in $\mathbb{S}^2$. We study the dynamic response of a structure consisting of two adherents connected by a thin adhesive layer which is subjected to a given loading. Let Ω be a domain of $\mathbb{R}^3$ with Lipschitz-continuous boundary $\partial\Omega$. The intersection of Ω with $\{x_3 = 0\}$ is a domain S of $\mathbb{R}^2$ with a positive two-dimensional Haussdorf measure $\mathcal{H}_2(S)$. Let ε be a positive number and $\Omega_\pm := \Omega \cap \{\pm x_3 > 0\}$, then adhesive and adherents occupy $B^\varepsilon := S \times (-\varepsilon, +\varepsilon)$ and $\Omega^\varepsilon_\pm := \Omega_\pm \pm \varepsilon e_3$ respectively; we define $\Omega^\varepsilon := \Omega^\varepsilon_+ \cup \Omega^\varepsilon_-$, $S^\varepsilon_\pm := S \pm \varepsilon e_3$ and $\mathcal{O}^\varepsilon := \Omega^\varepsilon \cup B^\varepsilon \cup S^\varepsilon_+ \cup S^\varepsilon_-$. We consider a partition (Γ_0, Γ_1) of $\partial\Omega$ and, for all Γ in $\{\Gamma_0, \Gamma_1\}$, the sets $\Gamma_\pm$, $\Gamma^\varepsilon_\pm$ and Γ^ε respectively denote $\Gamma \cap \{\pm x_3 > 0\}$, $\Gamma_\pm \pm \varepsilon e_3$ and $\Gamma^\varepsilon_+ \cup \Gamma^\varepsilon_-$. Moreover, we assume that $\mathcal{H}_2(\Gamma_{0+}) > 0$. The structure made of the adhesive and the two adherents, perfectly stuck together along $S^\varepsilon_\pm$, is clamped on Γ^ε_0 and subjected to body forces of density f^ε and to surface forces g^ε on Γ^ε_1. The adherents are modeled as linearly elastic materials with a strain energy density W^ε such that

$$W^\varepsilon(x, e) = \frac{1}{2} a^\varepsilon(x) e \cdot e, \quad a.e.\ x \in \Omega^\varepsilon, \quad \forall e \in \mathbb{S}^3 \tag{1}$$

The *thin* adhesive is assumed to be made of a homogeneous, isotropic and "viscoelastic of Kelvin–Voigt generalized type". Its strain energy density reads as $\mu\, w_I$, while its dissipation potential is denoted by $b\,\mathcal{D}$, where μ and b are positive scalars; w_I is a positive definite quadratic form on $\mathbb{S}^3$ and $\mathcal{D}$ a convex and positively homogeneous function of degree q, $1 \le q \le 2$.

Let $\rho > 0$, $\overline{\rho}_M > \overline{\rho}_m > 0$ and $\overline{\rho}^\varepsilon$ a measurable function. The density γ^ε of the structure is equal to $\overline{\rho}^\varepsilon$ in Ω^ε and to ρ in B^ε. Denoting by $\mathrm{Lin}(\mathbb{S}^3)$ the space of linear mappings from $\mathbb{S}^3$ into $\mathbb{S}^3$, we make the following assumptions on the data:

$$\begin{cases} \text{There exists } (f, g, a, \overline{\rho}) \text{ in } L^2(\Omega; \mathbb{R}^3) \times L^2(\Gamma_1; \mathbb{R}^3) \times L^\infty(\Omega; \mathrm{Lin}(\mathbb{S}^3)) \times L^\infty(\Omega) \text{ such that} \\ f^\varepsilon(x) = f(x \mp \varepsilon e_3) \quad a.e.\ x \in \Omega^\varepsilon_\pm, \quad f^\varepsilon(x) = 0 \quad a.e.\ x \in B^\varepsilon \\ g^\varepsilon(x) = g(x \mp \varepsilon e_3) \quad a.e.\ x \in (\Gamma_1)^\varepsilon_\pm, \quad g^\varepsilon(x) = 0 \quad a.e.\ x \in \partial S \times (-\varepsilon, \varepsilon) \\ a^\varepsilon(x) = a(x \mp \varepsilon e_3) \quad a.e.\ x \in \Omega^\varepsilon_\pm \\ \overline{\rho}^\varepsilon(x) = \overline{\rho}(x \mp \varepsilon e_3) \quad a.e.\ x \in \Omega^\varepsilon_\pm \\ \exists\, a_m, a_M > 0 \quad s.t. \quad a_m |e|^2 \le a(x) e \cdot e \le a_M |e|^2, \quad \forall e \in \mathbb{S}^3 \\ \exists\, \overline{\rho}_m, \overline{\rho}_M > 0 \quad s.t. \quad \overline{\rho}_m \le \overline{\rho}(x) \le \overline{\rho}_M, \quad a.e.\ x \in \Omega \end{cases} \tag{2}$$

