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Eigensolutions to a vibroacoustic interior coupled problem with a perturbation method

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ABSTRACT

In this paper, an efficient and robust numerical method is proposed to solve non-symmetric eigenvalue problems resulting from the spatial discretization with the finite element method of a vibroacoustic interior problem. The proposed method relies on a perturbation method. Finding the eigenvalues consists in determining zero values of a scalar that depends on angular frequency. Numerical tests show that the proposed method is not sensitive to poorly conditioned matrices resulting from the displacement–pressure formulation. Moreover, the computational times required with this method are lower than those needed with a classical technique such as, for example, the Arnoldi method.

1. Introduction

The study of linear vibrations of fluid–solid interaction (FSI) problems is encountered in many industrial applications. In this paper, the considered FSI problem consists of an elastic solid filled with an inviscid and compressible non-weighting fluid. A variational principle based on a displacement (for the solid)–pressure (for the fluid) formulation is adopted. In this approach, spatially discretized equations lead to a non-symmetric eigenvalue problem. Moreover, the latter can be poorly conditioned, mainly due to large difference of magnitude between the fluid and the solid stiffness and mass matrices [1]. This poor conditioning can harm the convergence of classical eigensolvers and sometimes give inaccurate results. So, in this paper, a very basic numerical method is proposed to solve the non-symmetric and poorly conditioned eigenvalue problem associated with an interior vibroacoustic problem. This numerical method is based on a perturbation method and has been initially proposed in Ref. [2] to solve symmetric eigenvalue problems arising from elastic linear solid framework. The method consists in introducing a right-hand side (rhs) in the initial problem. This rhs is a scalar multiplied by a random vector. The objective is then to find the angular frequency value for which this scalar becomes null. Indeed, this null value indicates a solution to the initial problem (*i.e.* an eigenvalue). To find the null values of this scalar, unknowns of the problem (solid displacement, fluid pressure and the scalar) are searched as asymptotic expansions of the angular frequency. The resolution of a set of linear problems permits to build an analytical expression, whose roots can be numerically computed. From these roots, the eigenvalues of the vibroacoustic problem are determined.

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The present paper is organized as follows. The discretized non-symmetric eigenvalue problem to be solved is introduced in Section 2. Section 3 is devoted to the presentation of the proposed numerical algorithm. In Section 4, numerical tests, a deformable cavity filled with air or water, permit to show efficiency and accuracy of the proposed method.

2. Governing equations

In this study, the problem of an inviscid compressible fluid contained in an elastic solid is considered. Using the finite element method to discretize the variational ⟨u–p⟩ formulation, the following classical matrix system is obtained:

$$\left(\begin{bmatrix} \mathrm{K_s} & -\mathrm{C} \\ 0 & \mathrm{K_f} \end{bmatrix} - \lambda \begin{bmatrix} \mathrm{M_s} & 0 \\ \rho_\mathrm{f}\mathrm{C}^\mathrm{t} & \mathrm{M_f} \end{bmatrix}\right) \begin{Bmatrix} \boldsymbol{u_s} \\ p \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{1}$$

In this equation $\boldsymbol{u_s}$ and p are the solid displacement and the fluid pressure, respectively. The subscripts s and f stand for the solid and the fluid. The matrices K, M and C represent the stiffness matrix, the mass matrix and the coupling matrix due to the fluid–solid interaction. The scalar ρ_f is the fluid density. The eigenvalue λ is the square of the angular frequency (*i.e.* $\lambda = \omega^2$). Formally, the previous equation (1) can be written as the following generalized eigenvalue problem:

$$(\mathrm{K} - \lambda \mathrm{M})\,\boldsymbol{U} = 0 \tag{2}$$

where $\boldsymbol{U}$ is a mixed unknown vector containing the displacement of the solid, $\boldsymbol{u_s}$, and the pressure, p, in the fluid. The previous system is non-symmetric due to the presence of the coupling matrix, C, in the mass and stiffness matrices, respectively M and K. Moreover, as the previous system is poorly conditioned (see [1] for more details), numerical preconditioning is strongly encouraged to ensure convergence and accuracy of the solutions. In this study, with a classical eigensolver, ARPACK [6], a preconditioning technique should be added to find eigensolutions. On the contrary, with the proposed method based on perturbation technique, no preconditioning will be required to find accurate solutions.

