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Complete band gaps including non-local effects occur only in the relaxed micromorphic model

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A B S T R A C T

In this paper, we substantiate the claim implicitly made in previous works that the relaxed micromorphic model is the *only linear, isotropic, reversibly elastic, nonlocal generalized continuum model* able to describe complete band-gaps on a phenomenological level. To this end, we recapitulate the response of the standard Mindlin–Eringen micromorphic model with the full micro-distortion gradient ∇P, the relaxed micromorphic model depending only on the $\operatorname{Curl} P$ of the micro-distortion P, and a variant of the standard micromorphic model, in which the curvature depends only on the divergence $\operatorname{Div} P$ of the micro distortion. The Div-model has size-effects, but the dispersion analysis for plane waves shows the incapability of that model to even produce a partial band gap. Combining the curvature to depend quadratically on $\operatorname{Div} P$ and $\operatorname{Curl} P$ shows that such a model is similar to the standard Mindlin–Eringen model, which can eventually show only a partial band gap.

1. Introduction

The micromorphic model [1–5] is a generalized continuum model suitable for the effective multi-scale-description of heterogeneous media with strong contrast of the mechanical properties at the microscopic level through the introduction of a *characteristic length scale* L_c. It allows us to incorporate new effects, which extend the classical linear elastic description, e.g., *size effects*, the *dispersion of waves* and the possibility of micro-motions, which are in principle independent of the macro motions. This model couples the *macroscopic displacement field* $u : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ with an *affine substructure deformation* attached to each macroscopic point encoded by the *micro-distortion field* $P : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$.

The curvature contribution in the micromorphic model conceptually determines how the substructure interacts with itself, and the associated characteristic length is a measure of the range of action of such micro-structure-related deformation modes. In this sense, we call the full-gradient contribution $\|\nabla P\|^2$ (or any other curvature term essentially controlling ∇P)

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of *strong-interaction* type: neighboring substructures feel the presence of each other, or, what is the same, the generated moment stresses depend on ∇P.

To the contrary, in the relaxed micromorphic model, the corresponding moment stresses depend only on Curl P, therefore there is some freedom between particles, but a connection of neighboring cells is still possible thanks to tangent micro-interactions. Certain substructure deformations are energetically free (in fact, all compatible parts $\nabla\vartheta$ in P are not taken into account), while the model remains reversibly elastic and energy-conservative. We may call this a *weak interaction*. As a matter of fact, the wording *relaxed* is motivated by this observation.

In the Div-model to be introduced below, a similar effect appears. The corresponding moment stresses depend only on Div P. Therefore, substructure deformations of the type $P = \mathrm{Curl}\,\zeta + \nabla\vartheta$, where $\zeta : \mathbb{R}^{3\times3} \to \mathbb{R}^{3\times3}$ is arbitrary and $\vartheta : \mathbb{R}^3 \to \mathbb{R}^3$ satisfies $\Delta\vartheta \equiv 0$, are energetically free. This model is, hence, also of weak-interaction type.

It is therefore intriguing that it is not simply the "*weak* versus *strong interaction*" duality which determines the possibility of band gaps, but that there is some further hidden mechanism in the relaxed micromorphic model, which, together with a positive Cosserat couple modulus $\mu_c > 0$, is decisive for the ability to model complete band gaps, whereas still being nonlocal.

In further contributions, we will provide more detailed arguments concerning the fact that the residual freedom which is peculiar to the relaxed micromorphic model is a key feature for allowing band-gap behaviors. In fact, internal variable models (i.e. models with no dependence on the derivatives of P at all) still allow the description of complete band gaps [6,7], but they loose any information concerning non-locality. Non-local effects are intrinsically present in micro-structured materials, even if in some particular cases their overall effect can be, in a first approximation, neglected. Nevertheless, as far as the contrast of mechanical properties between adjacent unit cells at the micro level becomes more pronounced, non-local effects are sensible enough to rapidly become non-negligible. In this optic, a model including non-locality is to be considered as the natural choice for modeling the mechanical behavior of metamaterials.

This paper is now structured as follows. First, we introduce the relaxed micromorphic model with an augmented curvature energy depending also on Div P. The governing equations are derived and the plane-wave ansatz is introduced to study wave propagation. Then we particularize the result for specific cases and show the resulting dispersion curves for each of them. Finally, we provide for completeness the standard Mindlin–Eringen micromorphic model together with its dispersion curves, thus recognizing that it is equivalent to a particular case of the augmented relaxed micromorphic model with Div P.

2. The relaxed micromorphic continuum with $\|\mathrm{Curl}\, P\|^2$ and $\|\mathrm{Div}\, P\|^2$

The *relaxed micromorphic model* [8–11] has been introduced in 2013 in [9] and endows the standard Mindlin–Eringen representation with more geometric structure by reducing the curvature energy term to depend *only* on the *second-order dislocation density tensor* $\alpha = -\mathrm{Curl}\, P$. Here, we additionally consider also a curvature term depending on Div P. The strain energy density for the resulting micromorphic continuum can be written as:

$$W = \underbrace{\mu_e \,\|\mathrm{sym}\,(\nabla u - P)\,\|^2 + \frac{\lambda_e}{2}\,(\mathrm{tr}\,(\nabla u - P))^2}_{\text{isotropic elastic energy}} + \underbrace{\mu_c \,\|\mathrm{skew}\,(\nabla u - P)\,\|^2}_{\text{rotational elastic coupling}} \tag{1}$$
$$+ \underbrace{\mu_{\mathrm{micro}}\,\|\mathrm{sym}\, P\|^2 + \frac{\lambda_{\mathrm{micro}}}{2}\,(\mathrm{tr}\, P)^2}_{\text{micro-self-energy}} + \underbrace{\frac{\mu\, L_c^2}{2}\,\|\mathrm{Curl}\, P\|^2 + \frac{\mu\, L_d^2}{2}\,\|\mathrm{Div}\, P\|^2}_{\text{simple isotropic curvature}}$$

where all the introduced elastic coefficients are assumed to be constant. This decomposition of the strain energy density, valid in the isotropic, linear-elastic case, has been proposed in [8,12] where well-posedness theorems have also been proved. It is clear that this decomposition introduces a limited number of elastic parameters and we will show how this may help in the physical interpretation of the latter. The positive definiteness of the potential energy implies the following simple relations on the introduced parameters

$$\mu_e > 0, \quad \mu_c \geq 0, \quad 3\lambda_e + 2\mu_e > 0, \quad \mu_{\mathrm{micro}} > 0, \quad 3\lambda_{\mathrm{micro}} + 2\mu_{\mathrm{micro}} > 0, \quad \mu\, L_c^2 > 0, \quad \mu\, L_d^2 > 0 \tag{2}$$

We need to remark that this model variant is not strictly positive definite in the sense of the standard Mindlin–Eringen model. One of the most interesting features of the proposed strain energy density is the reduced number of elastic parameters which are needed to fully describe the mechanical behavior of a micromorphic continuum. Indeed, each parameter can be easily related to specific micro- and macro-deformation modes.

Comparing classical linear elasticity with our new relaxed model for $L_c, L_d \to 0$, we can offer an *a priori relation* between μ_e, λ_e, μ_{micro} and λ_{micro} on the one side and the effective macroscopic elastic parameters λ_{macro} and μ_{macro} on the other side, which we call *macroscopic consistency condition* (see [13] for the fully anisotropic case and [14] for the isotropic case):

$$\mu_{\mathrm{macro}} := \frac{\mu_{\mathrm{micro}}\,\mu_e}{\mu_{\mathrm{micro}} + \mu_e}\,, \qquad 2\mu_{\mathrm{macro}} + 3\lambda_{\mathrm{macro}} := \frac{(2\mu_{\mathrm{micro}} + 3\lambda_{\mathrm{micro}})\,(2\mu_e + 3\lambda_e)}{(2\mu_{\mathrm{micro}} + 3\lambda_{\mathrm{micro}}) + (2\mu_e + 3\lambda_e)} \tag{3}$$

For $\mu_{\mathrm{micro}} \to \infty$, we recover the *Cosserat model or micropolar model*, which means that $P \in \mathfrak{so}(3)$, and for $L_c \to 0$ we obtain classical linear elasticity with μ_{macro}, λ_{macro} from (3).

For comparison, the standard isotropic Mindlin–Eringen model with $\mu_c > 0$ and curvature energy depending on $\|\nabla P\|^2$ tends to a *second gradient model* when $\mu_e, \mu_c \to \infty$.

The dynamical formulation is obtained defining the kinetic and strain energy densities of the considered mechanical system and postulating a stationary action principle. For this, we introduce a micro-inertia density contribution:

$$J\left(u_{,t},\ P_{,t}\right) = \frac{1}{2}\rho\left\|u_{,t}\right\|^2 + \frac{1}{2}\eta\left\|P_{,t}\right\|^2 \tag{4}$$

where η is the scalar *micro-inertia density* and ρ is the scalar *mean density*.

