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Variational discretizations for the dynamics of fluid-conveying flexible tubes

Discrétisations variationnelles pour l'étude de la dynamique de tubes flexibles avec écoulement interne

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ABSTRACT

We derive a variational approach for discretizing fluid–structure interactions, with a particular focus on the dynamics of fluid-conveying elastic tubes. Our method is based on a discretization of the fluid's back-to-labels map and a Lie group discretization of the tube's variables, coupled with an appropriately formulated discrete version of the fluid conservation law. This approach allows the development of geometric numerical schemes for the dynamics of fluid-conveying collapsible tubes, which preserve several intrinsic geometric properties of the continuous system, such as symmetries and symplecticity. In addition, our approach can also be used to derive simplified, but geometrically consistent, low-component models for further analytical and numerical analysis of the system.

RÉSUMÉ

Nous proposons une approche variationnelle pour la discrétisation d'interactions fluide–structure, en nous focalisant sur la dynamique de tubes élastiques avec écoulement interne. Notre approche est basée sur une discrétisation des trajectoires inverses du fluide et une discrétisation de type groupe de Lie des variables du tube élastique, couplée à une discrétisation appropriée de la contrainte de préservation du volume de fluide. Notre approche permet le développement de schémas numériques géométriques pour la dynamique des tubes souples avec écoulement interne, qui préservent plusieurs propriétés géométriques intrinsèques du système continu, telles que les symétries et la symplecticité. De plus, notre approche peut être utilisée pour produire des modèles simplifiés et

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géométriquement consistants, appropriés pour des études analytiques et numériques plus approfondies de ce système.

1. Background of the studies in dynamics of fluid-conveying flexible tubes

In this paper, we consider variational discretizations for systems coming from fluid–structure interactions, with a particular emphasis on the dynamics of fluid-conveying tubes. For such systems, the key to the dynamics lies in the interaction of a moving elastic body with a moving fluid. From a mathematical point of view, the Lagrangian description of these systems involves both left-invariant (elastic) and right-invariant (fluid) quantities. For the case of fluid-conveying tubes, an instability appears when the flow rate through the tube exceeds a certain critical value. While this phenomenon has been known for a very long time, the quantitative research in the field started around the middle of the 20th century [1]. Benjamin [2,3] was perhaps the first to formulate a quantitative theory for the 2D dynamics by considering a linked chain of fluid-conveying tubes and by using an augmented Hamilton principle of critical action that takes into account the momentum of the jet leaving the tube. A continuum equation for the linear disturbances was then derived as the limit of the discrete system. The same linearized equation, considered by Païdoussis and collaborators [4] from force balance considerations, formed the basis for further stability analysis of this problem for finite tubes [5–11]. This theory has shown a reasonable agreement with the onset of the instability observed experimentally [7,12–15]. Nonlinear deflection models were also considered in [11,16–18], and a more detailed 3D theory of motion was developed in [19], based on a modification of the full Cosserat rod dynamics. Previous works have also approached this problem as a paradigm for the *follower force approach*, which treats the system as an elastic beam with a follower force that remains tangent to the end of the tube and models the effect of the jet leaving the nozzle [20]. However, once the length of the tube becomes large, the validity of the follower force approach has been questioned, see [21] for a lively and thorough discussion.

The main drawback of the previous approaches lies in the difficulty to incorporate in a consistent way the change of the cross section in the dynamics, which is known as a challenging problem for the so-called collapsible tube. Previous works based on traditional frameworks have considered the change of cross section only through the quasi-static approximation: if $A(s,t)$ is the local cross-sectional area, and $u(s,t)$ is the local velocity of the fluid, with s being the coordinate along the tube and t the time, then the quasi-static assumption states that $uA = \text{const}$ [11,16,17,22]. Unfortunately, this simple law is not correct in general and should not be used unless there is a guarantee that the velocity equilibrates on a faster time scale than the change of A and of the tube properties. This problem has been addressed in our previous works [23,24], where we have developed a geometrically exact setting for dealing with a variable cross-section, and studied the important effects of cross-sectional changes on both linear and nonlinear dynamics. The nonlinear theory was derived from a variational principle in a rigorous geometric setting and for general Lagrangians. It can incorporate general boundary conditions and arbitrary deviations from equilibrium in the three-dimensional space.

The theory derived in [23,24] raised the question of writing consistent approximations of the solutions, both from the point of view of deriving simplified reduced models and of developing structure preserving numerical schemes. The present paper deriving such consistent approximations can be viewed as the development of ideas put forward by Benjamin [2,3] for the case of nonlinear dynamics in three dimensions and with cross-sectional dependence. We also note that nonlinear equations for deformations were derived as a limit of a linked chain in [8] and subsequently used in [13]; however, this derivation was not variational and does not generalize to the case of cross-sectional change we are interested in. In the limit of small spatial and/or temporal steps, our discrete equations converge to the continuous equations obtained in [23,24]. We will base our method on the recent works [25,26] on the multisymplectic discretization of an elastic beam in $\mathbb{R}^3$, which were in turn based on the geometric variational spacetime discretization of Lagrangian field theories first developed in [27]. We refer to [28] for an extensive review of time-variational integrators in Lagrangian mechanics. There are two major difficulties that we will need to address here, namely, the appropriate coupling of the fluid with the elastic tube, and the treatment of the constraint of fluid volume conservation. As it turns out, both of these considerations lead to interesting new concepts, which have not been addressed in the previous literature.

