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Flows induced by power-law stretching surface motion modulated by transverse or orthogonal surface shear

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A R T I C L E I N F O

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A B S T R A C T

Boundary-layer solutions to Banks' problem for the flow induced by power-law stretching of a plate are obtained for two generalizations that include arbitrary transverse plate shearing motion. In one extension an arbitrary transverse shearing motion is the product of the power-law stretching. In the other extension the streamwise stretching coordinate is added to an arbitrary transverse shearing and together raised to the power of stretching. In addition we find that Banks' power law stretching may be accompanied by orthogonal power-law shear. In all cases, the original boundary-value problem of Banks [1] is recovered. Results are illustrated with velocity profiles both at the plate and at fixed height in the fluid above the plate.

1. Introduction

Central to this work are the papers by Crane [2] and Banks [1]. Crane [2] reported an exact solution to the Navier–Stokes equations for the flow generated by a linearly-stretching surface beneath a quiescent fluid. Banks [1] found a family of solutions for power-law stretching of a surface in the context of the boundary-layer approximation.

Flows induced by flat impermeable surfaces executing both stretching and shearing motions are the focus of this study. The Cartesian coordinate system (x, y, z) is used with corresponding velocities (u, v, w). Here x is the streamwise direction, y is the spanwise direction and z is the plate-normal coordinate. Problems for the flow induced by a plate undergoing linear stretching with an attendant linear shear flow have been discussed by Weidman [3]. He proved that uniform surface shear flow $u(x, y, 0) = ay$, $v(x, y, 0) = 0$ cannot exist without uniform transpiration $w(x, y, z) = -W_0$. Solutions for certain linear combinations of plate stretching and shearing motions were shown to exist without transpiration, but the problem of orthogonal linear plate shearing motion $u(x, y, 0) = a\,y$, $v(x, y, 0) = b\,x$ has no solution satisfying zero horizontal motion in the far field, even with uniform transpiration.

Recently Weidman [4] considered in all generality the problem of orthogonal linear stretching superposed onto orthogonal linear shearing of a surface. It transpires that a certain relation between the orthogonal shearing motions is required for a similarity solution to exist. Weidman, et al. [5] considered the problem of flows induced by biaxial linear stretching and biaxial linear shearing of an impermeable surface in a uniformly rotating fluid system.

The current study generalizes the zero-pressure-gradient boundary-layer solutions for power-law stretching $u = a\,x^n$ reported by Banks [1] to include two types of *arbitrary* transverse shearing motion $\alpha(y)$. In these flows, although $v \equiv 0$,

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the flow is three-dimensional. It is further found that Banks' power law stretching may be accompanied by an orthogonal power-law shearing of the plate for which the transverse velocity v is finite, thus rendering the flow fully three-dimensional. Consequently, the governing boundary layer equations are the equation of continuity

$$u_x + v_y + w_z = 0 \tag{1a}$$

the streamwise momentum equation

$$u\,u_x + v\,u_y + w\,u_z = \nu\,u_{zz} \tag{1b}$$

and the spanwise momentum equation

$$u\,v_x + v\,v_y + w\,v_z = \nu\,v_{zz} \tag{1c}$$

The presentation is as follows. A first generalization of Banks' power-law stretching to include one form of arbitrary transverse plate shearing is presented in §2. This is followed by a second generalization in §3. In §4 we solve the problem of power-law stretching accompanied by an orthogonal power-law shearing motion of the plate. A summary and discussion of results are given in §5.

2. First extension of Banks' problem

Here we modulate Banks' power-law stretching by an arbitrary transverse shear $\alpha(y)$ with $v = 0$ in the form

$$u(x, y, 0) = a\,x^n \alpha(y), \qquad w(x, y, 0) = 0, \qquad u \to 0, \qquad (z \to \infty) \tag{2}$$

The solution *ansatz* for the streamwise motion is taken as

$$u(x, y, \eta) = a\,x^n \alpha(y) f'(\eta), \qquad \eta = \sqrt{\frac{a}{\nu}}\,\mu(x)\,\sigma(y)\,z \tag{3}$$

and the continuity equation (1a) gives the requisite form of the normal velocity, *viz.*

$$w(x, y, \eta) = -\frac{\sqrt{a\nu}\;\alpha(y)}{\mu(x)\sigma(y)}\left[n x^{n-1} f + x^n \frac{\mu_x}{\mu}(\eta f' - f)\right] \tag{4}$$