3. The proposed method

The numerical method designed to solve the linear problem (2) has been initially proposed to solve eigenvalue problems arising from vibrations of elastic solids [2]. So, to solve the problem (2), the latter is modified by introducing a right-hand side μF where μ is an unknown scalar and F is a random load vector. The problem (2) then becomes:

$$(\mathrm{K} - \lambda \mathrm{M})\,\boldsymbol{U} = \mu F \tag{3}$$

As the number of unknowns is greater than the number of equations, the following orthogonality condition is chosen to be the additional equation:

$$\langle \boldsymbol{U} - \boldsymbol{V}, \boldsymbol{V} \rangle = 0 \tag{4}$$

where $\langle \bullet, \bullet \rangle$ stands for the Euclidian scalar product of two vectors. The vector $\boldsymbol{V}$ is a known vector and will be clarified below.

Hence, according to Eq. (3), determining solutions $(\lambda, \boldsymbol{V})$ that satisfy the initial problem (2) consists in determining the values λ for which the scalar μ is equal to zero. A way to find these null values of μ is to compute the scalar μ for several discrete values of λ. This 'direct' method requires a lot of computations (*i.e.* a lot of matrix triangulations K) and then is not efficient in term of computational times. Instead of this 'direct' method, it is proposed to solve the problem (3) by using a perturbation method, λ being the perturbation parameter. At the end of the computations, an analytical expression of the scalar μ is then obtained. The determination of the value of λ for which the scalar μ is equal to zero should be then straightforward. In order to use the perturbation method, it is suggested to define the eigenvalue λ under the following form:

$$\lambda = \lambda_0 + \hat{\lambda} \tag{5}$$

where λ_0 is the initial value. For the first step of the presented method, λ_0 is chosen equal to zero. The problem (3) can be rewritten as:

$$\left(\mathrm{K} - (\lambda_0 + \hat{\lambda})\mathrm{M}\right) \boldsymbol{U} = \mu F \tag{6}$$

The unknowns (U, μ) are then sought as an integro-power series with respect to the parameter $\hat{\lambda}$:

$$\begin{Bmatrix} \boldsymbol{U} \\ \mu \end{Bmatrix} = \sum_{i=0}^{N} \hat{\lambda}^i \begin{Bmatrix} \boldsymbol{U}_i \\ \mu_i \end{Bmatrix} \tag{7}$$

where N is the truncation order of the asymptotic expansions. Inserting Eq. (7) into equations (4) and (6) and balancing terms with identical powers of $\hat{\lambda}$, a set of linear problems is obtained:

order 0:

$$\begin{cases} (\mathrm{K} - \lambda_0 \mathrm{M})\,\mathrm{U}_0 = \mu_0 F \\ \langle U_0, V \rangle = \langle V, V \rangle \end{cases} \tag{8}$$

order $1 \leq i \leq N$:

$$\begin{cases} (\mathrm{K} - \lambda_0 \mathrm{M})\,\mathrm{U}_i = \mu_i F + \mathrm{M}\mathrm{U}_{i-1} \\ \langle U_i, V \rangle = 0 \end{cases} \tag{9}$$