2. A brief introduction to the variational dynamics of a collapsible fluid-conveying tube in the continuous case

In this section, we briefly review the variational formulation for the dynamics of a fluid-conveying collapsible tube developed in [23,24], as it plays a fundamental role in the present paper. The interested reader may consult these articles for the complete treatment of the variational approach, as well as for detailed discussions on boundary conditions, linearized stability and fully nonlinear solutions.

To describe the tube's dynamics, we use the framework of geometrically exact rod theory [29], which is equivalent to the Kirchhoff rod theory for purely elastic rods. The configuration of the tube deforming in space is defined by: (i) the position of its line of centroids given by the map $(s,t) \mapsto \boldsymbol{r}(s,t) \in \mathbb{R}^3$, and (ii) the orientation of the tube's cross sections at the points $\boldsymbol{r}(s,t)$, defined by using a moving orthonormal basis $\mathbf{d}_i(s,t)$, $i = 1,2,3$. The moving basis is described by an orthogonal transformation $\Lambda(s,t) \in SO(3)$ such that $\mathbf{d}_i(s,t) = \Lambda(s,t)\mathbf{E}_i$, where $\mathbf{E}_i$, $i = 1,2,3$ is a fixed material frame. Note that the parameter s is not necessarily the arc length.

We assume that the fluid inside the tube is inviscid and incompressible, and we approximate the fluid motion by a one-dimensional mapping from the initial position of the fluid particle S to its current position at time t, denoted as $s = \varphi(S,t)$. The fluid is thus moving along the tube with velocity $u = \partial_t \varphi \circ \varphi^{-1}$ measured relative to the tube. The velocity of a point on the tube's centerline in space is given by $\boldsymbol{v}_r = \partial_t \boldsymbol{r}$, and the velocity of the fluid is $\boldsymbol{v}_f = \partial_t \boldsymbol{r} + u \partial_s \boldsymbol{r}$, as follows from time differentiation of the position of the tube and of a fluid particle at s. The physical variables describing the evolution of the tube are the local angular and linear velocities in the tube's frame, $\widehat{\boldsymbol{\omega}} = \Lambda^{-1}\partial_t \Lambda$ and $\boldsymbol{\gamma} = \Lambda^{-1}\partial_t \boldsymbol{r}$, and the corresponding deformations $\widehat{\boldsymbol{\Omega}} = \Lambda^{-1}\partial_s \Lambda$ (Darboux vector) and $\boldsymbol{\Gamma} = \Lambda^{-1}\partial_s \boldsymbol{r}$. In these definitions, we made use of the isomorphism $\widehat{\ }: \mathbb{R}^3 \to \mathfrak{so}(3)$, given by $\widehat{\mathbf{v}}_{ab} = -\epsilon_{abc}\mathbf{v}_c$, where ϵ_{abc} is the Levi-Civita symbol in 3D. The compatibility constraints coming from the equality of cross-derivatives in s and t read

$$\partial_t \boldsymbol{\Omega} = \boldsymbol{\Omega} \times \boldsymbol{\omega} + \partial_s \boldsymbol{\omega}\,, \quad \partial_t \boldsymbol{\Gamma} + \boldsymbol{\omega} \times \boldsymbol{\Gamma} = \partial_s \boldsymbol{\gamma} + \boldsymbol{\Omega} \times \boldsymbol{\gamma}\,. \tag{1}$$

We assume that the cross-sectional area A depends on the instantaneous tube configuration, *i.e.* is determined by Λ, $\partial_s \Lambda$ and $\partial_s \boldsymbol{r}$, but not on the tube's dynamic variables or fluid motion. Since the function defining the cross-sectional area A has to be invariant with respect to $SO(3)$ rotations, we can posit a function $A = A(\boldsymbol{\Omega}, \boldsymbol{\Gamma})$ that we consider arbitrary, but given. The variations in A thus come from the bending, twisting and stretching of the local element of the tube. While the assumption that the change of the cross-sectional area is described by a function of $\boldsymbol{\Omega}$ and $\boldsymbol{\Gamma}$ will break down for a tube with easily stretchable walls, in most applications, including the familiar case of the garden hose, this approximation is valid. With this physical approximation in mind and assuming that the fluid inside the tube is incompressible (in 3D), the volume conservation along the tube reads

$$\partial_t Q + \partial_s (Q u) = 0\,, \quad Q := A(\boldsymbol{\Omega}, \boldsymbol{\Gamma})|\boldsymbol{\Gamma}|\,, \tag{2}$$