Inserting the above velocity field into (1b) gives

$$\mu^2\sigma^2 f''' + \left[\left(n x^{n-1} - x^n \frac{\mu_x}{\mu}\right) f f'' - n x^{n-1} f'^2\right]\alpha(y) = 0 \tag{5}$$

in which it is clear that $\sigma(y) = \sqrt{\alpha(y)}$ for similarity. Also, for the coefficient of the ff'' term, we require

$$x^n \frac{\mu_x}{\mu} = K n x^{n-1} \tag{6}$$

and, after isolating f''', we set the coefficient of the ff'' term to unity to find

$$\mu(x) = C x^{nK}, \qquad K = \frac{n-1}{2n}, \qquad C = \sqrt{\frac{n+1}{2}} \tag{7}$$

This furnishes the boundary-value problem for an impermeable plate found by Banks [1], *viz.*

$$f''' + ff'' - \beta f'^2 = 0, \qquad f(0) = 0, \quad f'(0) = 1, \quad f'(\infty) = 0 \tag{8}$$

where $\beta = 2n/(n+1)$ and we note that n is not required to be an integer. Banks [1] presented numerical solutions to this problem for many values of β in the range $-1.9999 \le \beta \le 202$.

In summary, the velocity field is

$$u(x, y, \eta) = a\,x^n\,\alpha(y)\,f'(\eta), \qquad \eta = \sqrt{\frac{a(n+1)\,\alpha(y)}{2\nu}}\,x^{(n-1)/2} z \tag{9a}$$

$$w(x, y, \eta) = -\sqrt{\frac{a\,\nu(n+1)\,\alpha(y)}{2}}\,x^{(n-1)/2}\left[f + \left(\frac{n-1}{n+1}\right)\eta\,f'\right] \tag{9b}$$

Note that although $v \equiv 0$, the flow is three-dimensional, since the two remaining velocities u and w depend on all three coordinates.

To see some transverse wall shear motions of this type, we select $\alpha(y) = y$ and choose $a = 1$ and $n = 2$ to obtain the wall motion

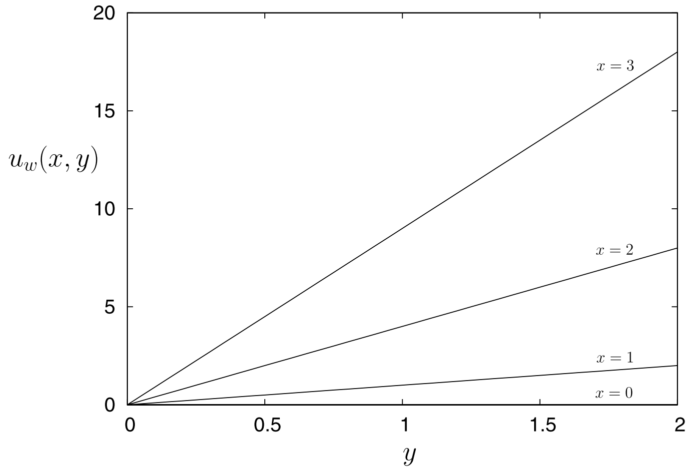

Fig. 1. Transverse wall shear motions $u_w(x, y) = x^2 y$ for the first extension of Banks' problem plotted for $x = \{0, 1, 2, 3\}$ over the range of the transverse coordinate as indicated.