For us, it is not at all surprising that the combination of Curl and Div in the curvature contribution at positive Cosserat couple modulus behaves similarly as does the full-micro gradient model. This is understandable, since after integration and imposing boundary conditions, we have the well-known inequality [15]:

$$\exists C^+ > 0 \qquad \forall P \in C_0^\infty(\Omega, \mathbb{R}^{3\times 3}): \quad \int_\Omega \|\mathrm{Curl}\, P\|^2 + \|\mathrm{Div}\, P\|^2 \mathrm{d}x \geq C^+(\Omega)\int_\Omega \|\nabla P\|^2 \mathrm{d}x \tag{5}$$

Equation (5) means that $\|\mathrm{Curl}\, P\|^2$ and $\|\mathrm{Div}\, P\|^2$ considered point-wise are not equivalent to the full gradient term $\|\nabla P\|^2$, but they become so after integration. Therefore, the Curl–Div-model effectively controls all first derivatives of P. In consequence, the dispersion relations are similar, as can clearly be seen comparing Figs. 7 and 8 with Figs. 1 and 2.

It should also be remarked that the well-posedness of the Div-model ($L_c = 0$) needs a strictly positive Cosserat couple modulus $\mu_c > 0$, since an inequality of the type:

$$\exists C^+ > 0 \qquad \forall P \in C_0^\infty(\Omega, \mathbb{R}^{3\times 3}): \quad \int_\Omega \|\mathrm{sym}\, P\|^2 + \|\mathrm{Div}\, P\|^2 \mathrm{d}x \geq C^+(\Omega)\int_\Omega \|P\|^2 \mathrm{d}x + \|\mathrm{Div}\, P\|^2 \mathrm{d}x \tag{6}$$

is *not true*. Then for $\mu_c > 0$, there is no need for any additional inequality, since the elastic energy density bounds a priori

$$\int_\Omega \|P\|^2 + \|\mathrm{Div}\, P\|^2 \mathrm{d}x \tag{7}$$

Therefore, the corresponding suitable space is a tensor-valued $H(\mathrm{Div})$-Sobolev space.

Both expressions $\mathrm{Div}\, P$ and $\mathrm{Curl}\, P$ can be used to formulate a complete anisotropic curvature energy. This is possible since $\mathrm{Div}\, P$ and $\mathrm{Curl}\, P$ are not arbitrary collections of partial derivatives of P, but satisfy the transformation laws:

$$\begin{aligned} &\mathrm{Curl}_\xi\, P^\#(\xi) = Q\,[\mathrm{Curl}_x\, P(x)]\, Q^T, \qquad \xi = Q^T x, \qquad \text{where } P^\#(\xi) := Q\, P(Q^T \xi)\, Q^T \\ &\mathrm{Div}_\xi\, P^\#(\xi) = Q\,[\mathrm{Div}_x\, P(x)] \end{aligned} \tag{8}$$

with respect to simultaneous rigid rotations Q of the spatial and referential frame [16, eq. (4.29)]. Therefore, we may make the ansatz:

$$\begin{aligned} W(\nabla P) &= W_{\mathrm{Curl}}(\mathrm{Curl}\, P) + W_{\mathrm{Div}}(\mathrm{Div}\, P) \\ &= \frac{\mu L_c^2}{2}\left\langle \mathbb{L}_{\mathrm{aniso}}\, \mathrm{Curl}\, P,\ \mathrm{Curl}\, P\right\rangle_{\mathbb{R}^{3\times 3}} + \frac{\mu L_c^2}{2}\left\langle \widetilde{\mathbb{C}}_{\mathrm{aniso}}\, \mathrm{Div}\, P, \mathrm{Div}\, P\right\rangle_{\mathbb{R}^3} \end{aligned} \tag{9}$$

where $\mathbb{L}_{\mathrm{aniso}} : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$ is a fourth-order tensor with in general 45 independent coefficients and $\widetilde{\mathbb{C}}_{\mathrm{aniso}} : \mathbb{R}^3 \to \mathbb{R}^3$ (for isotropy $\widetilde{\mathbb{C}}_{\mathrm{aniso}}$ has just 1 parameter [13]). In case of isotropy, this can be significantly reduced to:

$$W(\nabla P) = \frac{\mu L_c^2}{2}\left[\alpha_1 \|\mathrm{dev\,sym\,Curl}\, P\|^2 + \alpha_2 \|\mathrm{skew\,Curl}\, P\|^2 + \frac{\alpha_3}{3}(\mathrm{tr\,Curl}\, P)^2 + \alpha_4 \|\mathrm{Div}\, P\|^2\right] \tag{10}$$

2.1. Governing equations

The Lagrangian density $\mathscr{L}$ for the augmented relaxed model is defined as follows:

$$\mathscr{L}\left(u_{,t},\ P_{,t},\ \nabla u\ ,\ P,\ \mathrm{Curl}\, P, \mathrm{Div}\, P\right) = J\left(u_{,t},\ P_{,t}\right) - W\left(\nabla u\ ,\ P,\ \mathrm{Curl}\, P, \mathrm{Div}\, P\right) \tag{11}$$

In order to find the strong equations of motion, we have to perform the first variation of the action functional

$$\mathscr{A}[(u,\ P)] := \int_I \int_\Omega \mathscr{L}\left(u_{,t},\ P_{,t},\ \nabla u\ ,\ P,\ \mathrm{Curl}\, P, \mathrm{Div}\, P\right) \mathrm{d}x\, \mathrm{d}t \tag{12}$$

where $I = [a, b]$ is the time interval during which we observe the motion of our system. For the kinetic part we compute

$$\delta \int_I \int_\Omega J\left(u_{,t},\ P_{,t}\right) \mathrm{d}x\,\mathrm{d}t = \int_I \int_\Omega \left[D_{u_{,t}} J\left(u_{,t},\ P_{,t}\right) \cdot \delta u_{,t} + D_{P_{,t}} J\left(u_{,t},\ P_{,t}\right) \cdot \delta P_{,t}\right] \mathrm{d}x\,\mathrm{d}t \tag{13}$$
$$= \int_I \int_\Omega \frac{1}{2}\left[D_{u_{,t}}\left(\rho\left\langle u_{,t}, u_{,t}\right\rangle\right) \cdot \delta u_{,t} + D_{P_{,t}}\left(\eta\left\langle P_{,t},\ P_{,t}\right\rangle\right) \cdot \delta P_{,t}\right] \mathrm{d}x\,\mathrm{d}t$$
$$= \int_I \int_\Omega \left[\rho\left\langle u_{,t}, \delta u_{,t}\right\rangle + \eta\left\langle P_{,t}, \delta P_{,t}\right\rangle\right] \mathrm{d}x\,\mathrm{d}t$$
$$= \rho \int_\Omega \left(\left\langle u_{,t}, \delta u\right\rangle\Big|_a^b - \int_I \left\langle u_{,tt}, \delta u\right\rangle \mathrm{d}t\right) \mathrm{d}x + \eta \int_\Omega \left(\left\langle P_{,t}, \delta P\right\rangle\Big|_a^b - \int_I \left\langle P_{,tt}, \delta P\right\rangle \mathrm{d}t\right) \mathrm{d}x$$

So, considering only the bulk part, we find

$$\int_\Omega \int_I \left\langle -\rho\, u_{,tt}, \delta u\right\rangle \mathrm{d}t\,\mathrm{d}x + \int_\Omega \int_I \left\langle -\eta\, P_{,tt}, \delta P\right\rangle \mathrm{d}t\,\mathrm{d}x \tag{14}$$

For the potential part, we find

$$\delta \int_I \int_\Omega W\,\mathrm{d}x\,\mathrm{d}t = \int_I \int_\Omega \left[\left\langle D_{\nabla u} W, \delta \nabla u\right\rangle + \left\langle D_P W, \delta P\right\rangle + \left\langle D_{\mathrm{Curl}\,P} W, \delta\,\mathrm{Curl}\,P\right\rangle + \left\langle D_{\mathrm{Div}\,P} W, \delta\,\mathrm{Div}\,P\right\rangle\right] \mathrm{d}x\,\mathrm{d}t \tag{15}$$

Having already evaluated the part $\left\langle D_{\nabla u} W, \delta \nabla u\right\rangle + \left\langle D_P W, \delta P\right\rangle + \left\langle D_{\mathrm{Curl}\,P} W, \delta\,\mathrm{Curl}\,P\right\rangle$ in [11], we perform the explicit calculation only for the term in $\mathrm{Div}\,P$. So, we have

$$\delta \int_I \int_\Omega \frac{\mu L_\mathrm{d}^2}{2} \|\mathrm{Div}\,P\|^2 \,\mathrm{d}x\,\mathrm{d}t = \int_I \int_\Omega \frac{\mu L_\mathrm{d}^2}{2}\, \delta \|\mathrm{Div}\,P\|^2 \,\mathrm{d}x\,\mathrm{d}t = \int_I \int_\Omega \mu L_\mathrm{d}^2 \left\langle \mathrm{Div}\,P, \delta\,\mathrm{Div}\,P\right\rangle \mathrm{d}x\,\mathrm{d}t \tag{16}$$
$$= \int_I \int_\Omega \mu L_\mathrm{d}^2 \left\langle \mathrm{Div}\,P, \mathrm{Div}\,\delta P\right\rangle \mathrm{d}x\,\mathrm{d}t$$

with[1]

$$\left\langle \mathrm{Div}\,P, \mathrm{Div}\,\delta P\right\rangle = \mathrm{Div}\left(\mathrm{Div}\,P \cdot \delta P\right) - \left\langle \nabla\,\mathrm{Div}\,P, \delta P\right\rangle \tag{17}$$

which in index notation is

$$P_{ij,j}\delta P_{ih,h} = \left(P_{ij,j}\delta P_{ih}\right)_{,h} - P_{ij,jh}\delta P_{ih} \tag{18}$$