All linear problems (8) and (9) have the same operator and differ from their right-hand sides. So one single matrix triangulation and $(\mathrm{N}+1)$ backward and forward substitutions are required to compute all the unknowns (U_i, μ_i) of the polynomial approximation (7). At the end of the computation, asymptotic expansions are replaced by equivalent rational approximations, called Padé approximants [3,4],

$$\mathrm{X}_{\text{Padé},\,N}(\hat{\lambda}) - \mathrm{X}_0 = \sum_{k=1}^{N-1} \frac{R_{(N-1-k)}(\hat{\lambda})}{Q_{(N-1)}(\hat{\lambda})} \hat{\lambda}^k \mathrm{X}_k \tag{10}$$

where R_k, Q_k are polynoms of degree k and X is a mixed vector containing the vector V and the scalar μ. This rational representation permits to increase the validity range of the polynomial approximation. At the end of the computations, finding the couples (U, λ) that are solutions to the generalized eigenproblem (3) consists in determining the roots of numerators $R_{(N-1-k)}$ in rational approximations (10). These numerators are defined by the following expression:

$$R_{(J)} = 1 + \sum_{j=1}^{J} \hat{\lambda}^j d_j \text{ for } 0 \leq J \leq N-2 \tag{11}$$

where the scalar coefficients d_j are expressed as:

$$d_1 = -\frac{\alpha_{(N,N-1)}}{\alpha_{(N-1,N-1)}} \text{ and } d_j = -\frac{\alpha_{(N,N-j)}}{\alpha_{(N-j,N-j)}} - \sum_{l=1}^{j-1} \frac{\alpha_{(N-l,N-j)}}{\alpha_{(N-j,N-j)}} d_l \text{ for } 2 \leq j \leq N-1 \tag{12}$$

The coefficients $\alpha_{(I,J)}$ are determined by performing a modified Gram–Schmidt orthonormalization of the initial set of vectors X_N. The denominator $Q_{(N-1)}(\hat{\lambda})$, which is the same as for the rational fraction (Eq. (10)), is computed in the same way as the numerators $R_{(N-1-k)}$ by using expressions (11) and (12).

The roots of the numerators, $R_{(N-1-k)}$ are denoted in the following by (λ_r). Among all these roots, some of them do not lead to a solution to the initial problem (3). To check if a root, λ_r, gives a true and accurate eigenvalue, the following criterion is defined:

$$\frac{\left({}^{\mathrm{t}}\boldsymbol{U}_r K \boldsymbol{U}_r - \lambda_r {}^{\mathrm{t}}\boldsymbol{U}_r M \boldsymbol{U}_r\right)}{\lambda_r} \leq \epsilon \tag{13}$$

where $\boldsymbol{U}_r$ is computed by introducing the root λ_r into Padé's approximant (10). The parameter ϵ is an user parameter that specifies the quality of the numerical solutions. In this study, the value of ϵ is chosen equal to 10^{-8}.

Once all the roots have been carried out for one step (*i.e.* for a given value of λ_0), a new step of the method is done by defining a new value of the scalar λ_0. In this work, the latter is computed in two ways. If during the previous step, a solution to the eigenvalue problem has been found, the root λ_r verifying the criterion (13), a new starting value λ_0 is defined according to the following expression:

$$\lambda_0 = 1.1 * \lambda_r \tag{14}$$

Otherwise, if no eigenvalue has been found, then the range of validity of the Padé approximants is estimated by using a criterion introduced in Ref. [5] and leading to a new initial value λ_0.

Lastly, the method requires an additional condition (4) in which a vector V must be introduced. The first step of the current algorithm permits to define this vector. To get it, λ_0 is considered to be null and the initial value of the scalar μ is arbitrarily chosen equal to one ($\mu = 1$). Then, from equation (3), the vector V is obtained as a solution to:

$$\mathrm{K}\boldsymbol{V} = \boldsymbol{F} \tag{15}$$

4. Numerical results

To prove the efficiency of the proposed method, a steel cavity (linear elastic structure) is filled with an inviscid compressible fluid (air or water). This example is issued from Ref. [7]. The geometric and material characteristics are given in Fig. 1. Finite element discretization uses linear quadrilateral elements both for the fluid and the structure. The solid element

Steel:
$E = 1.44\,10^{11}$ Pa
$\nu = 0.35$
$\rho_s = 7.7\,10^3$ kg·m^{-3}
Properties of the air:
$c_f = 340$ m·s^{-1}
$\rho_f = 1$ kg·m^{-3}
Properties of water
$c_f = 1430$ m·s^{-1}
$\rho_f = 1000$ kg·m^{-3}