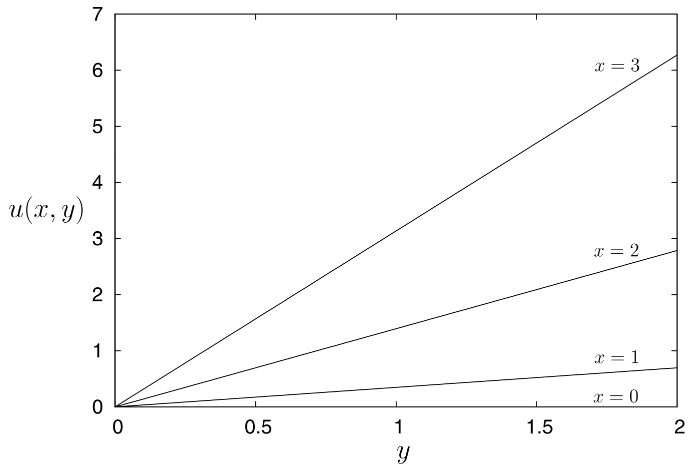

Fig. 2a. Streamwise velocities $u(x, y, \eta)$ for the first extension of Banks at the normalized position $\eta = 1$ above the plate plotted for $x = \{0, 1, 2, 3\}$ over the range of the transverse coordinate as indicated.

$$u(x, y, 0) \equiv u_w(x, y) = x^2 y \tag{10}$$

Sample shear flows at $x = \{0, 1, 2, 3\}$ are presented in Fig. 1 over the region $0 \le y \le 2$.

To visualize how the flow evolves above the plate, we provide plots of the $u(x, y)$ and $w(x, y)$ velocities for the same parameters $a = 1$, $n = 2$ and $\alpha(y) = y$ as in Fig. 1. The chosen value $n = 2$ gives $\beta = 4/3$ for evaluation of Banks's equation (8). The value $\eta = 1.0$ above the plate is chosen. At this position we find $f(1) = 0.613845$ and $f'(1) = 0.348382$, which gives the following horizontal velocities calculated from Eqs. (9a), (9b)

$$u(x, y) = 0.348382\, x^2 y, \qquad \frac{w(x, y)}{\sqrt{3\nu/2}} = -0.729973\, x^{1/2} y^{1/2} \tag{11}$$

in which the vertical velocity has been appropriately normalized. The results for the streamwise velocity $u(x, y)$ as a function of y at the same fixed values $x = \{0, 1, 2, 3\}$ as in Fig. 1 are shown in Fig. 2a. Comparable results for the vertical velocity are displayed in Fig. 2b.

A comparison of results in Fig. 2a with the wall motion $u_w(x, y)$ in Fig. 1 shows the streamwise velocities decrease with height above the plate, as expected to satisfy the far field condition $f'(\infty) = 0$. The results in Fig. 2b show that the induced velocity above the plate is negative, as expected in order for the flow to not separate from the plate.

3. Second extension of Banks' problem

In this problem, we posit a second generalization of Banks' power-law stretching with $v = 0$ in the form

$$u(x, y, 0) = a[x + \alpha(y)]^n, \qquad w(x, y, 0) = 0, \qquad u \to 0, \qquad (z \to \infty) \tag{12}$$

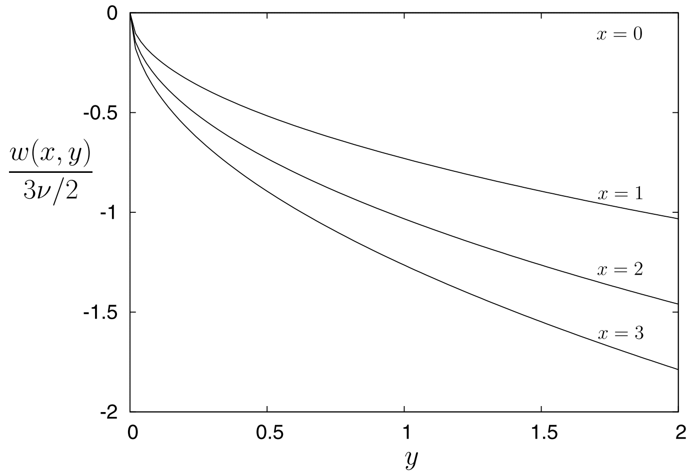

Fig. 2b. Normalized vertical velocities $w(x, y, \eta)$ for the first extension of Banks at the normalized position $\eta = 1$ above the plate plotted for $x = \{0, 1, 2, 3\}$ over the range of the transverse coordinate as indicated.

so the appropriate *ansatz* is now

$$u(x, y, \eta) = a[x + \alpha(y)]^n f'(\eta), \qquad \eta = \mu(x, y)\, z \tag{13}$$