we integrate by parts and find that

$$\delta \int_I \int_\Omega \frac{\mu L_\mathrm{d}^2}{2} \|\mathrm{Div}\,P\|^2 \,\mathrm{d}x\,\mathrm{d}t = \int_I \int_\Omega \mu L_\mathrm{d}^2 \left[\mathrm{Div}\left(\mathrm{Div}\,P \cdot \delta P\right) - \left\langle \nabla\,\mathrm{Div}\,P, \delta P\right\rangle\right] \mathrm{d}x\,\mathrm{d}t \tag{19}$$
$$= \int_I \int_{\partial\Omega} \mu L_\mathrm{d}^2 \left\langle \mathrm{Div}\,P \cdot \delta P,\ n\right\rangle \mathrm{d}s\,\mathrm{d}t + \int_I \int_\Omega \left\langle -\mu L_\mathrm{d}^2\, \nabla\,\mathrm{Div}\,P, \delta P\right\rangle \mathrm{d}x\,\mathrm{d}t$$

where n is the unit normal field to the boundary. Considering only the kinetic energy associated with P and the potential energy related to $\mathrm{Div}\,P$, we have

$$\int_I \int_\Omega \left(\frac{1}{2}\eta \left\| P_{,t}\right\|^2 - \frac{\mu L_\mathrm{d}^2}{2} \|\mathrm{Div}\,P\|^2\right) \mathrm{d}x\,\mathrm{d}t \tag{20}$$

and, with reference to equations (14) and (19), the bulk part of the first variation is

[1] Here and in the sequel, $\langle\cdot,\cdot\rangle$ denotes the scalar product between two tensor of orders greater than one (e.g., $\left\langle A, B\right\rangle = A_{ij}B_{ij}$). Moreover, a central dot stands for the simple contraction between two tensors of order greater than one. For example, $(A \cdot v)_i = A_{ij}v_j$. Finally, we use Einstein's convention of sum over repeated indexes if not differently specified.

$$\int_I \int_\Omega \left(\langle -\eta\, P_{,tt}, \delta P \rangle - \langle -\mu\, L_d^2\, \nabla \operatorname{Div} P, \delta P \rangle \right) \mathrm{d}x\, \mathrm{d}t = \int_I \int_\Omega \langle -\eta\, P_{,tt} + \mu\, L_d^2\, \nabla \operatorname{Div} P, \delta P \rangle \mathrm{d}x\, \mathrm{d}t \tag{21}$$

Altogether, see also [11], the strong equations in the bulk are

$$\begin{aligned} \rho\, u_{,tt} &= \operatorname{Div}[2\,\mu_e \operatorname{sym}(\nabla u - P) + \lambda_e \operatorname{tr}(\nabla u - P)\,\mathbb{1} + 2\,\mu_c \operatorname{skew}(\nabla u - P)] \\ \eta\, P_{,tt} &= 2\,\mu_e \operatorname{sym}(\nabla u - P) + \lambda_e \operatorname{tr}(\nabla u - P)\,\mathbb{1} + 2\,\mu_c \operatorname{skew}(\nabla u - P) \\ &\quad - 2\,\mu_{\text{micro}} \operatorname{sym} P - \lambda_{\text{micro}} \operatorname{tr}(P)\,\mathbb{1} - \mu\, L_c^2 \operatorname{Curl}\operatorname{Curl} P + \underbrace{\mu\, L_d^2\, \nabla \operatorname{Div} P}_{\text{new augmented term}} \end{aligned} \tag{22}$$

In our study of wave propagation in micromorphic media, we limit ourselves to the case of *plane waves* traveling in an *infinite domain*. We suppose that the space dependence of all introduced kinematic fields is limited to the component x_1 of x, which is also the direction of propagation of the wave. Therefore, we look for solutions to Eq. (22) in the form:

$$u(x,t) = \alpha\, \mathrm{e}^{\mathrm{i}(k\,x_1 - \omega\, t)}, \quad \alpha \in \mathbb{R}^3, \qquad P(x,t) = \beta\, \mathrm{e}^{\mathrm{i}(k\,x_1 - \omega\, t)}, \quad \beta \in \mathbb{R}^{3\times 3} \tag{23}$$

2.2. Decomposition of the equations of motion

Considering the system of PDEs found in (22), we can rewrite this system in a fashion more convenient for the study of the propagation of plane waves in a homogeneous isotropic medium. Our approach consists always in projecting the found relations in the three orthogonal sub vector spaces $\operatorname{Sym}(3) \cap \mathfrak{sl}(3), \mathfrak{so}(3), \langle \mathbb{1} \rangle$. In this way, a tensor $X \in \mathbb{R}^{3\times 3}$ is uniquely written by means of the Cartan–Lie decomposition as:

$$X = \operatorname{dev}\operatorname{sym}(X) + \operatorname{skew}(X) + \frac{1}{3}\operatorname{tr}(X)\,\mathbb{1} \tag{24}$$

where

$$\operatorname{dev}\operatorname{sym}(X) = \begin{pmatrix} X^{\mathrm{D}} & X_{(12)} & X_{(13)} \\ X_{(12)} & X_2^{\mathrm{D}} & X_{(23)} \\ X_{(13)} & X_{(23)} & X_3^{\mathrm{D}} \end{pmatrix}, \quad \operatorname{skew}(X) = \begin{pmatrix} 0 & X_{[12]} & X_{[13]} \\ -X_{[12]} & 0 & X_{[23]} \\ -X_{[13]} & -X_{[23]} & 0 \end{pmatrix} \tag{25}$$
$$\frac{1}{3}\operatorname{tr}(X)\,\mathbb{1} = X^{\mathrm{S}}\mathbb{1}$$

in which we set

$$\begin{aligned} &X^{\mathrm{S}} = \frac{1}{3}(X_{11} + X_{22} + X_{33}), && X_{[12]} = \frac{1}{2}(X_{12} - X_{21}), && X_{(12)} = \frac{1}{2}(X_{12} + X_{21}) \\ &X^{\mathrm{D}} = X_{11} - X^{\mathrm{S}}, && X_{[13]} = \frac{1}{2}(X_{13} - X_{31}), && X_{(13)} = \frac{1}{2}(X_{13} + X_{31}) \\ &X_\alpha^{\mathrm{D}} = X_{\alpha\alpha} - X^{\mathrm{S}}, \quad \alpha = 2,3, && X_{[23]} = \frac{1}{2}(X_{23} - X_{32}), && X_{(23)} = \frac{1}{2}(X_{23} + X_{32}) \end{aligned} \tag{26}$$

The components X_2^{D} and X_3^{D} are not independent, but are related by the following relation

$$X_2^{\mathrm{D}} - X_3^{\mathrm{D}} = X^V = P_{22} - P_{33} \tag{27}$$

In this way, applying the Cartan–Lie decomposition to the tensor $X = \operatorname{sym} P$ in the first equation and to all the tensors appearing in the second one, the equations (22) can be written as follows

$$\begin{aligned} \rho\, u_{,tt} &= \operatorname{Div}[2\,\mu_e \operatorname{sym}(\nabla u - P) + \lambda_e \operatorname{tr}(\nabla u - P)\,\mathbb{1} + 2\,\mu_c \operatorname{skew}(\nabla u - P)] \\ \eta\, (\operatorname{dev}\operatorname{sym} P_{,tt}) &= 2\,\mu_e \operatorname{dev}\operatorname{sym}(\nabla u - P) - 2\,\mu_{\text{micro}} \operatorname{dev}\operatorname{sym} P - \mu\, L_c^2 \operatorname{dev}\operatorname{sym}(\operatorname{Curl}\operatorname{Curl} P) \\ &\quad + \mu\, L_d^2 \operatorname{dev}\operatorname{sym}(\nabla \operatorname{Div} P) \\ \eta\, (\operatorname{skew} P_{,tt}) &= 2\,\mu_c \operatorname{skew}(\nabla u - P) - \mu\, L_c^2 \operatorname{skew}(\operatorname{Curl}\operatorname{Curl} P) + \mu\, L_d^2 \operatorname{skew}(\nabla \operatorname{Div} P) \\ \eta\, \frac{1}{3}\operatorname{tr}(P_{,tt})\,\mathbb{1} &= \left(\frac{2\,\mu_e + 3\,\lambda_e}{3}\right) \operatorname{tr}(\nabla u - P)\,\mathbb{1} - \left(\frac{2\,\mu_{\text{micro}} + 3\,\lambda_{\text{micro}}}{3}\right) \operatorname{tr}(P)\,\mathbb{1} \\ &\quad - \mu\, L_c^2\, \frac{1}{3}\operatorname{tr}(\operatorname{Curl}\operatorname{Curl} P)\,\mathbb{1} + \mu\, L_d^2\, \frac{1}{3}\operatorname{tr}(\nabla \operatorname{Div} P)\,\mathbb{1} \end{aligned} \tag{28}$$

where we have only five independent equations for the dev sym-part, three independent equations for the skew-part and one independent equation for the spherical part.