where the extra factor of $|\boldsymbol{\Gamma}|$ appears since s is not assumed to be the arc length. If $A = A(s)$ is independent of t, and $|\boldsymbol{\Gamma}| =$ const, (2) reduces to the conservation law $uA =$ const. This is the equation for velocity used in [11,16,17,22]; however, this approach is inexact as it neglects the time variation of A and stretch $\boldsymbol{\Gamma}$. Instead, the exact geometric variational approach taken in [23,24] is based on the critical action principle

$$\delta \iint \Big[\ell(\boldsymbol{\omega}, \boldsymbol{\gamma}, \boldsymbol{\Omega}, \boldsymbol{\Gamma}, u) + \mu \left((Q_0 \circ \varphi^{-1})\partial_s \varphi^{-1} - Q(\boldsymbol{\Omega}, \boldsymbol{\Gamma}) \right) \Big] \mathrm{d}s\, \mathrm{d}t = 0\,, \tag{3}$$

in which the transport equation integrating (2) is imposed with the help of a Lagrange multiplier $\mu(t,s)$ and with respect to the variations [30,31]

$$\begin{aligned} &\delta \boldsymbol{\omega} = \partial_t \boldsymbol{\Sigma} + \boldsymbol{\omega} \times \boldsymbol{\Sigma}, \qquad \delta \boldsymbol{\gamma} = \partial_t \boldsymbol{\Psi} + \boldsymbol{\gamma} \times \boldsymbol{\Sigma} + \boldsymbol{\omega} \times \boldsymbol{\Psi}, \qquad \delta u = \partial_t \eta + u \partial_s \eta - \eta \partial_s u, \\ &\delta \boldsymbol{\Omega} = \partial_s \boldsymbol{\Sigma} + \boldsymbol{\Omega} \times \boldsymbol{\Sigma}, \qquad \delta \boldsymbol{\Gamma} = \partial_s \boldsymbol{\Psi} + \boldsymbol{\Gamma} \times \boldsymbol{\Sigma} + \boldsymbol{\Omega} \times \boldsymbol{\Psi}\,, \end{aligned} \tag{4}$$

where $\widehat{\boldsymbol{\Sigma}} = \Lambda^{-1}\delta\Lambda$, $\boldsymbol{\Psi} = \Lambda^{-1}\delta \boldsymbol{r}$, and $\eta = \delta\varphi \circ \varphi^{-1}$ are free variations. In (3), Q_0 denotes the area in Lagrangian representation and the Lagrange multiplier in $\mu(s,t)$ has the dimensions of pressure. It can be viewed as (minus) the external pressure caused by the deformation of the tube, as suggested by a comparison of the fluid momentum equation in (5) with the Euler equation for an inviscid fluid. However, the precise physical meaning of $\mu(s,t)$ and its relationship to the physical pressure is still to be determined. The complete equations of motion for flexible tubes conducting fluid are [23,24]:

$$\begin{cases} D_t \dfrac{\partial \ell}{\partial \boldsymbol{\omega}} + \boldsymbol{\gamma} \times \dfrac{\partial \ell}{\partial \boldsymbol{\gamma}} + D_s \left(\dfrac{\partial \ell}{\partial \boldsymbol{\Omega}} - \dfrac{\partial Q}{\partial \boldsymbol{\Omega}} \mu \right) + \boldsymbol{\Gamma} \times \left(\dfrac{\partial \ell}{\partial \boldsymbol{\Gamma}} - \dfrac{\partial Q}{\partial \boldsymbol{\Gamma}} \mu \right) = 0\,, \\ D_t \dfrac{\partial \ell}{\partial \boldsymbol{\gamma}} + D_s \left(\dfrac{\partial \ell}{\partial \boldsymbol{\Gamma}} - \dfrac{\partial Q}{\partial \boldsymbol{\Gamma}} \mu \right) = 0\,, \quad \partial_t m + \partial_s (mu - \mu) = 0\,, \quad m := \dfrac{1}{Q} \dfrac{\partial \ell}{\partial u}\,, \end{cases} \tag{5}$$

together with (1) and (2), where $D_t := \partial_t + \boldsymbol{\omega} \times$ and $D_s := \partial_s + \boldsymbol{\Omega} \times$ are the full (material) derivatives. Equations (1), (2) and (5) form a closed system of equations for the problem, with the terms proportional to μ describing the effect of the cross-sectional dynamics, and are valid for an arbitrary cross-sectional area dependence $A(\boldsymbol{\Omega}, \boldsymbol{\Gamma})$ and Lagrangian density ℓ. As explained in [24], the variational principle (3)–(4) is rigorously justified by a reduction process applied to the Hamilton principle with holonomic constraint, written in terms of the Lagrangian variables $\Lambda, \dot{\Lambda}, \boldsymbol{r}, \dot{\boldsymbol{r}}, \varphi, \dot{\varphi}$, with free variations $\delta\Lambda$, $\delta \boldsymbol{r}$, $\delta\varphi$, vanishing at the temporal extremities.