The continuity equation (1a) then provides the plate-normal velocity

$$w(x, y, \eta) = -\left[\frac{n\, a[x + \alpha(y)]^{n-1}}{\mu} f + a[x + \alpha(y)]^n \frac{\mu_x}{\mu^2}(\eta f' - f)\right] \tag{14}$$

Inserting this velocity field into (1b) gives

$$n\, a[x + \alpha(y)]^{n-1} f'^2 + a\left[[x + \alpha(y)]^n \frac{\mu_x}{\mu} - n[x + \alpha(y)]^{n-1}\right] f f'' = \nu \mu^2 f''' \tag{15}$$

The coefficient of the ff'' term and similarity require

$$\mu(x, y) = C[x + \alpha(y)]^{nK(n)}, \qquad K = \left(\frac{n-1}{2n}\right) \tag{16}$$

Inserting this result into (15) and isolating the highest derivative gives

$$f''' + \frac{n\, a}{\nu\, C^2}\left[(1 - K) f f'' - f'^2\right] = 0 \tag{17}$$

Setting the coefficient of ff'' to unity then gives

$$C = \sqrt{\frac{a(n+1)}{2\nu}} \tag{18}$$

which renders (17) in the form

$$f''' + f f'' - \beta f'^2 = 0, \qquad \beta = \frac{2n}{n+1} \tag{19a}$$

to be solved with impermeable plate and far-field conditions

$$f(0) = 0, \qquad f'(0) = 1, \qquad f'(\infty) = 0 \tag{19b}$$

which is again recognized as the problem of Banks [1].

In summary, the velocity field is

$$u(x, y, \eta) = a[x + \alpha(y)]^n f'(\eta), \qquad \eta = \sqrt{\frac{a(n+1)}{2\nu}}\, [x + \alpha(y)]^{(n-1)/2} z \tag{20a}$$

$$w(x, y, \eta) = -\sqrt{\frac{a\,\nu}{2(n+1)}}\, [x + \alpha(y)]^{(n-1)/2} \left((n+1) f + (n-1)\eta f'\right) \tag{20b}$$

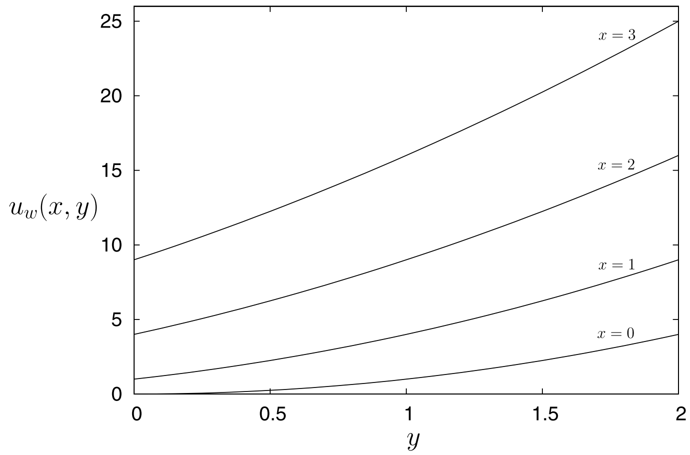

Fig. 3. Transverse wall shear motions $u_w(x, y) = (x + y)^n$ for the second extension of Banks' problem plotted for $x = \{0, 1, 2, 3\}$ over the range of the transverse coordinate as indicated.

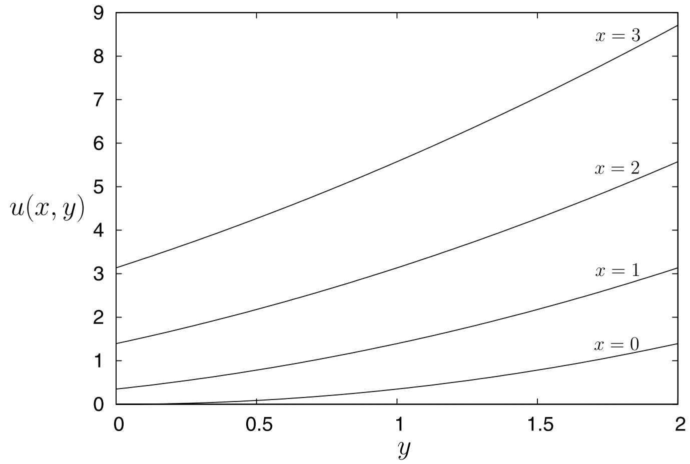

Fig. 4a. Streamwise velocities $u(x, y, \eta)$ for the second extension of Banks at the normalized position $\eta = 1$ above the plate plotted for $x = \{0, 1, 2, 3\}$ over the range of the transverse coordinate as indicated.