Thus, the problem $(\mathcal{P}_s)$ of determining the dynamic evolution of the assembly involves a quadruplet $s := (\varepsilon, \mu, b, \rho)$ of data so that all the fields will be hereafter indexed by s. In the following, t denotes the time, $e(u)$ is the linearized strain tensor associated with the field of displacement u, and $\partial J(v)$ denotes the subdifferential at v of any lower semi-continuous convex function J, while $DJ(v)$ stands for the differential at v of any Fréchet differentiable function J. If $U^0_s = (u^0_s, v^0_s)$ is the initial state, a formulation of $(\mathcal{P}_s)$ could be

$$(\mathcal{P}_s) \quad \begin{cases} \text{Find } u_s \text{ sufficiently smooth in } \Omega \times [0, T] \text{ such that } u_s = 0 \text{ on } \Gamma^\varepsilon_0 \times (0, T] \\ \left(u_s(\cdot, 0), \dfrac{\partial u_s}{\partial t}(\cdot, 0)\right) = U^s_0 \text{ and there exists } \zeta \text{ in } \partial\mathcal{D}(e(\dfrac{\partial u_s}{\partial t})) \text{ satisfying:} \\ \displaystyle\int_{\mathcal{O}^\varepsilon} \gamma^\varepsilon \frac{\partial^2 u_s}{\partial t^2} v \,\mathrm{d}x + \int_{\Omega^\varepsilon} a^\varepsilon e(u_s) \cdot e(v) \,\mathrm{d}x + \int_{B^\varepsilon} \Big(\mu\, D w_I(e(u_s)) + b\zeta\Big) \cdot e(v) \,\mathrm{d}x = \\ \qquad = \displaystyle\int_{\mathcal{O}^\varepsilon} f^\varepsilon \cdot v \,\mathrm{d}x + \int_{\Gamma^\varepsilon_1} g^\varepsilon \cdot v \,\mathrm{d}\mathcal{H}_2 \\ \text{for all } v \text{ sufficiently smooth in } \mathcal{O}^\varepsilon \text{ and vanishing on } \Gamma^\varepsilon_0 \end{cases}$$

2. Existence and uniqueness

We assume that

$$(f, g) \in BV\big(0, T; L^2(\Omega; \mathbb{R}^3)\big) \times BV^{(2)}\big(0, T; L^2(\Gamma_1; \mathbb{R}^3)\big) \tag{$\mathbf{H}_1$}$$

where, for any Banach space X, $BV(0,T;X)$ is the subspace of $L^1(0,T;X)$ consisting of all elements whose time derivative in the sense of distributions is a bounded X-valued measure on $(0,T)$, and $BV^{(2)}(0,T;X)$ is the subspace of $BV(0,T;X)$ consisting of all elements whose time derivative in the sense of distributions belongs to $BV(0,T;X)$.

We seek u_s in the form

$$u_s = u_s^e + u_s^r \tag{3}$$

where u_s^e is the unique solution to

$$u_s^e(t) \in H^1_{\Gamma_0^\varepsilon}(\mathcal{O}^\varepsilon;\mathbb{R}^3); \quad \varphi_s(u_s^e(t), v) = L(t)(v), \quad \forall v \in H^1_{\Gamma_0^\varepsilon}(\mathcal{O}^\varepsilon;\mathbb{R}^3),\ \forall t \in [0,T] \tag{4}$$

with

$$\begin{aligned}
\varphi_s(v,v') &:= \int_{\Omega^\varepsilon} a^\varepsilon e(v)\cdot e(v')\,\mathrm{d}x + \mu \int_{B^\varepsilon} D w_I\big(e(v)\big)\cdot e(v')\,\mathrm{d}x, \quad \forall v, v' \in H^1_{\Gamma_0^\varepsilon}(\mathcal{O}^\varepsilon;\mathbb{R}^3)\\
\Phi_s(v) &:= \varphi_s(v,v)\\
L^\varepsilon(t)(v) &:= \int_{\Gamma_1^\varepsilon} g^\varepsilon(x,t)\cdot v(x)\,\mathrm{d}\mathcal{H}_2, \quad \forall v \in H^1_{\Gamma_0^\varepsilon}(\mathcal{O}^\varepsilon;\mathbb{R}^3),\ \forall t\in[0,T]
\end{aligned} \tag{5}$$

and where $H^1_{\Gamma_0^\varepsilon}(\mathcal{O}^\varepsilon;\mathbb{R}^3)$ is the closed subspace of $H^1(\mathcal{O}^\varepsilon;\mathbb{R}^3)$ consisting of all elements with vanishing traces on Γ_0^ε. Note that this notation $H^1_g(G;\mathbb{R}^n)$ will be systematically used for any $G \subset \mathbb{R}^n$, $g \subset \partial G$ and Sobolev space $H^1(G;\mathbb{R}^n)$. As $g \mapsto u_s^e$ is linear continuous from $L^2(\Gamma_1;\mathbb{R}^3)$ into $H^1_{\Gamma_0^\varepsilon}(\mathcal{O}^\varepsilon;\mathbb{R}^3)$, we have:

$$u_s^e \in BV^{(2)}\big(0,T;H^1_{\Gamma_0^\varepsilon}(\mathcal{O}^\varepsilon;\mathbb{R}^3)\big) \tag{6}$$