3. Variational discretization in space

We now consider the spatial discretization of the variational approach outlined in Section 2 above. For simplicity, we consider a spatial discretization with equal space steps, $s_{i+1} - s_i = \Delta s$. The linear and angular deformation gradients are discretized by the elements $\lambda_i := \Lambda_i^{-1}\Lambda_{i+1} \in SO(3)$ and $\boldsymbol{\kappa}_i = \Lambda_i^{-1}(\boldsymbol{r}_{i+1} - \boldsymbol{r}_i) \in \mathbb{R}^3$, which is a standard discretization on Lie groups, see [32,33], adapted here to the spatial, rather than temporal, discretization, and for the special Euclidean group $SE(3)$. If the velocity of the fluid were constant during the evolution, which would happen when the cross-section of the tube is held constant, one could proceed in the established fashion to derive the variational discretization. In our case, the main mathematical difficulty lies in the appropriate discretization of the fluid velocity $u_i \simeq u(s_i, t)$ and in the derivation of a discrete fluid conservation law consistent with the variational description taken in [23,24]. As it turns out, the key to the solution lies in the discretization of the *inverse* of the Lagrangian mapping $\varphi(s,t)$, which is also known under the name of back-to-labels map and will be denoted as $\psi(s,t) := \varphi^{-1}(s,t) = S$. More precisely, we discretize $\psi(s,t)$ by its values at the points s_i by introducing the vector $\overline{\psi} = (\psi_1, \psi_2, \ldots, \psi_N)$, with ψ_i being functions of time t.

Consider a discretization of the derivative $\partial_s \psi(s_i, t)$ given by $D_i \overline{\psi}(t) := \sum_{j \in J} a_j \psi_{i+j}(t)$, where J is a finite set of integers in a neighborhood of $j = 0$ and D_i is the linear operator of differentiation acting on the vector $\overline{\psi} = (\psi_1, \ldots \psi_n)$. In the more general case of unequal spatial spacing, the discretization of the derivative may depend explicitly on the index i and is described by the formula $D_i \overline{\psi}(t) := \sum_{j \in i+J} A_{ij} \psi_j(t)$. Our approach generalizes to this case with no particular difficulty; however, this leads to quite cumbersome notation, especially for the spacetime discretization in Section 4. We shall thus only consider, for simplicity, the case of uniform spatial steps. In order to approximate the fluid velocity $u(s,t) = (\partial_t \varphi \circ \varphi^{-1})(s,t)$, we rewrite this relation in terms of ψ as

$$u(s,t) = (\partial_t \varphi \circ \psi)(s,t) = -\frac{\partial_t \psi(s,t)}{\partial_s \psi(s,t)}. \tag{6}$$

This relation is discretized as $u_i = -\dot{\psi}_i / D_i \overline{\psi}$. Taking the variation of u_i, we obtain

$$\delta u_i = -\frac{\delta \dot{\psi}_i}{D_i \overline{\psi}} + \frac{\dot{\psi}_i}{(D_i \overline{\psi})^2} \sum_{j \in J} a_j \delta \psi_{i+j} = -\frac{1}{D_i \overline{\psi}} \left(\delta \dot{\psi}_i + u_i \sum_{j \in J} a_j \delta \psi_{i+j} \right) = -\frac{1}{D_i \overline{\psi}} \left(\delta \dot{\psi}_i + u_i D_i \overline{\delta \psi} \right). \tag{7}$$

In what follows, we assume that at rest, the tube's cross section is constant and the tube is straight, which gives $Q_0 = \text{const}$. This assumption is not essential for the derivation, but does simplify the final expressions. In this case, the conservation law $(Q_0 \circ \varphi^{-1}) \partial_s \varphi^{-1} = Q(\boldsymbol{\Omega}, \boldsymbol{\Gamma})$ is discretized as $Q_0 D_i \overline{\psi} = F(\lambda_i, \boldsymbol{\kappa}_i) := F_i$. The appropriate discretization of the conservation law (2) is found by differentiating this discrete conservation with respect to time, and using the definition of Eulerian velocity $u_i = -\dot{\psi}_i / D_i \overline{\psi}$, leading to

$$\dot{F}_i + D_i \left(\overline{uF} \right) = 0. \tag{8}$$

One can see that (8) is consistent with the continuous conservation law (2). We assume that the Lagrangian density is spatially discretized as $\int_{s_i}^{s_{i+1}} \ell(\boldsymbol{\omega}, \boldsymbol{\gamma}, \boldsymbol{\Omega}, \boldsymbol{\Gamma}, u)\, \mathrm{d}s \simeq \ell_d(\boldsymbol{\omega}_i, \boldsymbol{\gamma}_i, \lambda_i, \boldsymbol{\kappa}_i, u_i)$. From the definition of the variables λ_i and $\boldsymbol{\kappa}_i$, we also have the discrete compatibility equations, cf. (1):

$$\dot{\lambda}_i = -\omega_i \lambda_i + \lambda_i \omega_{i+1}, \quad \dot{\boldsymbol{\kappa}}_i = -\boldsymbol{\omega}_i \times \boldsymbol{\kappa}_i + \lambda_i \boldsymbol{\gamma}_{i+1} - \boldsymbol{\gamma}_i. \tag{9}$$

We are now ready to formulate the main result of the paper regarding spatial discretization as follows.