If we demand that the kinematic fields u and P are plane waves in the x_1 direction as indicated in (23), we have equivalently the following expressions in index notation:

$$u_i\,(x,t) = u_i\,(x_1,t) = \alpha_i\,\mathrm{e}^{\mathrm{i}(kx_1-\omega t)} \qquad (29)$$
$$P_{ij}\,(x,t) = P_{ij}\,(x_1,t) = \beta_{ij}\,\mathrm{e}^{\mathrm{i}(kx_1-\omega t)}$$

In this way, it is easy to derive the expression in components of the projected equations. With respect to the article [11], we have to explicitly calculate only the new part in $\nabla\,\mathrm{Div}\,P$. We have that

$$\nabla\,\mathrm{Div}\,P = \nabla\,\mathrm{Div}\,\mathrm{dev\,sym}\,P + \nabla\,\mathrm{Div}\,\mathrm{skew}\,P + \nabla\,\mathrm{Div}\left(\frac{1}{3}\,\mathrm{tr}\,(P)\,\mathbb{1}\right)$$

so

$$\begin{aligned}
\mathrm{dev\,sym}\,\nabla\,\mathrm{Div}\,P &= \mathrm{dev\,sym}\left(\nabla\,\mathrm{Div}\,\mathrm{dev\,sym}\,P + \nabla\,\mathrm{Div}\,\mathrm{skew}\,P + \nabla\,\mathrm{Div}\,\frac{1}{3}\,\mathrm{tr}\,(P)\,\mathbb{1}\right)\\
\mathrm{skew}\,\nabla\,\mathrm{Div}\,P &= \mathrm{skew}\left(\nabla\,\mathrm{Div}\,\mathrm{dev\,sym}\,P + \nabla\,\mathrm{Div}\,\mathrm{skew}\,P + \nabla\,\mathrm{Div}\,\frac{1}{3}\,\mathrm{tr}\,(P)\,\mathbb{1}\right) \qquad (30)\\
\frac{1}{3}\,\mathrm{tr}\,(\nabla\,\mathrm{Div}\,P)\,\mathbb{1} &= \frac{1}{3}\,\mathrm{tr}\left(\nabla\,\mathrm{Div}\,\mathrm{dev\,sym}\,P + \nabla\,\mathrm{Div}\,\mathrm{skew}\,P + \nabla\,\mathrm{Div}\,\frac{1}{3}\,\mathrm{tr}\,(P)\,\mathbb{1}\right)\mathbb{1}
\end{aligned}$$

and finally, using the fact that P is assumed to depend only on the scalar space variable x_1, we obtain:

$$\mathrm{dev\,sym}\,\nabla\,\mathrm{Div}\,P = \begin{pmatrix}
\frac{2}{3}P^{\mathrm{D}}_{,11} + \frac{2}{3}P^{\mathrm{S}}_{,11} & \frac{1}{2}P_{(12),11} - \frac{1}{2}P_{[12],11} & \frac{1}{2}P_{(13),11} - \frac{1}{2}P_{[13],11}\\
\frac{1}{2}P_{(12),11} - \frac{1}{2}P_{[12],11} & -\frac{1}{3}P^{\mathrm{D}}_{,11} - \frac{1}{3}P^{\mathrm{S}}_{,11} & 0\\
\frac{1}{2}P_{(13),11} - \frac{1}{2}P_{[13],11} & 0 & -\frac{1}{3}P^{\mathrm{D}}_{,11} - \frac{1}{3}P^{\mathrm{S}}_{,11}
\end{pmatrix}$$

$$\mathrm{skew}\,\nabla\,\mathrm{Div}\,P = \frac{1}{2}\begin{pmatrix}
0 & -P_{(12),11} + P_{[12],11} & -P_{(13),11} + P_{[13],11}\\
P_{(12),11} - P_{[12],11} & 0 & 0\\
P_{(13),11} - P_{[13],11} & 0 & 0
\end{pmatrix} \qquad (31)$$

$$\frac{1}{3}\,\mathrm{tr}\,(\nabla\,\mathrm{Div}\,P)\,\mathbb{1} = \frac{1}{3}\left(P^{\mathrm{D}}_{,11} + P^{\mathrm{S}}_{,11}\right)\mathbb{1}$$

Introducing the quantities[2]

$$\begin{aligned}
c_m &= \sqrt{\frac{\mu\,L_{\mathrm{c}}^2}{\eta}}, & c_d &= \sqrt{\frac{\mu\,L_{\mathrm{d}}^2}{\eta}}, & c_s &= \sqrt{\frac{\mu_{\mathrm{e}} + \mu_{\mathrm{c}}}{\rho}}\\
c_p &= \sqrt{\frac{\lambda_{\mathrm{e}} + 2\mu_{\mathrm{e}}}{\rho}}, & \omega_s &= \sqrt{\frac{2\,(\mu_{\mathrm{e}} + \mu_{\mathrm{micro}})}{\eta}}, & \omega_p &= \sqrt{\frac{2\,(\mu_{\mathrm{e}} + \mu_{\mathrm{micro}}) + 3\,(\lambda_{\mathrm{e}} + \lambda_{\mathrm{micro}})}{\eta}} \qquad (32)\\
\omega_r &= \sqrt{\frac{2\mu_{\mathrm{c}}}{\eta}}, & \omega_l &= \sqrt{\frac{\lambda_{\mathrm{micro}} + 2\mu_{\mathrm{micro}}}{\eta}}, & \omega_t &= \sqrt{\frac{\mu_{\mathrm{micro}}}{\eta}}
\end{aligned}$$

the equations can be written as:

- a set of three equations only involving longitudinal quantities:

$$\ddot{u}_1 = c_p^2 u_{1,11} - \frac{2\mu_{\mathrm{e}}}{\rho}\,P^{\mathrm{D}}_{,1} - \frac{3\lambda_{\mathrm{e}} + 2\mu_{\mathrm{e}}}{\rho}\,P^{\mathrm{S}}_{,1} \qquad (33)$$
$$\ddot{P}^{\mathrm{D}} = \frac{4}{3}\,\frac{\mu_{\mathrm{e}}}{\eta}\,u_{1,1} + \frac{1}{3}\,c_m^2\,P^{\mathrm{D}}_{,11} - \frac{2}{3}\,c_m^2 P^{\mathrm{S}}_{,11} - \omega_s^2\,P^{\mathrm{D}} + \underbrace{\frac{2}{3}\,c_d^2\,P^{\mathrm{D}}_{,11} + \frac{2}{3}\,c_d^2\,P^{\mathrm{S}}_{,11}}_{\text{new augmented terms}} \qquad (34)$$

[2] Due to the chosen values of the parameters, which are supposed to satisfy (2), all the introduced characteristic velocities and frequencies are real. Indeed it can be checked that the condition $(3\,\lambda_{\mathrm{e}} + 2\,\mu_{\mathrm{e}}) > 0$ together with the condition $\mu_{\mathrm{e}} > 0$ imply $(\lambda_{\mathrm{e}} + 2\mu_{\mathrm{e}}) > 0$.

$$\ddot{P}^{\rm S} = \frac{3\lambda_{\rm e}+2\mu_{\rm e}}{3\eta}\,u_{1,1} - \frac{1}{3}\,c_m^2 P^{\rm D}_{,11} + \frac{2}{3}\,c_m^2 P^{\rm S}_{,11} - \omega_p^2\,P^{\rm S} + \underbrace{\frac{1}{3}\,c_d^2\,P^{\rm D}_{,11} + \frac{1}{3}\,c_d^2\,P^{\rm S}_{,11}}_{\text{new augmented terms}} \tag{35}$$

- two sets of three equations only involving transverse quantities in the ξ-th direction, with $\xi = 2, 3$:

$$\ddot{u}_\xi = c_s^2 u_{\xi,11} - \frac{2\mu_{\rm e}}{\rho}\,P_{(1\xi),1} + \frac{\eta}{\rho}\,\omega_r^2 P_{[1\xi],1} \tag{36}$$

$$\ddot{P}_{(1\xi)} = \frac{\mu_{\rm e}}{\eta}\,u_{\xi,1} + \frac{1}{2}\,c_m^2\,P_{(1\xi),11} + \frac{1}{2}\,c_m^2\,P_{[1\xi],11} - \omega_s^2\,P_{(1\xi)} + \underbrace{\frac{1}{2}\,c_d^2\,P_{(1\xi),11} - \frac{1}{2}\,c_d^2\,P_{[1\xi],11}}_{\text{new augmented terms}} \tag{37}$$

$$\ddot{P}_{[1\xi]} = -\frac{1}{2}\,\omega_r^2\,u_{\xi,1} + \frac{1}{2}\,c_m^2\,P_{(1\xi),11} + \frac{1}{2}\,c_m^2 P_{[1\xi],11} - \omega_r^2\,P_{[1\xi]} - \underbrace{\frac{1}{2}\,c_d^2\,P_{(1\xi),11} + \frac{1}{2}\,c_d^2\,P_{[1\xi],11}}_{\text{new augmented terms}} \tag{38}$$

- one equation only involving the variable $P_{(23)}$:

$$\ddot{P}_{(23)} = -\omega_s^2 P_{(23)} + c_m^2 P_{(23),11} \tag{39}$$

- one equation only involving the variable $P_{[23]}$:

$$\ddot{P}_{[23]} = -\omega_r^2 P_{[23]} + c_m^2 P_{[23],11} \tag{40}$$

- one equation only involving the variable P^V:

$$\ddot{P}^V = -\omega_s^2 P^V + c_m^2 P^V_{,11} \tag{41}$$

In what follows, we will refer to the dispersion curves stemming from the last three equations as “uncoupled waves”. This nomenclature has been chosen because in these equations each variable is not coupled with the others, so that such waves propagate independently of the others. Due to the non-locality of the considered micromorphic model, such modes, even if independent one from the other, show a dispersive behavior which is completely due to the existence of a characteristic length $L_{\rm c}$. From a phenomenological point of view, this means that such modes do not propagate at a constant speed, since they are affected by what is occurring in the adjacent cells. Such a phenomenon is more intuitively understandable if one thinks to a strongly contrasted medium.