The remaining part u_s^r of u_s will therefore satisfy an evolution equation governed by a maximal monotone operator A_s defined in a Hilbert space H_s of possible states with finite total mechanical (kinetic + strain) energy. The space of velocities $L^2(\mathcal{O}^\varepsilon;\mathbb{R}^3)$ is equipped with the following inner product k_s and the square of norm K_s associated with kinetic energy:

$$k_s(v,v') := \int_{\mathcal{O}^\varepsilon} \gamma^\varepsilon(x) v(x)\cdot v'(x)\,\mathrm{d}x, \quad K_s(v) := k_s(v,v), \quad \forall v, v' \in L^2(\mathcal{O}^\varepsilon;\mathbb{R}^3) \tag{7}$$

while the space of displacements, $H^1_{\Gamma_0^\varepsilon}(\mathcal{O}^\varepsilon;\mathbb{R}^3)$, is equipped with the inner product φ_s defined in (5), which is equivalent to the usual one by Korn inequality. Hence

$$H_s := H^1_{\Gamma_0^\varepsilon}(\mathcal{O}^\varepsilon;\mathbb{R}^3) \times L^2(\mathcal{O}^\varepsilon;\mathbb{R}^3) \tag{8}$$

where, for all $U=(u,v)$ and $U'=(u',v')$ in H_s, the inner product and norm are

$$(U,U')_s := \varphi_s(u,u') + k_s(v,v'), \quad |U|_s^2 := (U,U)_s \tag{9}$$

while A_s is defined by

$$\begin{cases}
D(A_s) = \left\{ U=(u,v)\in H_s;\ \begin{cases} \text{i) } v \in H^1_{\Gamma_0^\varepsilon}(\mathcal{O}^\varepsilon;\mathbb{R}^3)\\ \text{ii) } \exists\,(w,\xi) \in L^2(\mathcal{O}^\varepsilon;\mathbb{R}^3)\times\partial\mathcal{D}(e(v)) \text{ with}\end{cases} \right.\\
\qquad\qquad \left. k_s(w,v') + \varphi_s(u,v') + b\int_{B^\varepsilon} \xi\cdot e(v')\,\mathrm{d}x = 0, \quad \forall v' \in H^1_{\Gamma_0^\varepsilon}(\mathcal{O}^\varepsilon;\mathbb{R}^3) \right\}\\
A_s U = (-v,0) + \{(0,-w);\ w \text{ satisfies ii) of definition of } D(A_s)\}
\end{cases} \tag{10}$$

Proceeding as in [1], one has the following.

Proposition 2.1. *The operator A_s is a maximal monotone operator and, for all $\psi = (\psi^1,\psi^2)$ in H_s,*

$$\begin{cases} \overline{U}_s = (\overline{u}_s, \overline{v}_s) \text{ s.t.}\\ \overline{U}_s + A_s\overline{U}_s \ni \psi \end{cases} \Leftrightarrow \begin{cases} J_s(\overline{v}_s) \le J_s(v) \quad \forall v \in H^1_{\Gamma_0^\varepsilon}(\mathcal{O}^\varepsilon;\mathbb{R}^3)\\ J_s(v) := \frac{1}{2}K_s(v) - k_s(\psi^2,v) + \frac{1}{2}\phi_s(v) + \varphi_s(\psi^1,v) + b\int_{B^\varepsilon}\mathcal{D}\big(e(v)\big)\,\mathrm{d}x\\ \overline{u}_s = \overline{v}_s + \psi^1 \end{cases} \tag{11}$$

Then, taking into account ($\mathbf{H}_1$), (3), (4), (6), (10), we check straightforwardly that $(\mathcal{P}_s)$ is "formally equivalent" to

$$\begin{cases} \dfrac{\mathrm{d}U_s^r}{\mathrm{d}t} + A_s U_s^r \ni F_s \\ U_s^r(0) = U_s^o - \big(u_s^e(0), 0\big) \end{cases} \tag{12}$$

where

$$F_s = \Big(-\frac{\mathrm{d}u_s^e}{\mathrm{d}t}, f^\varepsilon/\gamma^\varepsilon\Big) \tag{13}$$

A result of [2] therefore yields Theorem 2.1.

Theorem 2.1. *If* (f, g) *satisfies* ($\mathbf{H}_1$) *and* $U_s^o \in \big(u_s^e(0), 0\big) + D(A_s)$*, then* (12) *has a unique solution such that* U_s^r *belongs to* $W^{1,\infty}(0, T; H_s)$ *and the first line of* (12) *is satisfied almost everywhere in* $[0, T]$*. Hence, there exists a unique* u_s *in* $W^{1,\infty}\big(0, T; H^1_{\Gamma_0^\varepsilon}(\mathcal{O}^\varepsilon; \mathbb{R}^3)\big) \cap W^{2,\infty}\big(0, T; L^2(\mathcal{O}^\varepsilon; \mathbb{R}^3)\big)$ *which does satisfy*

$$\begin{cases} \exists \xi \in \partial\mathcal{D}\Big(e\Big(\dfrac{\mathrm{d}u_s}{\mathrm{d}t}\Big)\Big) \text{ such that} \\ \int_{\mathcal{O}^\varepsilon} \gamma^\varepsilon \dfrac{\mathrm{d}^2 u_s}{\mathrm{d}t^2} v\,\mathrm{d}x + \displaystyle\int_{\Omega^\varepsilon} a^\varepsilon e(u_s)\cdot e(v)\,\mathrm{d}x + \mu \int_{B^\varepsilon} D w_I\big(e(u_s)\big)\cdot e(v)\,\mathrm{d}x + b\int_{B^\varepsilon} \xi\cdot e(v)\,\mathrm{d}x \\ \qquad = \displaystyle\int_{\mathcal{O}^\varepsilon} f^\varepsilon\cdot v\,\mathrm{d}x + \int_{\Gamma_1^\varepsilon} g^\varepsilon\cdot v\,\mathrm{d}\mathcal{H}_2, \quad \forall v \in H^1_{\Gamma_0^\varepsilon}(\mathcal{O}^\varepsilon; \mathbb{R}^3),\ \text{a.e. } t \in (0, T] \\ u_s(0) = u_s^0,\ \dfrac{\mathrm{d}u_s}{\mathrm{d}t}(0) = v_s^0 \end{cases} \tag{14}$$