Theorem 3.1. *The critical action principle for the flexible collapsible tube conveying an incompressible fluid for discrete space and continuous time reads* $\delta \int \sum_i \left[\ell_d(\boldsymbol{\omega}_i, \boldsymbol{\gamma}_i, \lambda_i, \boldsymbol{\kappa}_i, u_i) + \mu_i \left(Q_0 D_i \overline{\psi} - F(\lambda_i, \boldsymbol{\kappa}_i) \right) \right] \mathrm{d}t = 0$ *and yields the equations of motion:*

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \boldsymbol{\omega}_i \times \right) \frac{\partial \ell_d}{\partial \boldsymbol{\omega}_i} + \boldsymbol{\gamma}_i \times \frac{\partial \ell_d}{\partial \boldsymbol{\gamma}_i} + \left[\left(\frac{\partial \ell_d}{\partial \lambda_i} - \mu_i \frac{\partial F}{\partial \lambda_i} \right) \lambda_i^{\mathsf{T}} - \lambda_{i-1}^{\mathsf{T}} \left(\frac{\partial \ell_d}{\partial \lambda_{i-1}} - \mu_{i-1} \frac{\partial F}{\partial \lambda_{i-1}} \right) \right]^{\vee}$$
$$+ \boldsymbol{\kappa}_i \times \left(\frac{\partial \ell_d}{\partial \boldsymbol{\kappa}_i} - \mu_i \frac{\partial F}{\partial \boldsymbol{\kappa}_i} \right) = \mathbf{0} \tag{10}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \boldsymbol{\omega}_i \times \right) \frac{\partial \ell_d}{\partial \boldsymbol{\gamma}_i} + \left(\frac{\partial \ell_d}{\partial \boldsymbol{\kappa}_i} - \mu_i \frac{\partial F}{\partial \boldsymbol{\kappa}_i} \right) - \lambda_{i-1}^{\mathsf{T}} \left(\frac{\partial \ell_d}{\partial \boldsymbol{\kappa}_{i-1}} - \mu_{i-1} \frac{\partial F}{\partial \boldsymbol{\kappa}_{i-1}} \right) = \mathbf{0} \tag{11}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{F_i} \frac{\partial \ell_d}{\partial u_i} \right) + D_i^+ \left(\overline{\frac{u}{F} \frac{\partial \ell_d}{\partial u} - \mu} \right) = 0, \quad \text{where} \quad D_i^+ \overline{X} := -\sum_{j \in J} a_j X_{i-j}, \quad \text{and} \quad (m^{\vee})_c := -\frac{1}{2} \sum_{ab} \epsilon_{abc} m_{ab}, \tag{12}$$

together with conservation law (8)*. Equations* (10)*,* (11) *and* (12) *are the discrete equivalents of conservation laws for the angular, linear and fluid momenta.*

Proof. In a way similar to the continuous case in [24], the variational principle that we use is deduced from the standard Hamilton principle with holonomic constraint enforced through a Lagrangian multiplier. This principle is first expressed in terms of the variables $\Lambda_i, \dot{\Lambda}_i, \boldsymbol{r}_i, \dot{\boldsymbol{r}}_i, \psi_i, \dot{\psi}_i$, with free variations $\delta\Lambda_i$, $\delta \boldsymbol{r}_i$, $\delta \psi_i$. It is then rewritten in terms of the variables $\boldsymbol{\omega}_i, \boldsymbol{\gamma}_i, \lambda_i, \boldsymbol{\kappa}_i, u_i$. The variations $\delta\lambda_i$ and $\delta\boldsymbol{\kappa}_i$ are found to be $\delta\lambda_i = -\xi_i \lambda_i + \lambda_i \xi_{i+1}$ and $\delta\boldsymbol{\kappa}_i = -\xi_i \boldsymbol{\kappa}_i + \lambda_i \boldsymbol{\eta}_{i+1} - \boldsymbol{\eta}_i = -\boldsymbol{\xi}_i \times \boldsymbol{\kappa}_i + \lambda_i \boldsymbol{\eta}_{i+1} - \boldsymbol{\eta}_i$, where $\xi_i := \Lambda_i^{-1} \delta\Lambda_i \in \mathfrak{so}(3)$, $\boldsymbol{\eta}_i := \Lambda_i^{-1} \delta \boldsymbol{r}_i \in \mathbb{R}^3$, and $\xi = \widehat{\boldsymbol{\xi}}$. The variations of $\widehat{\boldsymbol{\omega}}_i = \Lambda_i^{-1} \dot{\Lambda}_i$ and $\boldsymbol{\gamma}_i = \Lambda_i^{-1} \dot{\boldsymbol{r}}_i$ are computed in a way similar to (4). In order to apply the variational principle, we also need to use the variations δu_i in terms of $\delta\psi_i$ computed in (7). Let us now consider the term enforcing the conservation law (8) with the Lagrange multiplier $\mu_i(t)$, which will give us the discrete analogue of pressure. A tedious but straightforward calculation gives