As with the first extension of Banks in §2, the flow is three-dimensional since the velocities u and w depend on all three coordinates.

To view sample transverse wall shear motions of this type we choose $\alpha(y) = y$, $a = 1$ and $n = 2$ to obtain the wall motion

$$u(x, y, 0) \equiv u_w(x, y) = (x + y)^2 \tag{21}$$

and plot results at $x = \{0, 1, 2, 3\}$ in Fig. 3 over the region $0 \le y \le 2$.

To see how the flow evolves above the plate, we provide plots of the $u(x, y)$ and $w(x, y)$ velocities for the same parameters $a = 1$, $n = 2$ and $\alpha(y) = y$ as in Fig. 3. The value $\eta = 1.0$ above the plate is chosen and again $\beta = 4/3$ in Banks' equation (19). Again $f(1) = 0.613845$ and $f'(1) = 0.348382$ which gives the following horizontal velocities calculated from Eq. (20)

$$u(x, y) = 0.348382\,(x + y)^2, \qquad \frac{w(x, y)}{\sqrt{\nu/6}} = -2.189917\,(x + y)^{1/2} \tag{22}$$

in which the vertical velocity has been appropriately normalized. The results for the streamwise velocity $u(x, y)$ as a function of y at the same fixed values $x = \{0, 1, 2, 3\}$ as in Fig. 3 are shown in Fig. 4a. Comparable results for the vertical velocity are displayed in Fig. 4b.

Again Fig. 4a shows that the streamwise velocities decrease with height above the plate to satisfy the far field condition $f'(\infty) = 0$. The results in Fig. 4b show that the induced velocity above the plate is negative, as expected.

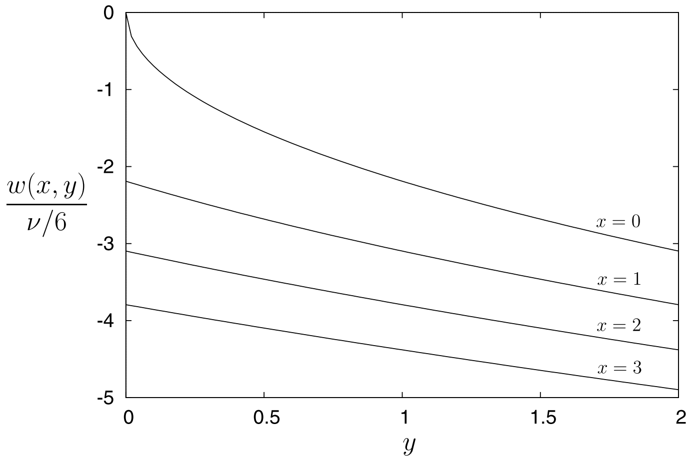

Fig. 4b. Normalized vertical velocities $w(x, y, \eta)$ for the second extension of Banks at the normalized position $\eta = 1$ above the plate plotted for $x = \{0, 1, 2, 3\}$ over the range of the transverse coordinate as indicated.

4. Third extension of Banks' problem

A similarity solution also exists for a plate where an orthogonal power-law shearing motion is added to the streamwise power-law stretching motion, provided that the exponents are equal. In this case the plate motion is

$$u(x, y, 0) = a x^n, \qquad v(x, y, 0) = bx^n, \qquad w(x, y, 0) = 0 \tag{23}$$

Inserting the solution *ansatz*

$$u(x, y, \eta) = a x^n f'(\eta), \qquad v(x, y, \eta) = bx^n g'(\eta) \qquad \eta = \sqrt{\frac{a}{\nu}}\,\mu(x)\, z \tag{24}$$

into the continuity equation (1a) provides the plate-normal velocity

$$w(x, y, \eta) = -\frac{\sqrt{a\nu}}{\mu}\left[nx^{n-1} f + x^n \frac{\mu_x}{\mu}(\eta f' - f)\right] \tag{25}$$