We set

$$U_s^e = \Big(u_s^e, 0\Big), \quad U_s = U_s^r + U_s^e \tag{15}$$

3. Asymptotic behavior

Now we regard the quadruplet s of geometrical and mechanical *data* as a quadruplet of *parameters* taking values in a countable subset of $(0, +\infty)^4$ with a single cluster point $\bar{s}$ and study the asymptotic behavior of U_s in order to obtain a simplified but accurate enough model for the genuine physical situation. We will show that two different models indexed by $p \in \{1, 2\}$ appear at the limit depending on the relative behavior of ε and μ. We make the following assumptions:

$$\begin{cases} \text{i)} \quad \bar{s}_p \in \{0\} \times \{+\infty\}^2 \times [0, +\infty] \\ \text{ii)} \quad \exists (\bar{\mu}_p, \bar{b}_p) \in (0, +\infty)^2 \text{ s.t.} \lim\limits_{s\to\bar{s}} 2\varepsilon\Big(\dfrac{\varepsilon^{2(p-1)}}{2p-1}\mu, \dfrac{\varepsilon^{q(p-1)}}{1+(p-1)q} b\Big) = (\bar{\mu}_p, \bar{b}_p) \\ \text{iii)} \quad \lim\limits_{s\to\bar{s}} 2\varepsilon\rho = 0 \\ \text{iv)} \quad w_I \text{ is an even function of } x_3 \\ \text{v)} \quad \exists \varepsilon_0 > 0 \text{ s.t. } S \times (0, \varepsilon_0) \subset \Omega_+ \end{cases} \tag{$\mathbf{H}_2$}$$

3.1. A candidate for the limit behavior

This candidate could be determined by studying the asymptotic behavior of sequences with bounded total mechanical energy. Let

$$\begin{aligned} &{}^pH := {}^pH_d \times {}^pH_v \\ &{}^1H_d := \{u \in H^1(\Omega; \mathbb{R}^3);\ \widehat{u} \in H^1(S; \mathbb{R}^2)\} \\ &{}^2H_d := \{u \in H^1(\Omega; \mathbb{R}^3);\ \widehat{e(u)} = 0 \text{ in } S \text{ and } u_3 \in H^2(S)\} \\ &{}^pH_v := L^2(\Omega; \mathbb{R}^3), \quad p = 1, 2 \end{aligned} \tag{16}$$

We introduce

$${}^p\varphi(u, u') := \int_\Omega a e(u)\cdot e(u')\,\mathrm{d}x + \bar{\mu}_p \int_S D w_I^{KL}(\widehat{e_p(u)})\cdot \widehat{e_p(u')}\,\mathrm{d}x, \quad {}^p\Phi(u) = {}^p\varphi(u, u),\ \forall u \in {}^pH_d \tag{17}$$

with $w_I^{KL}(\xi) = \mathrm{Inf}\{w_I(q); \widehat{q} = \xi\}$ for all ξ in $\mathbb{S}^2$ and $\widehat{e_1(u')} = \widehat{e(u')}$, $\widehat{e_2(u')} = \widehat{D^2 u'_3}$ for all u' in ${}^pH_d$, where D^2 stands for the second derivative in the distributional sense. We also define

$$ {}^pk(v, v') = \int_\Omega \overline{\rho} v \cdot v' \,\mathrm{d}x, \quad {}^pK(v) = {}^pk(v, v), \quad \forall v \in {}^pH_v, \quad p = 1, 2 \tag{18} $$

so that, for all $U^i = (u^i, v^i)$ in ${}^pH$, the inner product and norm are given by

$$ ((U^1, U^2))_p := {}^p\varphi(u^1, u^2) + {}^pk(v^1, v^2), \quad ||U||_p^2 := ((U, U))_p \tag{19} $$

Let T^ε be the mapping from $L^2(\mathcal{O}^\varepsilon; \mathbb{R}^3)$ into $L^2(\Omega; \mathbb{R}^3)$ defined by

$$ (T^\varepsilon w)(x) := w(x \pm \varepsilon x_3), \quad \forall x \in \Omega_\pm \tag{20} $$

Note that if w belongs to $H^1_{\Gamma_0^\varepsilon}(\mathcal{O}^\varepsilon; \mathbb{R}^3)$ then $T^\varepsilon w$ belongs to $H^1_{\Gamma_0}(\Omega \setminus S; \mathbb{R}^3)$. For any w in $H^1(\Omega_\pm; \mathbb{R}^3)$, we denote the trace of w on S by $\gamma_S^\pm(w)$. Thus, for any w in $H^1(\Omega \setminus S; \mathbb{R}^3)$, the jump of w across S, denoted by $[\![w]\!]$, is $\gamma_S^+(w^+) - \gamma_S^-(w^-)$, $w^\pm$ being the restriction of w to $\Omega_\pm$. Moreover, for any element w of $H^1(\Omega; \mathbb{R}^3)$, its trace on S is denoted by $\gamma_S(w)$.