3. Particularization for specific energies

In what follows, we will present the results obtained with particular energies and the numerical values of the elastic coefficients are chosen as in Table 1 if not differently specified.

We explicitly mention that the numerical values of the present parameters are chosen with the only constraint of respecting positive definiteness of the strain energy density.

In particular, the value $L_{\rm c} = 1$ mm is chosen as representative of the non-locality of the considered metamaterial. This means that $L_{\rm c}$ represents the distance at which the deformation of a unit cell is “sensed” by the neighboring cells. Such a characteristic length can be smaller than the size of the cell when the neighboring cells are weakly influenced by what happens in the considered unit cell or can even be much larger than the size of the unit cell for highly non-local metamaterials. Hence, $L_{\rm c}$ should not be a priori confused with the characteristic size of the cell itself. This means that the value of $L_{\rm c}$ cannot be used to decide for which wavelength the continuum model starts losing its physical meaning. Indeed, it is clear that for wavelengths that are smaller than the unit cell, a continuum model is not reasonable anymore, since the

Table 1
Values of the parameters used in the numerical simulations (left) and corresponding values of the Lamé parameters and of the Young modulus and Poisson ratio as obtained with formula (3) (right).

Parameter	Value	Unit
$\mu_{\rm e}$	200	MPa
$\lambda_{\rm e} = 2\mu_{\rm e}$	400	MPa
$\mu_{\rm c} = 5\mu_{\rm e}$	1000	MPa
$\mu_{\rm micro}$	100	MPa
$\lambda_{\rm micro}$	100	MPa
$L_{\rm c}$	1	mm
ρ	2000	kg/m^3
η	10^{-2}	kg/m

Parameter	Value	Unit
$\lambda_{\rm macro}$	82.5	MPa
$\mu_{\rm macro}$	66.7	MPa
$E_{\rm macro}$	170	MPa
$\nu_{\rm macro}$	0.28	–

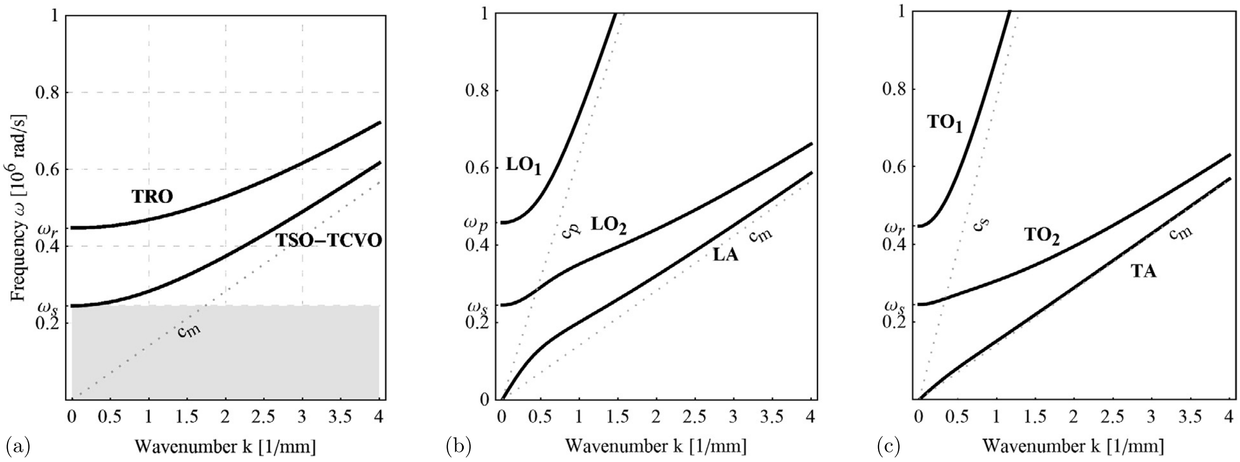

Fig. 1. Dispersion relations $\omega = \omega(k)$ for the *micromorphic model with* $\|\mathrm{Div}\, P\|^2 + \|\mathrm{Curl}\, P\|^2$ and non-vanishing Cosserat couple modulus $\mu_c > 0$: only a *partial band gap* on the uncoupled waves can be modeled.

discreteness of the metamaterial cannot be treated in an "averaged" sense. In this paper, we decided not to choose a specific topology for the microstructure of the considered metamaterial, this being the object of future work. We hence trace the dispersion diagrams by choosing the interval for the wave number k in such a way to disclose the asymptotic properties of the curves. Whether the value $k = 4/\text{mm}$ is such that the continuum model has already lost is physical meaning or not would be intimately connected to the microstructural topologies.

3.1. The micromorphic model with $\|\mathrm{Div}\, P\|^2$ and $\|\mathrm{Curl}\, P\|^2$ ($L_c = L_d \neq 0$)

We consider now the model obtained considering $L_c = L_d$ with energy:

$$W = \underbrace{\mu_e \,\|\mathrm{sym}\,(\nabla u - P)\,\|^2 + \frac{\lambda_e}{2}\,(\mathrm{tr}\,(\nabla u - P))^2}_{\text{isotropic elastic energy}} + \underbrace{\mu_c \,\|\mathrm{skew}\,(\nabla u - P)\,\|^2}_{\text{rotational elastic coupling}} \qquad (42)$$
$$+ \underbrace{\mu_{\text{micro}}\,\|\mathrm{sym}\, P\,\|^2 + \frac{\lambda_{\text{micro}}}{2}\,(\mathrm{tr}\, P)^2}_{\text{micro-self-energy}} + \underbrace{\frac{\mu\, L_c^2}{2}\left(\|\mathrm{Div}\, P\,\|^2 + \|\mathrm{Curl}\, P\,\|^2\right)}_{\text{augmented isotropic curvature}}$$

The dynamical equilibrium equations are:

$$\rho\, u_{,tt} = \mathrm{Div}\,\sigma = \mathrm{Div}\,[2\,\mu_e\,\mathrm{sym}\,(\nabla u - P) + 2\,\mu_c\,\mathrm{skew}\,(\nabla u - P) + \lambda_e\,\mathrm{tr}\,(\nabla u - P)\,\mathbb{1}]$$
$$\eta\, P_{,tt} = 2\,\mu_e\,\mathrm{sym}\,(\nabla u - P) + 2\,\mu_c\,\mathrm{skew}\,(\nabla u - P) + \lambda_e\,\mathrm{tr}\,(\nabla u - P)\,\mathbb{1} \qquad (43)$$
$$- [2\mu_{\text{micro}}\,\mathrm{sym}\, P + \lambda_{\text{micro}}\,\mathrm{tr}(\,P)\mathbb{1}] + \mu\, L_c^2\,\underbrace{(\nabla\,(\mathrm{Div}\, P) - \mathrm{Curl}\,\mathrm{Curl}\, P)}_{\mathrm{Div}\nabla\, P = \Delta\, P}$$

Note that the structure of the equation is equivalent to the one obtained in the standard micromorphic model with curvature $\frac{1}{2}\|\nabla P\|^2$, see equation (50) in section 4.

We present the *dispersion relations* obtained with a non-vanishing Cosserat couple modulus $\mu_c > 0$ (Fig. 1) and for a vanishing Cosserat couple modulus $\mu_c = 0$ (Fig. 2). In all the figures, we consider uncoupled waves (a), longitudinal waves (b), and transverse waves (c). The nomenclature adopted is the following: TRO: transverse rotational optic, TSO: transverse shear optic, TCVO: transverse constant-volume optic, LA: longitudinal acoustic, LO_1–LO_2: first and second longitudinal optic, TA: transverse acoustic, TO_1–TO_2: first and second transverse optic.

We conclude that when considering the model with micromorphic medium with $\|\mathrm{Div}\, P\|^2 + \|\mathrm{Curl}\, P\|^2$ and vanishing Cosserat couple modulus μ_c, there always exist waves that propagate inside the considered medium independently of the value of frequency, even if considering separately longitudinal, transverse, and uncoupled waves. The only effect obtainable switching on the Cosserat couple modulus μ_c is to obtain a partial band gap for the uncoupled waves.