$$\delta\int_0^T \sum_i \mu_i\left(Q_0 D_i\overline{\psi} - F(\lambda_i,\boldsymbol{\kappa}_i)\right)\mathrm{d}t = \int\sum_i \delta\mu_i\left(Q_0 D_i\overline{\psi} - F(\lambda_i,\boldsymbol{\kappa}_i)\right)\mathrm{d}t + \int\sum_i Q_0 D_i\overline{\delta\psi}$$
$$+\left\langle\left(\mu_{i-1}\lambda_{i-1}^{\mathsf{T}}\frac{\partial F}{\partial\lambda_{i-1}} - \mu_i\frac{\partial F}{\partial\lambda_i}\lambda_i^{\mathsf{T}}\right)^{\vee} - \mu_i\boldsymbol{\kappa}_i\times\frac{\partial F}{\partial\boldsymbol{\kappa}_i},\boldsymbol{\xi}_i\right\rangle + \left\langle -\mu_i\frac{\partial F}{\partial\boldsymbol{\kappa}_i} + \mu_{i-1}\lambda_{i-1}^{\mathsf{T}}\frac{\partial F}{\partial\boldsymbol{\kappa}_{i-1}},\boldsymbol{\eta}\right\rangle\mathrm{d}t.$$

Proceeding by taking appropriate variational derivatives of the action functional and collecting the terms proportional to $\boldsymbol{\xi}_i$ and $\boldsymbol{\eta}_i$ gives the conservation law for the angular momentum (10) and linear momentum (11), respectively. The fluid momentum equation (12) is obtained by using (7), collecting the terms proportional to $\delta\psi_i$ and performing one integration by parts to remove the time derivative on $\delta\psi_i$ where necessary. Finally, we substitute $D_i\overline{\psi} = F_i/Q_0$ to remove $\overline{\psi}$ from the system and write all the equations in terms of the variables $\boldsymbol{\omega}_i, \boldsymbol{\gamma}_i, \lambda_i, \boldsymbol{\kappa}_i, u_i, \mu_i$ only. □

It is interesting to notice that (12) is a direct analogue of the fluid momentum equation $\partial_t m + \partial_s(mu-\mu) = 0$, see (5). Note that we have chosen the minus sign in the definition of D_i^+ in (12) so that D_i^+ is a discrete approximation of the derivative, and not minus the derivative. For example, for equal spacing Δs between the neighboring points s_i, and taking $D_i f = (f_i - f_{i-1})/\Delta s$ as the first-order backward derivative, we have $D_i^+ f = (f_{i+1} - f_i)/\Delta s$, the forward derivative.

4. Variational integrator in space and time

Let us now turn our attention to the derivation of a variational integrator for this problem. The important novel step here is to provide a discretization of the fluid part. The rest of the analysis can be obtained according to the methods of spacetime multisymplectic variational discretization [27]. In particular, the recent work [26] derived a variational discretization based on the multisymplectic nature of the exact geometric rods, which we shall use as the foundation of our approach. Let us start with the discretization of the fluid part. Select a rectangular lattice of points in space and time (s_i, t_j) and define the discretization of the back-to-labels map $\psi(s,t)$ as $\overline{\psi}$ to have discrete values $\psi_{i,j}$ at the spacetime points (s_i, t_j). Assume that the spatial and time derivatives are given by the discrete operators

$$D^s_{i,j}\overline{\psi} := \sum_{k\in K} a_j\psi_{i,j+k}\,, \quad D^t_{i,j}\overline{\psi} := \sum_{m\in M} b_m\psi_{i+m,j}\,, \tag{13}$$

where K and M are discrete finite sets of indices in a neighborhood of 0. Using (6), we obtain the following connection between the velocity and back-to-labels map

$$u_{i,j} = -\frac{D^t_{i,j}\overline{\psi}}{D^s_{i,j}\overline{\psi}}. \tag{14}$$