Lastly, for any $\eta > 0$, let $V_{KL}(B^\eta)$ be the space of Kirchhoff–Love displacements defined by:

$$ \begin{aligned} V_{KL}(B^\eta) &:= \{u \in H^1(B^\eta; \mathbb{R}^3); e_{i3}(u) = 0, 1 \le i \le 3\} \\ &= \{u \in H^1(B^\eta; \mathbb{R}^3); \exists (u^M, u^F) \in H^1(S; \mathbb{R}^2) \times H^2(S)\, s.t. \\ &\qquad \widehat{u}(\widehat{x}, x_3) = u^M(\widehat{x}) - x_3 \nabla u^F(\widehat{x}), u_3(\widehat{x}, x_3) = u^F(\widehat{x})\} \end{aligned} \tag{21} $$

We have

Lemma 3.1. *For all sequences $U_s = (u_s, v_s)$ in H_s such that $|U_s|_s^2$ is bounded, there exists ${}^pU = ({}^pu, {}^pv)$ in ${}^pH$ and a not relabeled subsequence such that*

i) – *$T^\varepsilon u_s$ weakly converges in $H^1(\Omega \setminus S; \mathbb{R}^3)$ toward ${}^pu$,*
  – *$\frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \widehat{u_s}\,\mathrm{d}x_3$ weakly converges in $H^1(S; \mathbb{R}^2)$ toward $\widehat{{}^pu}$,*
  – *$\frac{1}{\varepsilon^3} \int_{-\varepsilon}^{\varepsilon} x_3 \widehat{e(u_s)}\,\mathrm{d}x_3$ weakly converges in $L^2(S)$ toward $-\frac{2}{3}\widehat{D^2({}^2u_3)}$ when $p = 2$,*
  – *${}^p\Phi({}^pu) \le \underline{\lim}_{s \to \bar{s}} \Phi_s(u_s)$,*

ii) – *$T^\varepsilon v_s$ weakly converges in $L^2(\Omega; \mathbb{R}^3)$ toward ${}^pv$,*
  – *${}^pK({}^pv) \le \underline{\lim}_{s \to \bar{s}} K_s(v_s)$.*

Proof. First, the boundedness of $\Phi_s(u_s)$ implies that there exists w in $H^1_{\Gamma_0}(\Omega \setminus S; \mathbb{R}^3)$ and a sequence ρ_s in the space $\mathcal{R}$ of rigid displacements such that $((T^\varepsilon u_s)^+, (T^\varepsilon u_s)^- + \rho_s)$ converges weakly in $H^1_{\Gamma_{0+}}(\Omega_+; \mathbb{R}^3) \times H^1_{\Gamma_{0-}}(\Omega_-; \mathbb{R}^3)$ toward (w^+, w^-). As $[\![T^\varepsilon u_s]\!] = \int_{-\varepsilon}^{\varepsilon} \partial_3 u_s \,\mathrm{d}x_3$, $(\gamma_S^-((T^\varepsilon u_s)^-))_3$ converges strongly in $L^2(S)$ to $(\gamma_S^+(w^+))_3$ due to the boundedness of $\Phi_s(u_s)$, which, combined with $\partial_3(u_s)_\alpha = 2e_{\alpha 3} - \partial_\alpha(u_s)_3$ and

$$ \int_{B^\varepsilon} (u_s)_3^2 \,\mathrm{d}x \le 2\varepsilon \left(\int_S |\gamma_s^+(T^\varepsilon u_s)|^2 \,\mathrm{d}\widehat{x} + 2\varepsilon \int_{B^\varepsilon} |\partial_3(u_s)_3|^2 \,\mathrm{d}x \right) $$

implies the convergence in the sense of distributions of $\gamma_S^-\widehat{((T^\varepsilon u_s)^-)}$ toward $\gamma_S^+\widehat{(w^+)}$. As $\rho_s = (T^\varepsilon u_s)^- + \rho_s - (T^\varepsilon u_s)^-$ lives in a finite dimensional space, $\gamma_S^-(\rho_s)$ converges strongly in $L^2(S; \mathbb{R}^3)$ toward $\gamma_S^-(w^-) - \gamma_S^+(w^+)$ and, consequently, $\gamma_S^-(u_s^-)$ converges strongly in $L^2(S; \mathbb{R}^3)$ toward $\gamma_S^+(w^+)$. This implies that $T^\varepsilon u_s$ converges weakly in $H^1_{\Gamma_0}(\Omega \setminus S; \mathbb{R}^3)$ toward some ${}^pu$ and $[\![{}^pu]\!]$, the strong limit in $L^2(S; \mathbb{R}^3)$ of $[\![T^\varepsilon u_s]\!]$, vanishes, that is to say ${}^pu$ belongs to $H^1_{\Gamma_0}(\Omega; \mathbb{R}^3)$. Of course, the proof is simpler when $\mathcal{H}_2(\Gamma_{0-}) > 0$!