3.2. The micromorphic model with only $\|\mathrm{Div}\, P\|^2$ obtained as a special case of the augmented relaxed model with $L_c = 0$

The isotropic micromorphic model with $\|\mathrm{Div}\, P\|^2$ is obtained from the model with $\|\mathrm{Curl}\, P\|^2$ and $\|\mathrm{Div}\, P\|^2$ by considering $L_c = 0$ obtaining as standard energy:

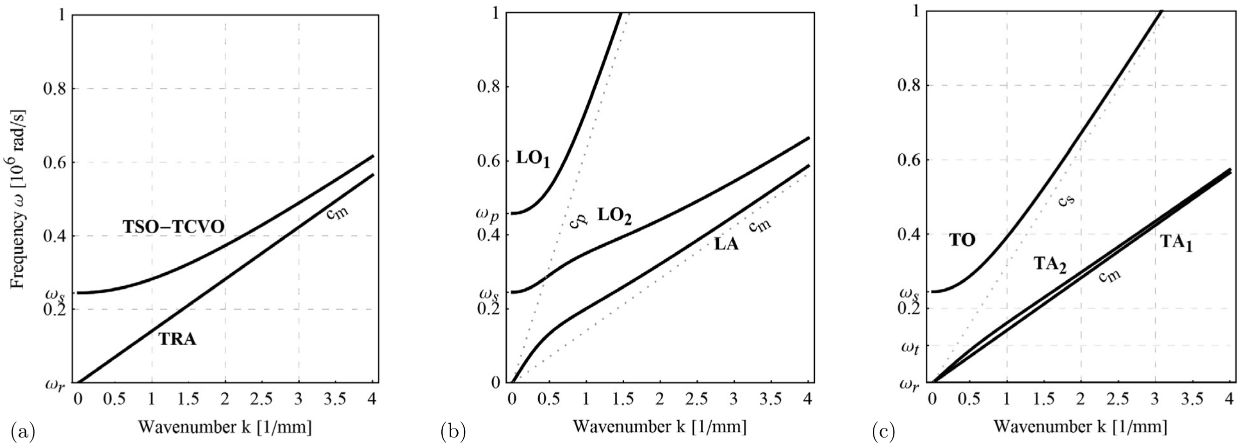

Fig. 2. Dispersion relations $\omega = \omega(k)$ for the *micromorphic model with* $\|\mathrm{Div}\, P\|^2 + \|\mathrm{Curl}\, P\|^2$ and vanishing Cosserat couple modulus $\mu_c = 0$: *no band gap* at all.

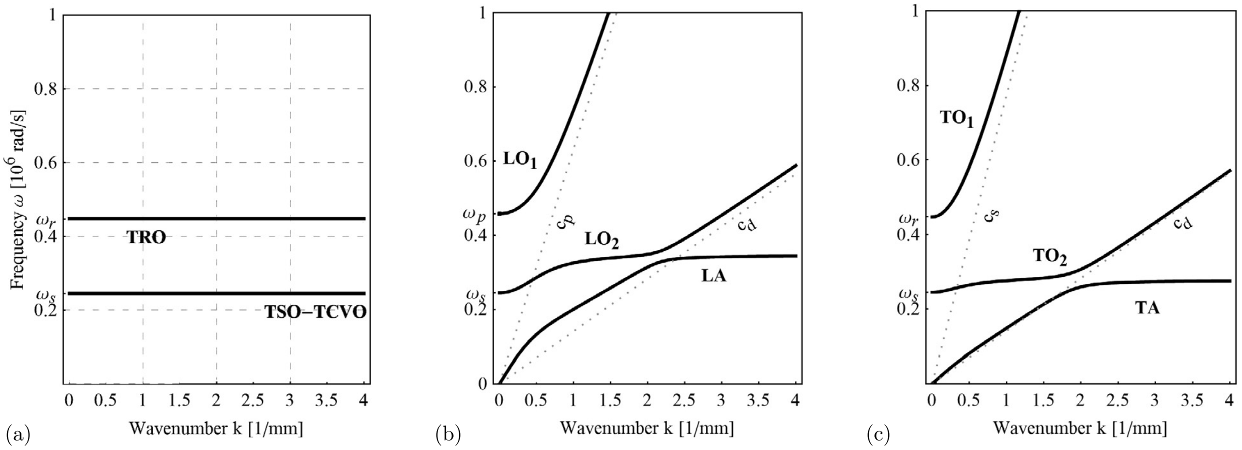

Fig. 3. Dispersion relations $\omega = \omega(k)$ for the *micromorphic model with* $\|\mathrm{Div}\, P\|^2$ and non-vanishing Cosserat couple modulus $\mu_c > 0$: no *band gap* on the longitudinal and transverse waves can be modeled and the uncoupled waves have fixed frequencies.

$$
\begin{aligned}
W = \; & \underbrace{\mu_e \, \|\mathrm{sym}\,(\nabla u - P)\,\|^2 + \frac{\lambda_e}{2}\,(\mathrm{tr}\,(\nabla u - P))^2}_{\text{isotropic elastic energy}} + \underbrace{\mu_c \, \|\mathrm{skew}\,(\nabla u - P)\,\|^2}_{\text{rotational elastic coupling}} \\
& + \underbrace{\mu_{\mathrm{micro}} \, \|\mathrm{sym}\, P\|^2 + \frac{\lambda_{\mathrm{micro}}}{2}\,(\mathrm{tr}\, P)^2}_{\text{micro-self-energy}} + \underbrace{\frac{\mu \, L_d^2}{2}\,\|\mathrm{Div}\, P\|^2}_{\text{isotropic curvature}}
\end{aligned}
\tag{44}
$$

The dynamical equilibrium equations are:

$$
\begin{aligned}
\rho\, u_{,tt} &= \mathrm{Div}\,\sigma = \mathrm{Div}\,[2\,\mu_e \,\mathrm{sym}\,(\nabla u - P) + 2\,\mu_c\,\mathrm{skew}\,(\nabla u - P) + \lambda_e\,\mathrm{tr}\,(\nabla u - P)\,\mathbb{1}] \\
\eta\, P_{,tt} &= 2\,\mu_e\,\mathrm{sym}\,(\nabla u - P) + 2\,\mu_c\,\mathrm{skew}\,(\nabla u - P) + \lambda_e\,\mathrm{tr}\,(\nabla u - P)\,\mathbb{1} \\
&\quad - [2\mu_{\mathrm{micro}}\,\mathrm{sym}\, P + \lambda_{\mathrm{micro}}\,\mathrm{tr}(\,P)\mathbb{1}] + \mu\, L_d^2\,\nabla\,(\mathrm{Div}\, P)
\end{aligned}
\tag{45}
$$

We present the *dispersion relations* obtained with a non-vanishing Cosserat couple modulus $\mu_c > 0$ (Fig. 3) and for a vanishing Cosserat couple modulus $\mu_c = 0$ (Fig. 4). In the figures, we consider uncoupled waves (a), longitudinal waves (b), and transverse waves (c). TRO: transverse rotational optic, TSO: transverse shear optic, TCVO: transverse constant-volume optic, LA: longitudinal acoustic, LO_1–LO_2: first and second longitudinal optic, TA: transverse acoustic, TO_1–TO_2: first and second transverse optic.

We can conclude that, when considering the micromorphic model with only $\|\mathrm{Div}\, P\|^2$ for every value of μ_c, there always exist waves that propagate inside the considered medium independently of the value of the frequency. The uncoupled waves

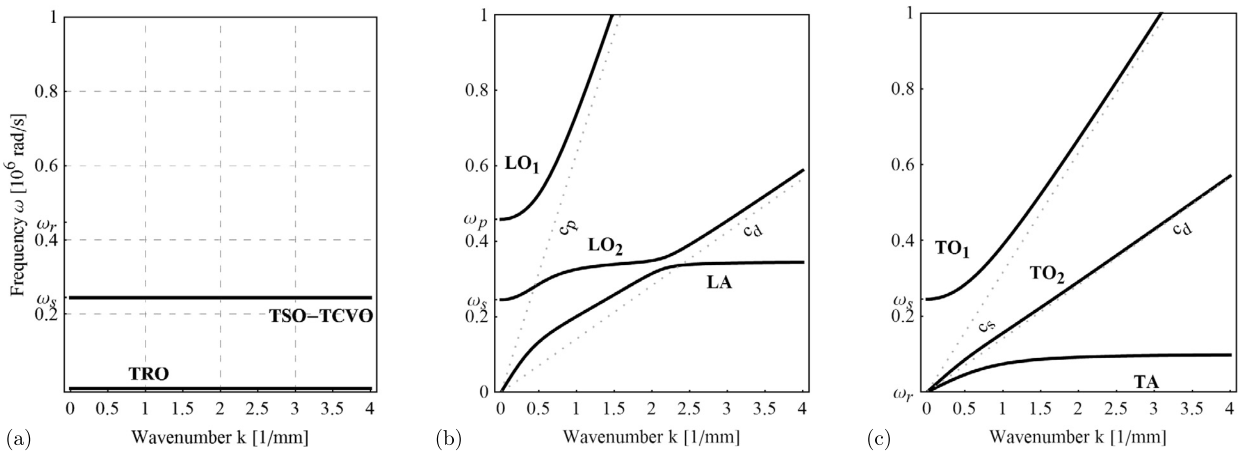

Fig. 4. Dispersion relations $\omega = \omega(k)$ for the *micromorphic model with* $\|\mathrm{Div}\, P\|^2$ and vanishing Cosserat couple modulus $\mu_c = 0$: no *band gap* on the longitudinal and transverse waves can be modeled, and the uncoupled waves have fixed frequencies.

assume a peculiar behavior in which the frequency is independent of the wavenumber k. This is due to the fact that $c_m = 0$ in Eqs. (39), (40) and (41), so that the modes for uncoupled waves become non-dispersive.