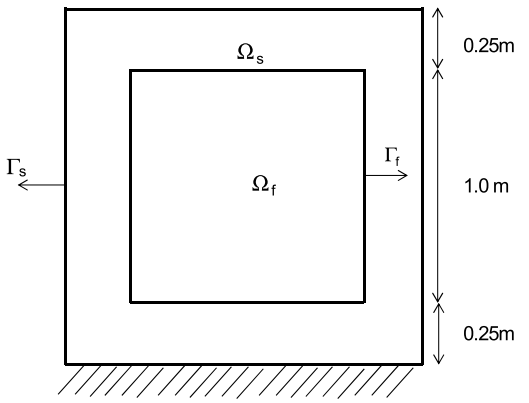
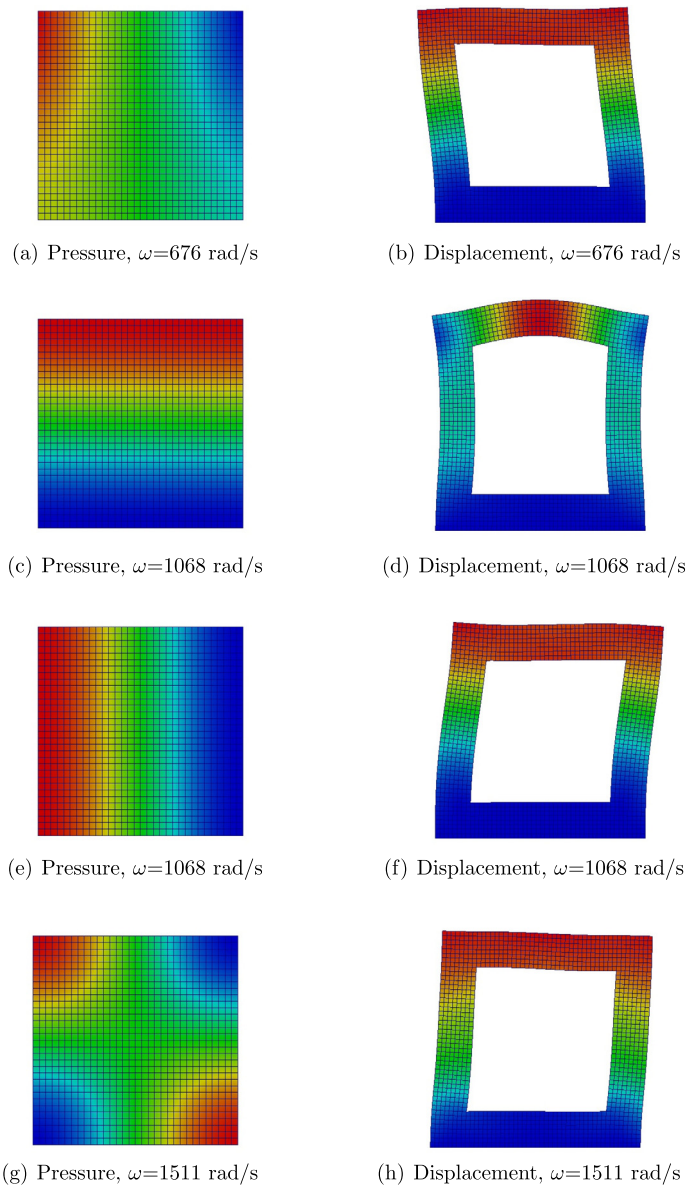


Fig. 1. Geometry and mechanical properties for the steel cavity [7].

Table 1
Comparison of the first ten eigenvalues obtained with the proposed method (PM) and those given in Ref. [7].