Next, the boundedness of $\Phi_s(u_s)$ allows us to easily identify the weak limit in $H^1(S; \mathbb{R}^2)$ of

$$ \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \widehat{u}_s \,\mathrm{d}x_3 = \gamma_s^+\widehat{((T^\varepsilon u_s)^+)} - \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \int_{x_3}^{\varepsilon} \partial_3 \widehat{u}_s \,\mathrm{d}t \,\mathrm{d}x_3 $$

which implies that ${}^pu$ belongs to ${}^pH_d$. Concerning $\frac{1}{\varepsilon^3} \int_{-\varepsilon}^{\varepsilon} x_3 \widehat{e(u_s)}\,\mathrm{d}x$, we may proceed as in [3] or as follows. Let S_ε be the mapping from $H^1(B^\varepsilon; \mathbb{R}^3)$ into $H^1(B^1; \mathbb{R}^3)$ defined by $\widehat{S_\varepsilon w}(\widehat{x}, x_3) = \varepsilon \widehat{w}(\widehat{x}, \varepsilon x_3)$, $(S_\varepsilon w)_3(\widehat{x}, x_3) = w_3(\widehat{x}, \varepsilon x_3)$, for all $x = (\widehat{x}, x_3)$ in B^1. Then, the boundedness of $\Phi_s(u_s)$ implies the boundedness of $\int_{B^1} |e(\varepsilon, S_\varepsilon u_s)|^2 \,\mathrm{d}x$, where $e(\varepsilon, w) = I_\varepsilon e(w) I_\varepsilon$

and $I_\varepsilon = e_1 \otimes e_1 + e_2 \otimes e_2 + \frac{1}{\varepsilon} e_3 \otimes e_3$. This implies that there exists u in $V_{KL}(B^1)$ and ρ_s in $\mathcal{R}$ such that, up to a subsequence, $S_\varepsilon u_s + \rho_s$ weakly converges toward u in $H^1(B^1;\mathbb{R}^3)$. As for all τ in $L^2(S;\mathbb{S}^2)$ one has

$$\int_S \tau(\widehat{x}) \frac{1}{\varepsilon^3} \int_{-\varepsilon}^{\varepsilon} x_3 \widehat{e(u_s)} \, \mathrm{d}x_3 \, \mathrm{d}\widehat{x} = \int_{B^1} \tau(\widehat{x}) x_3 \widehat{e(S_\varepsilon u_s)} \, \mathrm{d}x$$

one deduces that $\frac{1}{\varepsilon^3}\int_{-\varepsilon}^{\varepsilon} x_3 \widehat{e(u_s)} \, \mathrm{d}x_3$ converges weakly in $L^2(S;\mathbb{S}^2)$ toward

$$\int_{-1}^{+1} \widehat{e(\mathrm{u})} \, \mathrm{d}x = \int_{-1}^{+1} x_3 (\widehat{e}(\mathrm{u}^M) - x_3 D^2(\mathrm{u}^F)) \, \mathrm{d}x_3 = -\frac{2}{3} D^2(\mathrm{u}^F)$$

But the trace on $S + e_3$ of $(S_\varepsilon w)_3$ being equal to $(\gamma_S^+((T^\varepsilon u_s)^+)_3$, one deduces that $\mathrm{u}^F = (\gamma_s(^2 u))_3$.

Finally, the lower bound for $\Phi_s(u_s)$ is obtained by a simple use of the Jensen inequality and a standard lower semi-continuity argument, which is the source of the term w_I^{KL}.

The point *ii*) is obvious. □

We can now define the limit evolution operator ${}^pA$ through

$$\begin{cases} D({}^pA) = \left\{ U = (u,v) \in {}^pH; \begin{cases} \text{i)} & v \in {}^pH_\mathrm{d} \\ \text{ii)} & \exists (w,\xi) \in L^2(\Omega;\mathbb{R}^3) \times \partial \mathcal{D}^{KL}(\widehat{e_p(v)}) \text{ s.t.} \end{cases} \right. \\ \qquad {}^pk(w,v') + {}^p\varphi(u,v') + \overline{b}_p \int_S \xi \cdot \widehat{e}_p(v') \, \mathrm{d}\hat{x} = 0, \quad \forall v' \in {}^pH_d \Big\} \\ {}^pAU = (-v, 0) + \{(0,-w);\ w \text{ satisfying ii)}\} \end{cases} \tag{22}$$

$\mathcal{D}^{KL}$ being defined in the same way as w_I^{KL}.