3.3. The relaxed micromorphic model obtained as a special case of the augmented relaxed model with $L_d = 0$

The relaxed micromorphic model is obtained by the model with $\|\mathrm{Curl}\, P\|^2$ and $\|\mathrm{Div}\, P\|^2$ by considering $L_d = 0$, obtaining the energy:

$$W = \underbrace{\mu_e \,\|\mathrm{sym}\,(\nabla u - P)\,\|^2 + \frac{\lambda_e}{2}\,(\mathrm{tr}\,(\nabla u - P))^2}_{\text{isotropic elastic energy}} + \underbrace{\mu_c \,\|\mathrm{skew}\,(\nabla u - P)\,\|^2}_{\text{rotational elastic coupling}} \tag{46}$$
$$+ \underbrace{\mu_{\mathrm{micro}}\,\|\mathrm{sym}\, P\,\|^2 + \frac{\lambda_{\mathrm{micro}}}{2}\,(\mathrm{tr}\, P)^2}_{\text{micro-self-energy}} + \underbrace{\frac{\mu\, L_c^2}{2}\,\|\mathrm{Curl}\, P\,\|^2}_{\text{isotropic curvature}}$$

The dynamical equilibrium equations are, see also [11]:

$$\begin{aligned} \rho\, u_{,tt} &= \mathrm{Div}\,\sigma = \mathrm{Div}\,[2\,\mu_e\,\mathrm{sym}\,(\nabla u - P) + 2\,\mu_c\,\mathrm{skew}\,(\nabla u - P) + \lambda_e\,\mathrm{tr}\,(\nabla u - P)\,\mathbb{1}] \\ \eta\, P_{,tt} &= 2\,\mu_e\,\mathrm{sym}\,(\nabla u - P) + 2\,\mu_c\,\mathrm{skew}\,(\nabla u - P) + \lambda_e\,\mathrm{tr}\,(\nabla u - P)\,\mathbb{1} \\ &\quad - [2\mu_{\mathrm{micro}}\,\mathrm{sym}\, P + \lambda_{\mathrm{micro}}\,\mathrm{tr}(\,P)\mathbb{1}\,] - \mu\, L_c^2\,\mathrm{Curl}\,\mathrm{Curl}\, P \end{aligned} \tag{47}$$

We present the *dispersion relations* obtained with a non-vanishing Cosserat couple modulus $\mu_c > 0$ (Fig. 5) and for a vanishing Cosserat couple modulus $\mu_c = 0$ (Fig. 6). In the figures, we consider uncoupled waves (a), longitudinal waves (b) and transverse waves (c). TRO: Transverse rotational optic, TSO: transverse shear optic, TCVO: transverse constant-volume optic, LA: longitudinal acoustic, LO_1–LO_2: 1st and 2nd longitudinal optic, TA: transverse acoustic, TO_1–TO_2: 1st and 2nd transverse optic.

We can conclude that, in general, when considering the relaxed micromorphic medium with vanishing Cosserat couple modulus μ_c, there always exist waves that propagate inside the considered medium independently of the value of the frequency. Nevertheless, if one considers a particular case (obtained by imposing suitable kinematical constraints) in which only longitudinal waves can propagate, then in the frequency range (ω_s, ω_l), only standing waves exist, which does not allow for wave propagation.

On the other hand, switching on the Cosserat couple modulus μ_c allows for the description of complete frequency band gaps in which no propagation can occur.

4. The standard Mindlin–Eringen model with $\|\nabla P\|^2$

The elastic energy of the general anisotropic centro-symmetric micromorphic model in the sense of Mindlin–Eringen (see [4] and [3, p. 270, eq. 7.1.4]) can be represented as:

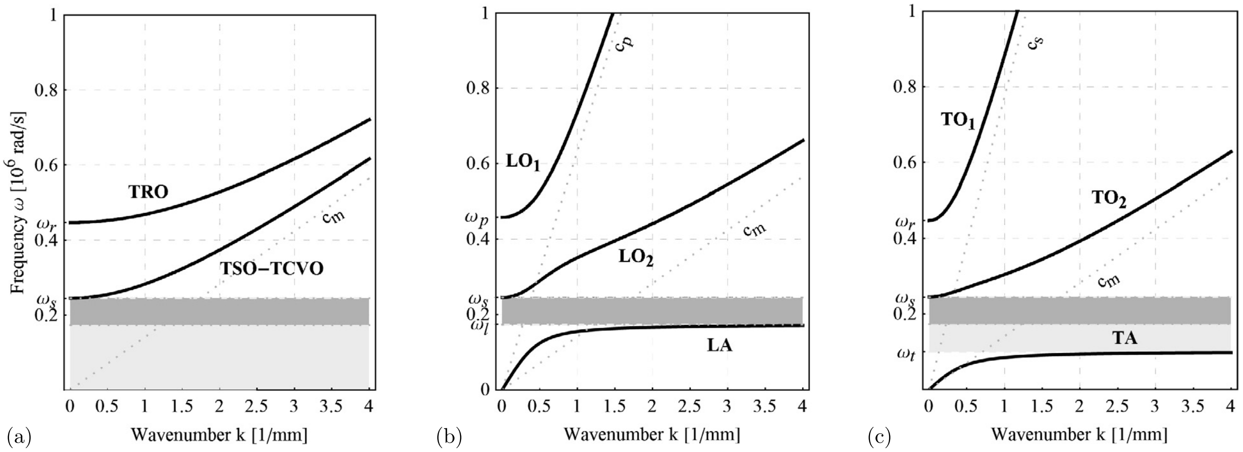

Fig. 5. Dispersion relations $\omega = \omega(k)$ for the *relaxed micromorphic model* with non-vanishing Cosserat couple modulus $\mu_c > 0$. The *complete frequency band gap* is the shaded intersected domain bounded from the maximum between ω_l and ω_t and the minimum between ω_r and ω_s. The existence of the band gap is related to $\mu_c > 0$ via the cut-off frequency $\omega_r = \sqrt{\frac{2\mu_c}{\eta}}$ of the uncoupled waves TRO and TO_1.

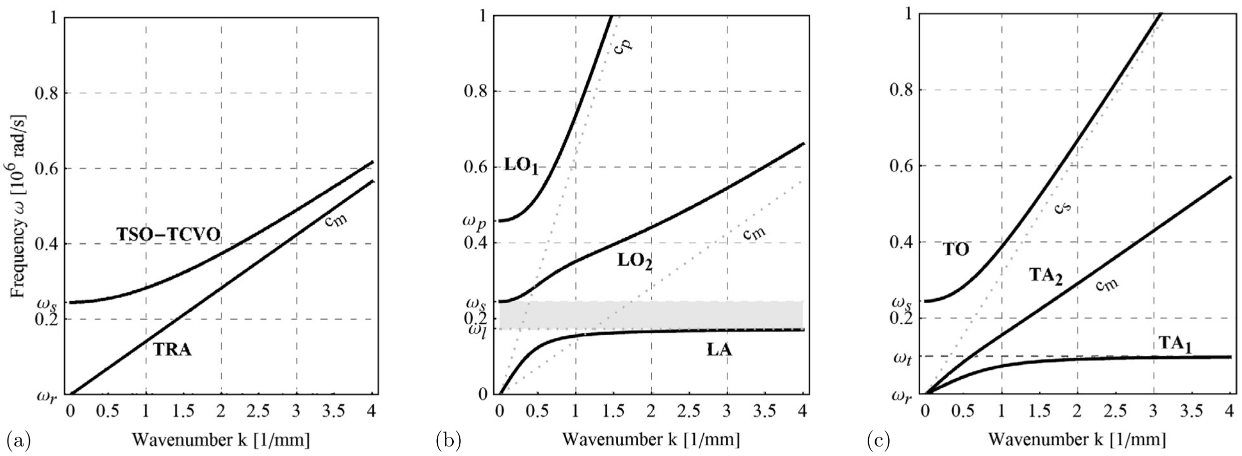

Fig. 6. Dispersion relations $\omega = \omega(k)$ for the *relaxed micromorphic model* with vanishing Cosserat couple modulus $\mu_c = 0$: only a *partial band gap* can be modeled.

$$
\begin{aligned}
W = {} & \underbrace{\frac{1}{2}\left\langle \overline{\mathbb{C}}_e\,(\nabla u - P), (\nabla u - P)\right\rangle_{\mathbb{R}^{3\times 3}}}_{\text{full anisotropic elastic energy}} + \underbrace{\frac{1}{2}\left\langle \mathbb{C}_{\text{micro}}\,\text{sym}\,P,\ \text{sym}\,P\right\rangle_{\mathbb{R}^{3\times 3}}}_{\text{micro-self-energy}} \\
& + \underbrace{\frac{1}{2}\left\langle \overline{\mathbb{E}}_{\text{cross}}\,(\nabla u - P),\ \text{sym}\,P\right\rangle_{\mathbb{R}^{3\times 3}}}_{\text{anisotropic cross-coupling}} + \underbrace{\frac{\mu\,L_c^2}{2}\left\langle \widehat{\mathbb{L}}_{\text{aniso}}\,\nabla P, \nabla P\right\rangle_{\mathbb{R}^{3\times 3\times 3}}}_{\text{full anisotropic curvature}}
\end{aligned}
\tag{48}
$$

where $\overline{\mathbb{C}}_e : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$ is a fourth-order micromorphic elasticity tensor which has at most 45 independent coefficients and which acts on the *non-symmetric elastic distortion* $e = \nabla u - P$; $\overline{\mathbb{E}}_{\text{cross}} : \mathbb{R}^{3\times 3} \to \text{Sym}(3)$ is a fourth-order cross-coupling tensor with the symmetry $\left(\overline{\mathbb{E}}_{\text{cross}}\right)_{ijkl} = \left(\overline{\mathbb{E}}_{\text{cross}}\right)_{jikl}$ having at most 54 independent coefficients. The fourth-order tensor $\mathbb{C}_{\text{micro}} : \text{Sym}(3) \to \text{Sym}(3)$ has the classical 21 independent coefficients of classical elasticity, while $\widehat{\mathbb{L}}_{\text{aniso}} : \mathbb{R}^{3\times 3\times 3} \to \mathbb{R}^{3\times 3\times 3}$ is a sixth-order tensor that shows astonishing 378 parameters. The parameter $\mu > 0$ is a typical shear modulus and $L_c > 0$ is one characteristic length, while $\widehat{\mathbb{L}}_{\text{aniso}}$ is, accordingly, dimensionless.