The discretization of the conservation law $Q_0\partial_s\psi = F$ is then given as $Q_0 D^s_{i,j}\overline{\psi} = F_{i,j}$. Applying D^t_{ij} to both sides of this conservation law, noticing that the operators D^t_{ij} and D^s_{ij} defined by (13) commute on a rectangular lattice in (t,s), and using (14) to eliminate $D^t_{i,j}\overline{\psi}$ from the equations, we obtain the discrete conservation law:

$$D^t_{i,j}\overline{F} + D^s_{i,j}(\overline{uF}) = 0\,. \tag{15}$$

Let us now define the spacetime discrete versions of the continuous deformations $(\boldsymbol{\Omega},\boldsymbol{\Gamma})$ and velocities $(\boldsymbol{\omega},\boldsymbol{\gamma})$. If $(\Lambda_{i,j},\boldsymbol{r}_{i,j})\in SE(3)$ are the orientation and position at (t_i, s_j), we define

$$\lambda_{i,j} := \Lambda_{i,j}^{-1}\Lambda_{i+1,j}\,, \quad \boldsymbol{\kappa}_{i,j} := \Lambda_{i,j}^{-1}\left(\boldsymbol{r}_{i+1,j} - \boldsymbol{r}_{i,j}\right)\,, \quad q_{i,j} := \Lambda_{i,j}^{-1}\Lambda_{i,j+1}\,, \quad \boldsymbol{\gamma}_{i,j} := \Lambda_{i,j}^{-1}\left(\boldsymbol{r}_{i,j+1} - \boldsymbol{r}_{i,j}\right)\,. \tag{16}$$

Taking the variations of (16) and (14) leads to

$$\delta\lambda_{i,j} = -\xi_{i,j}\lambda_{i,j} + \lambda_{i,j}\xi_{i+1,j}\,,\ \delta\boldsymbol{\kappa}_{i,j} = -\boldsymbol{\xi}_{i,j}\times\boldsymbol{\kappa}_{i,j} + \lambda_{i,j}\boldsymbol{\eta}_{i+1,j} - \boldsymbol{\eta}_{i,j}\,,\ \delta u_{i,j} = -\frac{1}{D^s_{i,j}\overline{\psi}}\left(D^t_{i,j}\overline{\delta\psi} + u_{i,j}D^s_{i,j}\overline{\delta\psi}\right)\,,$$
$$\delta q_{i,j} = -\xi_{i,j}q_{i,j} + q_{i,j}\xi_{i,j+1}\,,\ \ \delta\boldsymbol{\gamma}_{i,j} = -\boldsymbol{\xi}_{i,j}\times\boldsymbol{\kappa}_{i,j} + q_{i,j}\boldsymbol{\eta}_{i,j+1} - \boldsymbol{\eta}_{i,j}\,, \tag{17}$$

where $\xi_{i,j} := \Lambda_{i,j}^{-1}\delta\Lambda_{i,j}$ and $\boldsymbol{\eta}_{i,j} := \Lambda_{i,j}^{-1}\delta\boldsymbol{r}_{i,j}$. We assume that the Lagrangian density is discretized in space and time as $\int_{t_j}^{t_{j+1}}\int_{s_i}^{s_{i+1}}\ell\big(\boldsymbol{\omega},\boldsymbol{\gamma},\boldsymbol{\Omega},\boldsymbol{\Gamma},u\big)\,\mathrm{d}s\,\mathrm{d}t \simeq \mathcal{L}_d(\lambda_{i,j},\boldsymbol{\kappa}_{i,j},q_{i,j},\boldsymbol{\gamma}_{i,j},u_{i,j})$. We now formulate the main result concerning the spacetime discretization of a fluid-conveying elastic tube.

Theorem 4.1. *The discrete critical action principle* $\delta\left[\sum_{i,j}\mathcal{L}_d\left(\lambda_{i,j},\boldsymbol{\kappa}_{i,j},q_{i,j},\boldsymbol{\gamma}_{i,j},u_{i,j}\right)+\mu_{i,j}\left(Q_0 D^s_{i,j}\overline{\psi}-F(\lambda_{i,j},\boldsymbol{\kappa}_{i,j})\right)\right]=0$ *leads to the following discrete equations of motion:*

$$\left[\frac{\partial\mathcal{L}_d}{\partial q_{i,j}}q^{\mathsf{T}}_{i,j}-q^{\mathsf{T}}_{i,j-1}\frac{\partial\mathcal{L}_d}{\partial q_{i,j-1}}\right]^{\vee}+\left[\left(\frac{\partial\mathcal{L}_d}{\partial\lambda_{i,j}}-\mu_{i,j}\frac{\partial F}{\partial\lambda_{i,j}}\right)\lambda^{\mathsf{T}}_{i,j}-\lambda^{\mathsf{T}}_{i-1,j}\left(\frac{\partial\mathcal{L}_d}{\partial\lambda_{i-1,j}}-\mu_{i-1,j}\frac{\partial F}{\partial\lambda_{i-1,j}}\right)\right]^{\vee}$$
$$+\boldsymbol{\gamma}_{i,j}\times\frac{\partial\mathcal{L}_d}{\partial\boldsymbol{\gamma}_{i,j}}+\boldsymbol{\kappa}_{i,j}\times\frac{\partial\mathcal{L}_d}{\partial\boldsymbol{\kappa}_{i,j}}=\mathbf{0} \tag{18}$$