Air		Water	
PM	Reference [7]	PM	Reference [7]
676.926	664.121	654.159	641.837
1068.562	1068.129	2159.301	2116.398
1068.607	1068.152	3445.498	3201.475
1511.191	1510.589	3907.321	3804.124
2139.448	2136.102	4221.192	4211.62
2139.707	2136.240	4710.677	4687.927
2304.012	2258.686	5168.735	5155.246
2391.688	2388.418	5454.176	5385.805
2391.734	2388.539	6280.978	6239.332
3026.00	–	7597.443	–

has eight degrees of freedom (dofs) corresponding to two displacements and four nodes. The fluid element has four dofs (pressure at each node of the fluid element). Finally, the interface element has twelve dofs (eight for the solid displacement and four for the fluid pressure). In this study, Fig. 1, the finest mesh is composed of 2304 quadrilateral elements (1280 solid elements and 1024 for the fluid domain) leading to 2401 nodes (1312 solid nodes, 961 fluid nodes, and 128 nodes on the fluid–solid interface). Then the number of dofs for this mesh is close to 4000. The first ten eigenvalues obtained with the proposed method are reported in Table 1 and compared to values given in the work of Bermúdez and Rodríguez [7]. The results in this table show an excellent agreement. In Fig. 2, fluid and solid modes corresponding to the first four eigenfrequencies given in Table 1 are presented. For the proposed method, the results given in Table 1 are obtained with a truncation order, parameter N in Eq. (7), equal to 15 and an accuracy parameter, ϵ in Eq. (13), chosen equal to 10^{-8}. For these parameters, 37 continuation steps are required to compute the first ten eigenfrequencies for a cavity filled with air and 43 steps for a cavity filled with water. In Fig. 3, the evolution of the number of steps is plotted versus the truncation order to get the first ten eigenvalues (Table 1). The truncation order varies between 10 and 35, whereas the accuracy parameter is fixed at 10^{-8}. This plot shows that the number of continuation steps is relatively constant regardless of the truncation order chosen. In fact, the key point of the proposed algorithm is the root computation of the numerator R_k of the Padé approximants (10). So, to get accurate values of these roots, two numerical methods are used [8]. The first one is Bairstow's method and the second one consists in finding the eigenvalues of a matrix whose characteristic polynomial is the numerator of the Padé approximants (10). With these two techniques, accurate roots are found whatever the truncation order varying between 10 and 35. Nevertheless, the truncation order has an influence on the computational time required to obtain eigenvalues. Indeed, a truncation order equal to N requires a single matrix triangulation but $(N+1)$ backward and forward substitutions to compute the $(N+1)$ couples of unknowns (U_i, μ_i). So, the evolution of the computational time needed to get the ten first eigenvalues (given in Table 1) when the truncation order varies between 10 and 35 is plotted in Fig. 4. This plot shows that the lowest computational times is obtained for a truncation order close to 15. The performance of the present algorithm is now compared to a classical numerical method to compute the eigenvalues of a non-symmetric problem. In this study, the Arnoldi method is chosen by using the ARPACK solution [6]. Unfortunately, ARPACK is unable to compute the eigenvalues of the problem (2). Indeed, this problem is poorly conditioned due to the fact that components of matrices $\mathrm{M_s}$ and $\rho_f \mathrm{C^t}$ are very large compared to entries in matrix $\mathrm{M_f}$ [1]. So, the generalized eigenvalue problem



(a) Pressure, ω=676 rad/s (b) Displacement, ω=676 rad/s

(c) Pressure, ω=1068 rad/s (d) Displacement, ω=1068 rad/s

(e) Pressure, ω=1068 rad/s (f) Displacement, ω=1068 rad/s

(g) Pressure, ω=1511 rad/s (h) Displacement, ω=1511 rad/s

Fig. 2. First four modes of the coupled system, fluid (air) pressure mode (a, c, e and g) and solid displacement mode (b, d, f and h).