One of the major obstacles in using the micromorphic approach for specific materials is the impossibility to determine such multitude of new material coefficients. Not only is the huge number a technical problem, but also the interpretation of coefficients is problematic [17–19]. Some of these coefficients are size-dependent, while others are not. A purely formal approach, as it is often done, cannot be the final answer.

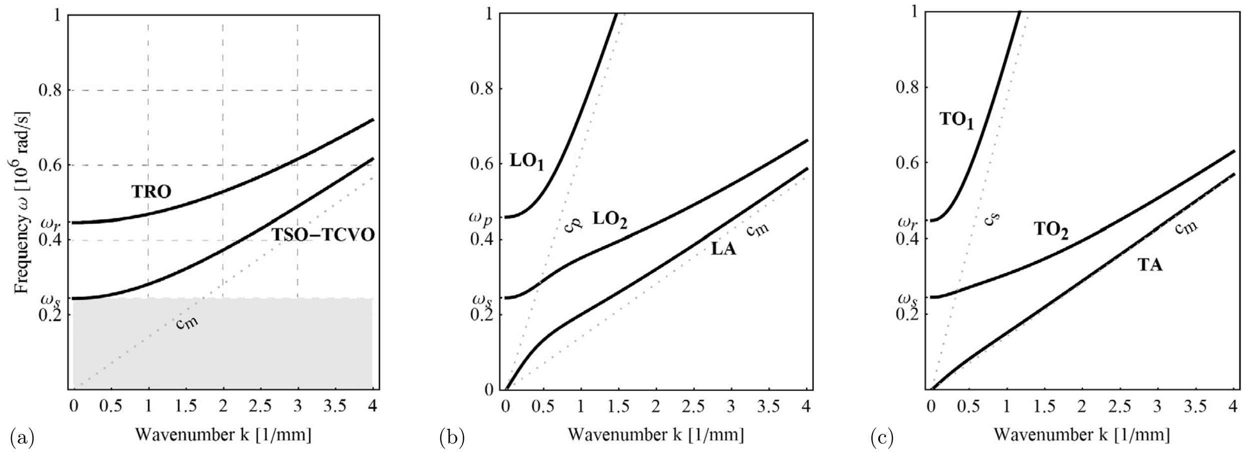

Fig. 7. Dispersion relations $\omega = \omega(k)$ for the *standard micromorphic model with* $\|\nabla P\|^2$ with a non-vanishing Cosserat couple modulus $\mu_c > 0$: only a *partial band gap* can be modeled for uncoupled waves.

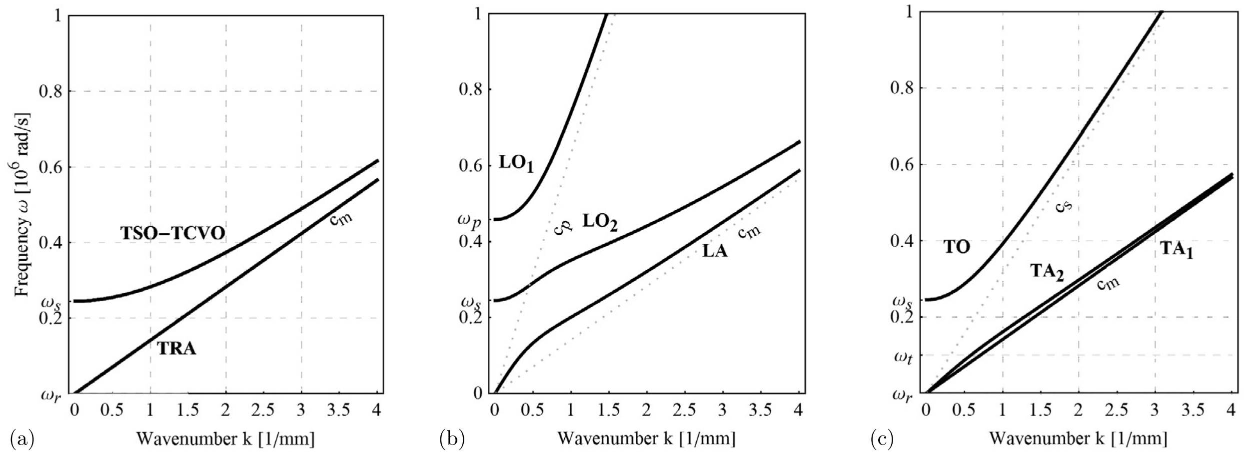

Fig. 8. Dispersion relations $\omega = \omega(k)$ for the *standard micromorphic model with* $\|\nabla P\|^2$ and a vanishing Cosserat couple modulus $\mu_c = 0$: *no band gap* at all.

In what follows, we will consider a simplified isotropic energy:

$$W = \underbrace{\mu_e \,\|\mathrm{sym}\,(\nabla u - P)\,\|^2 + \frac{\lambda_e}{2}\,(\mathrm{tr}\,(\nabla u - P))^2}_{\text{isotropic elastic energy}} + \underbrace{\mu_c \,\|\mathrm{skew}\,(\nabla u - P)\,\|^2}_{\text{rotational elastic coupling}} + \underbrace{\mu_{\mathrm{micro}} \,\|\mathrm{sym}\, P\|^2 + \frac{\lambda_{\mathrm{micro}}}{2}\,(\mathrm{tr}\, P)^2}_{\text{micro-self-energy}} + \underbrace{\frac{\mu\, L_c^2}{2}\,\|\nabla P\|^2}_{\text{isotropic curvature}} \tag{49}$$

The dynamical equilibrium equations are:

$$\begin{aligned} \rho\, u_{,tt} &= \mathrm{Div}\,\sigma = \mathrm{Div}\,[2\,\mu_e\,\mathrm{sym}\,(\nabla u - P) + 2\,\mu_c\,\mathrm{skew}\,(\nabla u - P) + \lambda_e\,\mathrm{tr}\,(\nabla u - P)\,\mathbb{1}] \\ \eta\, P_{,tt} &= 2\,\mu_e\,\mathrm{sym}\,(\nabla u - P) + 2\,\mu_c\,\mathrm{skew}\,(\nabla u - P) + \lambda_e\,\mathrm{tr}\,(\nabla u - P)\,\mathbb{1} \\ &\quad - [2\mu_{\mathrm{micro}}\,\mathrm{sym}\, P + \lambda_{\mathrm{micro}}\,\mathrm{tr}(\,P)\mathbb{1}] + \mu\, L_c^2\, \underbrace{\mathrm{Div}\,\nabla\, P}_{\Delta\, P} \end{aligned} \tag{50}$$

We present the *dispersion relations* obtained with a non-vanishing Cosserat couple modulus $\mu_c > 0$ (Fig. 7) and for a vanishing Cosserat couple modulus $\mu_c = 0$ (Fig. 8). In the figures, we consider uncoupled waves (a), longitudinal waves (b), and transverse waves (c). TRO: transverse rotational optic, TSO: transverse shear optic, TCVO: transverse constant-volume optic, LA: longitudinal acoustic, LO_1–LO_2: first and second longitudinal optic, TA: transverse acoustic, TO_1–TO_2: first and second transverse optic.

In a way completely equivalent to the case of $\|\mathrm{Div}\, P\|^2$ and $\|\mathrm{Curl}\, P\|^2$ (see section 3.1), we can conclude that when considering the standard Mindlin–Eringen model with vanishing Cosserat couple modulus μ_c, there always exist waves that propagate inside the considered medium independently of the value of frequency even if considering separately longitudinal, transverse and uncoupled waves.

The only effect obtainable switching on the Cosserat couple modulus μ_c is to obtain a partial band gap for the uncoupled waves.

5. Conclusion

Metamaterials are artifacts composed by *microstructural elements* assembled in periodic or quasi-periodic patterns, giving rise to materials with *unorthodox properties*. For some of these metamaterials, the presence of a microstructure allows for *macroscopic wave-inhibition*. More particularly, this means that, given the topology of the microstructure, when the material is solicited at frequencies that fall in the band-gap region, any of the possible micro-motions is activated at such frequencies. Hence, this results in the impossibility of waves to travel in the considered metamaterial.

The *relaxed micromorphic model* is the *only linear, isotropic, reversibly elastic, non-local generalized continuum model* known to date able to *predict complete frequency band gaps*. It is decisive to use Curl P instead of the full micro-distortion gradient ∇P and to take a positive Cosserat couple modulus $\mu_c > 0$.

Future work will be devoted to the fitting of some of the introduced parameters on real band-gap metamaterials. Moreover, the effect of extra micro-inertia terms besides $\eta\|P_{,t}\|^2$ will be also investigated.

Considering that non-locality is an intrinsic characteristic feature of micro-structured materials, especially when high contrasts of the mechanical properties occur at the micro-level, models that allow for its description are a necessary requirement. The relaxed micromorphic model is the only generalized continuum model that is simultaneously able to account for non-locality and for band-gaps onset in metamaterials.

Acknowledgements

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