Using these velocities the x-momentum equation (1b) takes the form

$$\mu^2 f''' + \left(nx^{n-1} - x^n \frac{\mu_x}{\mu}\right) ff'' - nx^{n-1} f'^2 = 0 \tag{26}$$

and for similarity one must have

$$\frac{\mu_x}{\mu} = K\frac{n}{x} \qquad \Rightarrow \qquad \mu(x) = Cx^{Kn} \tag{27}$$

for some constants K and C. The momentum equation (26) may now be written

$$f''' + \left(\frac{n(1-K)x^{n-1}}{C^2 x^{2Kn}}\right) ff'' - \left(\frac{nx^{n-1}}{C^2 x^{2Kn}}\right) f'^2 = 0 \tag{28}$$

Setting the coefficient of ff'' to unity requires

$$K = \frac{n-1}{2n}, \qquad C = \sqrt{\frac{n+1}{2}} \tag{29}$$

thus rendering (28) as the problem of Banks [1], *viz.*

$$f''' + ff'' - \left(\frac{2n}{n+1}\right) f'^2 = 0, \qquad f(0) = 0, \quad f'(0) = 1, \quad f'(\infty) = 0 \tag{30}$$

Turning to the y-momentum equation (1c) with $\mu(x)$ now determined, one readily finds the governing equation for the transverse motion

$$g''' + fg'' - \left(\frac{2n}{n+1}\right) f'g' = 0, \qquad g(0) = 0, \quad g'(0) = 1, \quad g'(\infty) = 0 \tag{31}$$

Since the boundary conditions for (30) and (31) are identical, it is clear that the solution to (31) is given by $g(\eta) = f(\eta)$. Consequently, the power-law stretching of Banks [1]

$$u(x,\eta) = a\,x^n f'(\eta), \qquad \eta = \sqrt{\frac{a(n+1)}{2\nu}}\,x^{(n-1)/2} z \tag{32a}$$

admits the concomitant orthogonal shear flow

$$v(x,\eta) = b x^n f'(\eta) \tag{32b}$$

for arbitrary shear strength b.

5. Discussion and conclusion

Three new solutions to Banks' problem on the flow induced by the power-law plate stretching motion $u(x,0) = a\,x^n$ in a quiescent fluid are found which include some form of transverse plate shearing motion or an orthogonal shearing of the plate. In all cases the problem reduces to that of Banks [1], independent of the additional shearing motions imposed. In the first extension for two-dimensional flow, $v \equiv 0$ and solutions are found for plate motions of the form

$$u(x,y,0) = a\,x^n \alpha(y), \qquad w(x,y,0) = 0 \tag{33}$$

for arbitrary $\alpha(y)$. Sample transverse plate motions for $\alpha(y) = y$, $a = 1$ and $n = 2$ are plotted in Fig. 1. Also, sample velocities $u(x,y,\eta)$ and $w(x,y,\eta)$ are plotted as a function of the transverse coordinate at the nondimensional height $\eta = 1$ above the plate for fixed values of the streamwise coordinate in Figs. 2a, 2b.

In the second extension, solutions are found for plate motions of the form

$$u(x,y,0) = a[x + \alpha(y)]^n, \qquad w(x,y,0) = 0 \tag{34}$$

for arbitrary $\alpha(y)$. Sample transverse plate motions for $\alpha(y) = y$, $a = 1$ and $n = 2$ are plotted in Fig. 3. Sample velocities $u(x,y,\eta)$ and $w(x,y,\eta)$ are also plotted as a function of the transverse coordinate at the nondimensional height $\eta = 1$ above the plate for fixed values of the streamwise coordinate in Figs. 4a, 4b.

The third extension corresponds to three-dimensional flow with orthogonal power-law shearing concomitant with the power-law stretching; these have plate motions

$$u(x,y,0) = a\,x^n, \qquad v(x,y,0) = b\,x^n, \qquad w(x,y,0) = 0 \tag{35}$$

for arbitrary stretching strengths a and b. Here the vertical variation of flow is determined directly by the Banks [1] solution.