(Eq. (1)) needs to be preconditioned to give fast and accurate eigenvalues. According to the analysis reported in Ref. [1], the preconditioned generalized eigenvalue problem to be solved with ARPACK is rewritten:

$$\left(\begin{bmatrix} \mathrm{K_s} & -\mathrm{C}_k \\ 0 & \hat{\mathrm{K}}_\mathrm{f} \end{bmatrix} - \lambda \begin{bmatrix} \mathrm{M_s} & 0 \\ \rho_\mathrm{f}\mathrm{C}_m^\mathrm{t} & \hat{\mathrm{M}}_\mathrm{f} \end{bmatrix}\right) \begin{Bmatrix} \boldsymbol{u_s} \\ \hat{p} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{16}$$

with the following definitions:

$$\begin{cases} \hat{\mathrm{K}}_\mathrm{f} = a\,b\,\mathrm{K_f} \quad \text{and} \quad \hat{\mathrm{M}}_\mathrm{f} = a\,b\,\mathrm{M_f} \\ \mathrm{C}_k = a\,\mathrm{C} \quad \text{and} \quad \mathrm{C}_m = b\,\mathrm{C} \\ \hat{\mathrm{p}} = \frac{1}{a}\,\mathrm{p} \\ a = \sqrt{\frac{\|\mathrm{K_s}\|_\mathrm{F}}{\|\mathrm{M_f}\|_\mathrm{F}}} \quad \text{and} \quad b = \sqrt{\frac{\|\mathrm{M_s}\|_\mathrm{F}}{\|\mathrm{K_f}\|_\mathrm{F}}} \end{cases} \tag{17}$$

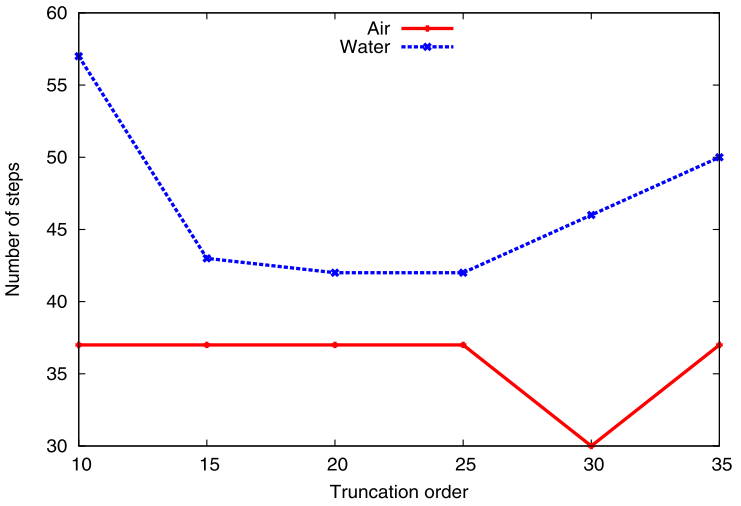


Fig. 3. Number of continuation steps of the proposed method versus truncation order to get the first ten eigenvalues shown in Table 1. Elastic cavity filled with air or water, 4000 dof.

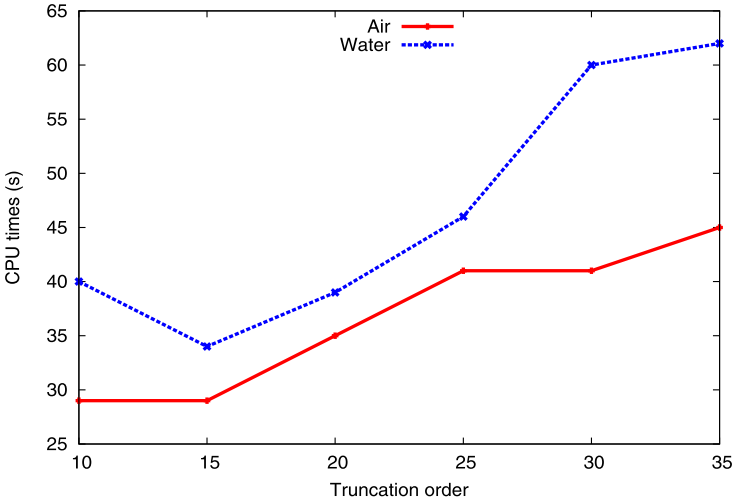


Fig. 4. Evolution of the CPU times (s) versus truncation order to get the first ten eigenvalues shown in Table 1. Elastic cavity filled with air or water, 4000 dof. Accuracy parameter ϵ is equal to 10^{-8}.