Similar to the case of A_s, it can be checked easily that ${}^pA$ is maximal monotone and, more specifically, that for all $\psi = (\psi^1, \psi^2)$ in ${}^pH$:

$$\begin{cases} {}^p\overline{U} = ({}^p\overline{u}, {}^p\overline{v}) \text{s.t.} \\ {}^p\overline{U} + {}^pA{}^p\overline{U} \ni \psi \end{cases} \Leftrightarrow \begin{cases} {}^pJ({}^p\overline{v}) \le {}^pJ(v) := \frac{1}{2}{}^pK(v) - {}^pk(\psi^2, v) + \frac{1}{2}{}^p\Phi(v) + \\ \qquad + {}^p\varphi(\psi^1, v) + \overline{b}_p \int_S \mathcal{D}^{KL}(\widehat{e_p(v)}) \, \mathrm{d}\hat{x}, \quad \forall v \in {}^pH_d \\ {}^p\overline{u} = {}^p\overline{v} + \psi^1 \end{cases} \tag{23}$$

Consequently, the same statement as that of Theorem 2.1 is valid for the following equation, which will be shown to describe the asymptotic behavior of u_s:

$$\frac{\mathrm{d}{}^pU^r}{\mathrm{d}t} + {}^pA{}^pU^r \ni {}^pF := \left(-\frac{\mathrm{d}{}^pu^e}{\mathrm{d}t}, f/\overline{\rho}\right), \quad {}^pU^r(0) = {}^pU^{r0} \tag{24}$$

with

$$\begin{aligned} & {}^pu^e \in BV^{(2)}(0,T;{}^pH_d);\ {}^p\varphi({}^pu^e(t), u') = L(t)(u'), \quad \forall u' \in {}^pH_d, \forall t \in [0,T] \\ & L(t)(u') = \int_{\Gamma_1} g(x,t) \cdot v(x) \, \mathrm{d}\mathcal{H}_2 \end{aligned} \tag{25}$$

We set

$${}^pU^e = \left({}^pu^e, 0\right), \quad {}^pU = {}^pU^e + {}^pU^r \tag{26}$$

3.2. Convergence

As in [1], to prove the convergence of u_s toward ${}^pu = {}^pu^e + {}^pu^r$, we will use the framework of a nonlinear version of Trotter's theory of convergence of semigroups acting on *variable spaces* (see [4,5] and Appendix of [6]) because u_s^r and ${}^pu^r$ *do not inhabit the same space*. To establish the convergence of the mechanical state, we need to compare the elements of ${}^pH$ to those of H_s. We therefore define ${}^pP_s$ by:

$$(u,v) \in {}^pH \mapsto {}^pP_s(u,v) = (u_s^*, v_s^*) \in H_s \tag{27}$$

with

– $u_s^* \in H^1_{\Gamma_0^\varepsilon}(\mathcal{O}^\varepsilon;\mathbb{R}^3)$; $\varphi_s(u_s^*, u') = {}^p l_\varepsilon(u, u')$, $\forall u' \in H^1_{\Gamma_0^\varepsilon}(\mathcal{O}^\varepsilon;\mathbb{R}^3)$

$$^p l_\varepsilon(u,u') = \int_\Omega a^\varepsilon e(u)\cdot e(u')\,\mathrm{d}x + \mu \int_{B^\varepsilon} D w_I^{KL}(\widehat{e(^p\nu)})\cdot \widehat{e(u')}\,\mathrm{d}x$$

where ${}^1\nu, {}^2\nu \in V_{KL}(B^\varepsilon)$ with $({}^1\nu^M, {}^1\nu^F) = (\gamma_S(\widehat{u}), 0)$, $({}^2\nu^M, {}^2\nu^F) = (0, \gamma_S(u_3))$,

– $v_s^*(x) = v(x \mp \varepsilon x_3)$, a.e. $x \in \Omega^\varepsilon_\pm$, $v_s^*(x) = 0$ a.e. $x \in B^\varepsilon$.

Taking advantage of the *variational definition* of u_s^*, Lemma 3.1 and the classical procedure of mathematical justification of Kirchhoff–Love theory of plates (cf. [7]) imply that ${}^pP_s$ enjoys the following fundamental property.

Proposition 3.1.

i) *There exists a strictly positive constant C such that* $|{}^pP_sU|_s \le C||U||_p$, $\forall U \in {}^pH$.
ii) *When s tends to* $\bar{s}$, $|{}^pP_sU|_s$ *converges toward* $\|U\|_p$ *for all U in* ${}^pH$.

Next we state that:

$$U_s \text{ in } H_s \text{ converges in the sense of Trotter toward } U \text{ in } {}^pH \text{ if } \lim_{s\to\bar{s}} |{}^pP_sU - U_s|_s = 0 \tag{28}$$

Even if this is the right mechanical notion, it could be of interest to consider this convergence with respect to some classical conventional notions.

Proposition 3.2. *For all* $U = (u, v)$ *in* ${}^pH$, $U_s = (u_s, v_s)$ *in* H_s *converges in the sense of Trotter toward U if and only if:*

i) $T^\varepsilon u_s$ *converges strongly in* $H^1(\Omega \setminus S;\mathbb{R}^3)$ *toward* u,
ii) $\frac{1}{2\varepsilon}\int_\varepsilon^\varepsilon \widehat{u}_s\,\mathrm{d}x_3$ *converges strongly in* $H^1(S;\mathbb{R}^2)$ *toward* $\widehat{u}$,
iii) ${}^p\Phi(u) = \lim_{s\to\bar{s}} \Phi_s(u_s)$,
iv) $T^\varepsilon v_s$ *converges strongly in* $L^2(\Omega;\mathbb{R}^2)$ *toward* v,
v) ${}^pK(v) = \lim_{s\to\bar{s}} K_s(v_s)$.