As a fourth possibility, plate motions of the form

$$u(x,y,0) = a\,x^n + \alpha(y), \qquad w(x,y,0) = 0 \tag{36a}$$

for which the similarity ansatz taken as

$$u(x,y,\eta) = (a\,x^n + \alpha(y))\,f'(\eta), \qquad \eta = \mu(x,y)z \tag{36b}$$

were posited, but no self-similar solutions were found for the following reason. Although setting the coefficient of ff'' to unity gives $\mu(x,y) = C(a\,x^n + \alpha(y))^K$ for some constants C and K, when one looks at the coefficient of f'^2 one finds the condition for similarity is given by

$$x^{n-1} = (a\,x^n + \alpha(y))^{2K} \tag{37}$$

which is possible only if $\alpha(y) = 0$. Other scenarios were tried, but it appears that (33) and (34) are the only plate motions for which an *arbitrary* transverse shearing motion of the plate may be added to the power-law stretching.

An interesting feature of (34) exists for positive integer values of the stretching exponent n. Denoting Banks' solutions as B_n, *viz.*

$$B_n = a\,x^n \tag{38}$$

then all plate motions of the form (34) are linear combinations of these power-law stretching solutions. For example, taking $n = 3$ gives the plate motion

$$\begin{aligned} u(x,y,0) &= a[x + \alpha(y)]^3 \\ &= a[x^3 + 3\,x^2\,\alpha(y) + 3\,x\,\alpha(y)^2 + \alpha(y)^3] \\ &= B_3 + 3\,B_2\,\alpha(y) + 3\,B_1\,\alpha(y)^2 + B_0\,\alpha(y)^3 \end{aligned} \tag{39}$$

The following observation for a linearly stretching surface is noted. The exponent $n = 1$ corresponding to $\beta = 1$ reduces Banks' equation (8) to the linear stretching problem of Crane [2], with solution $f(\eta) = 1 - \mathrm{e}^{-\eta}$ being an exact solution to the

Navier–Stokes equation. However, the inclusion of arbitrary transverse plate shearing $\alpha(y)$ in Eqs. (2) and (12) precludes an exact solution to the Navier–Stokes equation; only through the use of the boundary-layer formulation is a similarity solution possible.

It is of interest to compare the transverse shearing generalizations of Banks' problem with the nonlinear sheared plate solutions reported in Section 3 of Weidman [3]. In that study, there was no streamwise stretching of any kind, only nonlinear streamwise shearing. Those solutions were admissible only with the superposition of uniform suction at the surface of the plate. In the present study, solutions are found for which no suction needs to be applied.

In conclusion, it should be noted that the generalizations (33) and (34) of Banks' power-law stretching to include arbitrary transverse shearing of a plate are only possible because the flow, although three dimensional with $u(x, y, \eta)$ and $w(x, y, \eta)$, has no transverse velocity component. As noted by a referee, the imposition of transverse variations of boundary conditions uniquely serves the mathematical purpose of defining new self-similarity variables $\eta(x, y)$ that account, when possible for this dependence on the transverse coordinate y. In the two generalized extensions of Banks' power-law stretching solution, a dilation and translation of the y-solution slices, both proportional to the transverse coordinate y, permit the two new extension found in this study.

References

[1] W.H.H. Banks, Similarity solutions of the boundary-layer equations for a stretching wall, J. Méc. Théor. Appl. 2 (1983) 375–392.
[2] L.J. Crane, Flow past a stretching sheet, Z. Angew. Math. Phys. 21 (1970) 645–647.
[3] P.D. Weidman, Flows induced by flat surfaces sheared in their own plane, Fluid Dyn. Res. 45 (2013) 015506.
[4] P.D. Weidman, The motion induced by the orthogonal stretching and shearing of a membrane beneath a quiescent fluid, Acta Mech. 226 (2015) 3307–3316.
[5] P.D. Weidman, S. Mansur, A. Ishak, Biorthogonal stretching and shearing of an impermeable surface in a uniformly rotating fluid, Meccanica (2016), http://dx.doi.org/10.1007/s11012-016-0507-y, in press.