$$\frac{\partial\mathcal{L}_d}{\partial\boldsymbol{\gamma}_{i,j}}-q^{\mathsf{T}}_{i,j-1}\frac{\partial\mathcal{L}_d}{\partial\boldsymbol{\gamma}_{i,j-1}}+\left(\frac{\partial\mathcal{L}_d}{\partial\boldsymbol{\kappa}_{i,j}}-\mu_{i,j}\frac{\partial F}{\partial\boldsymbol{\kappa}_{i,j}}\right)-\lambda^{\mathsf{T}}_{i-1,j}\left(\frac{\partial\mathcal{L}_d}{\partial\boldsymbol{\kappa}_{i-1,j}}-\mu_{i-1,j}\frac{\partial F}{\partial\boldsymbol{\kappa}_{i-1,j}}\right)=\mathbf{0} \tag{19}$$

$$D^{t,+}_{i,j}\overline{m}+D^{s,+}_{i,j}\left(\overline{um}-\overline{\mu}\right)=0\,,\quad m_{i,j}:=\frac{1}{F_{i,j}}\frac{\partial\mathcal{L}_d}{\partial u_{i,j}}\,,\quad D^{s,+}_{i,j}\overline{X}:=-\sum_{k\in K}a_k X_{i,j-k}\,,\quad D^{t,+}_{i,j}\overline{X}:=-\sum_{m\in M}b_j X_{i-m,j}\,. \tag{20}$$

Proof. The variational principle that we use is deduced from the spacetime-discretized Hamilton principle with constraint. This principle is first expressed in terms of the variables $\Lambda_{i,j},\boldsymbol{r}_{i,j},\psi_{i,j}$ and with respect to arbitrary variations $\delta\Lambda_{i,j},\delta\boldsymbol{r}_{i,j},\delta\psi_{i,j}$. It is rewritten here in terms of the variables $\lambda_{i,j},\boldsymbol{\kappa}_{i,j},q_{i,j},\boldsymbol{\gamma}_{i,j},u_{i,j}$, and their corresponding constrained variations are derived in (17). Taking the variation of the discrete action functional and collecting the terms proportional to $\xi_{i,j}$, $\boldsymbol{\eta}_{i,j}$, and $\delta\psi_{i,j}$, we obtain the conservation laws of the angular (18), linear (19) and fluid (20) momenta, respectively. The term proportional to $\delta\mu_{i,j}$ provides the conservation law $Q_0 D^s_{i,j}\overline{\psi}=F_{i,j}$, which upon applying the discrete time derivative leads to (15) and eliminates $\psi_{i,j}$ from the equations. □

The system (18), (19), (20) is solved by taking into account the relations (16), which play the role of the compatibility conditions (9) in the spacetime discretized case.

Remark 4.2 *(On the inclusion of boundary conditions).* In this paper, we did not consider the important issue of the choice of appropriate boundary conditions at the inlet and the outlet of the tube. Consequently, we have assumed that all the boundary terms arising from discrete integration by parts vanish at the spatial extremities in the variational principles. As was discussed in [24], the boundary conditions can be consistently included in the variational principle by considering the Lagrange–d'Alembert approach. The variational approach in [24] also allows us to naturally obtain the expressions of the boundary forces exerted on the linear and angular momentum equation for the tube and on the momentum equation for the fluid. Being purely variational, the approach developed in [24] thus admits spatial and spacetime discrete versions, which can be naturally incorporated in the developments of the present paper. We shall address these important questions in the subsequent work.

5. Conclusions and further directions

We have derived spatial and spacetime variational discretizations for the dynamics of a flexible collapsible tube conveying an ideal fluid, such as prescribed fluid velocity at one extremity, and free boundary condition at the other extremity. The result summarized in Theorem 3.1 allows for a consistent discretization of the spatial coordinates well suited to the study of structure-preserving simplified models of fluid-conveying tubes. In particular, by construction, these discrete models automatically preserve the left invariance of the system as well as the symplectic character of the dynamics on the constraint subset determined by fluid volume preservation. Such discretization is useful for the derivation of consistent simplified models of the system. The result summarized in Theorem 4.1 extends the discretization in both time and space and thus provides a structure-preserving numerical scheme for a fluid-conveying tube. By construction, this scheme is also compatible with the consistent simplified models obtained in Section 3.

The main mathematical difficulty we have overcome came from the presence of both left-invariant dynamics coming from the elastic part, and right-invariant dynamics coming from the fluid part, additionally coupled through the Lagrangian and the constraint of the conservation of fluid volume. As it turns out, our scheme derived in Theorem 4.1 is not just variational, but also verifies an extended notion of discrete multisymplecticity inherited from the corresponding extended notion of multisymplecticity verified by the continuous problem. Such property can be shown by using the general spacetime variational discretization of field theories derived in [27], extended to account for the presence of the constraint. The general variational approach to discretization derived here will further allow us to exploit the symmetries of the problem in order to derive discrete Noether theorems for the dynamics.

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