Lastly, we conclude by using a suitable nonlinear version (see [5,6]) of Trotter's theory of convergence of semigroups, where it suffices to make an additional assumption ($\mathbf{H}_3$) about the initial states and to establish the following "static" result.

Proposition 3.3. *We have*

i) $\forall \psi \in {}^pH$, $\lim_{s\to\bar{s}} |{}^pP_s(I + {}^pA)^{-1}\psi - (I + A_s)^{-1}\,{}^pP_s\psi|_s = 0$,
ii) $\lim_{s\to\bar{s}} |{}^pP_s{}^pU^e(t) - U_s^e(t)|_s = 0$ *uniformly on* $[0, T]$,
iii) $\lim_{s\to\bar{s}} \int_0^T |{}^pP_s{}^pF(t) - F_s(t)|_s\,\mathrm{d}t = 0$.

As regards point *i*), we use the same strategy as in [1] by due account of (11) and (23). Taking advantage of Lemma 3.1 and the variational definition of P_s, we obtain that a subsequence of minimizers of $\widetilde{J}_s$ defined by

$$\widetilde{J}_s(v) = \frac{1}{2}K_s(v) - k_p(\psi^2, T^\varepsilon v) + \frac{1}{2}\Phi_s(v) + \varphi_s(\psi_s^{1^*}, v) + b\int_{B^\varepsilon} \mathcal{D}(e(v))\,\mathrm{d}x \tag{29}$$

converges to an element $\overline{v}$ in H_d satisfying ${}^pJ(\overline{v}) \le \underline{\lim}_{s\to\bar{s}} J_s(v_s)$. Indeed, $\overline{v}$ is the unique minimizer of ${}^pJ$ because, due to Proposition 3.1, for all w in ${}^pH_d$, one has $\lim_{s\to\bar{s}} J_s(w_s^*) = {}^pJ(w)$. Similar arguments as those of [1] establish *ii*) and *iii*). Thus, we deduce the convergence uniformly on $[0, T]$ in the sense of Trotter of the solution to (12) toward that to (24) with ${}^pU^{r0} := {}^pU^0 - {}^pU^e(0)$ and the additional conditions of convergence and compatibility between the initial state and loading:

$$\exists {}^pU^0 \in {}^pU^e(0) + D({}^pA); \quad U_s^0 \in U_s^e(0) + D(A_s) \text{ and } \lim_{s\to\bar{s}} |{}^pP_s{}^pU^0 - U_s^0|_s = 0 \tag{$\mathbf{H}_3$}$$

This can be rephrased in a more explicit way with respect to $(\mathcal{P}_s)$:

Theorem 3.1. *The solution to*

$$\frac{\mathrm{d}U_s}{\mathrm{d}t} + A_s(U_s - U_s^e) \ni (0, f^\varepsilon/\overline{\rho}^\varepsilon), \quad U_s(0) = U_s^0 \tag{30}$$

converges toward the solution to

$$\frac{\mathrm{d}^pU}{\mathrm{d}t} + {}^pA({}^pU - {}^pU^e) \ni (0, f/\bar{\rho}), \quad {}^pU(0) = {}^pU^0 \tag{31}$$

in the sense $\lim_{s\to\bar{s}} |{}^pP_s{}^pU(t) - U_s(t)|_s = 0$, $\lim_{s\to\bar{s}} |U_s(t)|_s = \|{}^pU(t)\|_p$ *uniformly on* $[0, T]$.

4. Concluding remarks

It is worthwhile to write (31) in a variational form:

$\exists \xi \in \partial\mathcal{D}^{KL}\Big(\widehat{e}_p(v)\Big)$ such that

$$\int_\Omega \overline{\rho}\, \frac{\mathrm{d}^2\,{}^pu}{\mathrm{d}t^2} \cdot \varphi\, \mathrm{d}x + \int_\Omega ae({}^pu) \cdot e(\varphi)\, \mathrm{d}x + \overline{\mu}_p \int_S Dw_I^{KL}\big(\widehat{e_p({}^pu)}\big) \cdot \big(\widehat{e_p(\varphi)}\big)\, \mathrm{d}\hat{x} + \overline{b}_p \int_S \xi \cdot \big(\widehat{e_p(\varphi)}\big)\, \mathrm{d}\hat{x}$$
$$= \int_\Omega f \cdot \varphi\, \mathrm{d}x + \int_{\Gamma_1} g \cdot \varphi\, \mathrm{d}\mathcal{H}_2, \quad \forall \varphi \in {}^pH_d$$

where ${}^pH_d$ and $\widehat{e_p(\cdot)}$ are defined in (16) and (17), respectively. Hence, the limit behavior describes the dynamic response to the real loads (f, g) of a structure consisting of two linearly elastic adherents occupying $\Omega_\pm$, which are perfectly bonded to a material deformable flat surface whose behavior is of the same kind as the genuine adhesive (*i.e.* non-linear viscoelasticity of Kelvin–Voigt generalized type). Moreover, the mass of the adhesive being evanescent, there is no inertial term in the interface condition. The case $p = 1$ corresponds to membrane deformations, whereas the case $p = 2$ corresponds to flexural deformations.

The Proposition 3.3 covers the static situation which has been considered in [8]. Our limit interface condition agrees with the one of [3], which studied a resembling problem (the adherents occupying the complementary of B^ε in a fixed domain Ω).

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