where the symbol $\|\mathrm{K}\|_{\mathrm{F}}$ stands for the Frobenius norm of the matrix K, defined by $\|\mathrm{K}\|_{\mathrm{F}} = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}|K_{ij}|}$. A key feature of this study is that the proposed method gives accurate eigenvalues with or without preconditioned matrices. So, in Fig. 5, the computational times required to get the first ten eigenvalues of the numerical tests (Table 1) are plotted versus the number of unknowns (from 500 to 4000). These times are given for both numerical methods (ARPACK and the proposed numerical method). This figure shows that the proposed algorithm needs computational times lower than those required with ARPACK whatever the size of the problem. Hence, with the finest mesh and in the case of a cavity filled with air, the proposed algorithm demands a computational time four times lower than the ARPACK computational time.

5. Conclusion

In this paper, it is proposed to use a perturbation method to compute the eigenvalues of a vibroacoustic interior coupled problem. Eigenvalues are determined in analyzing the roots of the numerator of a fraction (Padé approximants). The proposed method is easy to implement in a computational software because it only requires a linear solver and a subroutine to realize matrix–vector products. This algorithm is efficient even if two eigenvalues are very close to each other, which is the case when considering an elastic cavity filled with air. The proposed numerical method does not require the use of a preconditioning technique, contrary to a classical eigensolver such as the Arnoldi method. The optimal truncation order for the polynomial approximation is close to 15 according to the analysis carried out in this study. The computational times needed

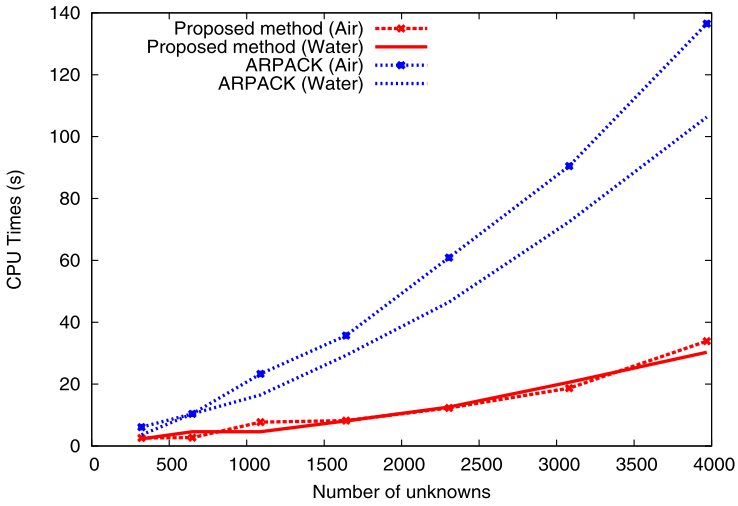


Fig. 5. Comparison of CPU times obtained with ARPACK and the proposed method to get the ten first eigenvalues (Table 1). Elastic cavity filled with air or water [7]. Parameters used for the proposed method are $N = 15$ and $\epsilon = 10^{-8}$.

with the proposed numerical method are lower than the ones required with the classical ARPACK solver. Nevertheless, the number of steps to get ten eigenvalues is relatively important (approximately 40 for the considered examples), leading to a great number of matrix triangulations. So, for a large-scale problem, involving several thousands of dofs, the presented algorithm can require too large computational times. Future works concern the use of reduced order models based on the perturbation method (recently proposed to analyze fluid bifurcation problems [9] and vibrations of plates [10]) to decrease computational times. Application of this kind of method to analyze vibrations of damped structures [11,12] are also in progress.

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