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Comments on ‘Large deflection and rotation of Timoshenko beams with frictional end supports under three-point bending’ [Comptes rendus Mecanique 344 (8) (2016) 556–568]

Milan Batista

University of Ljubljana, Faculty of Maritime Studies and Transport, Slovenia

A R T I C L E I N F O

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The paper discusses certain limitations of the analytical solution presented in this article [1]. It is demonstrated that the solution is valid only for small deflections of an inextensible beam.

1. Discussion

In the article [1], the authors present an analytical solution for the large deflection and rotation problem of the Timoshenko-type beam under three-point bending. The first question that arises is how do the authors successfully obtain the solution in terms of elementary functions—[1], Eqs. (26) and (39)—when dealing with large deflections and rotations? All known analytical solutions of large deflections of beams, whether shear deformable or not, include elliptical functions in some form [2–8]. When shear is neglected, the author's solution should lead to the well-known elastica solution, but it does not. Also their solution does not predict the beam slips between supports. This is seen from the graphs of midspan deflection versus the normalised applied loading in Figs. 5 and 7. In these graphs, the deflection increases at least to a factor of 2.5 times the span between the supports, while for elastica it increases only to approximately $0.8 \times \ell$ (ℓ is beam length between supports) before it slips between the supports [7]. In the case of Reissner large-deflection finite-strain beam, this increase is up to approximately $1.2 \times \ell$, before the beam slips between the supports [6]. This indicates that the solution involves some approximation and cannot be valid for large deflections. To understand this phenomenon, we reviewed the authors' derivation of the governing equations and then considered some numerical comparisons. In the following text, we mostly used the authors' notation, and for simplifying the problem, we will consider frictionless supports.

The authors' start with the well-known assumption of the components of displacement.

$$u^* = u - z\sin\theta \qquad w^* = w - z(1-\cos\theta) \tag{1}$$

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E-mail address: milan.batista@fpp.uni-lj.si.

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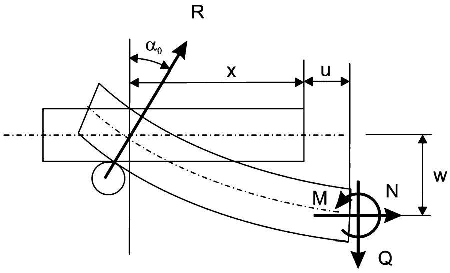

Fig. 1. Beam equilibrium.

where $u = u(x)$ and $w = w(x)$ are the displacement components of the beam base curve and $\theta = \theta(x)$ is the angle of rotation of the beam cross section. (The authors term the beam base curve as the neutral axis, which is a bit inappropriate since they assume $u \neq 0$.) Then, they tacitly assume that the infinitesimal strain tensor is valid. Hence, by definition, they obtain the following—[1], Eqs. (3) and (4):

$$\varepsilon_{xx} = \frac{\partial u^*}{\partial x} = \frac{\mathrm{d}u}{\mathrm{d}x} - z\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}x} \tag{2}$$

$$\gamma_{xz} = \frac{\partial w^*}{\partial x} + \frac{\partial u^*}{\partial z} = \frac{\mathrm{d}w}{\mathrm{d}x} - \left(1 + z\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)\sin\theta \tag{3}$$

By using Hook's law $\sigma_x = E\varepsilon_x$ and $\tau_{xz} = G\gamma_{xz}$, where E and G are the Young's modulus and shear modulus, respectively, they calculate the stress resultants as follows—[1], Eqs. (8)–(13):

$$N = \int_\Omega \sigma_x \mathrm{d}A = EA\frac{\mathrm{d}u}{\mathrm{d}x} \tag{4}$$

$$Q = \int_\Omega \tau_{xz}\mathrm{d}A = kGA\left(\frac{\mathrm{d}w}{\mathrm{d}x} - \sin\theta\right) \tag{5}$$

$$M = \int_\Omega z\sigma_x \mathrm{d}A = -EI\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}x} \tag{6}$$

where, according to the authors, N, Q, and M are the axial force, shear force, and bending moment, respectively; A is the cross-sectional area; I is the moment of inertia; and k is the shear correction coefficient.

From the equilibrium of the beam, the reaction forces R at the supports are $R = \frac{P}{\cos\alpha_0}$, where P is half load, and α_0 is angle of deflection at the supports. Then, for equilibrium at an arbitrary cross section, we found the following relations—[1], Eqs. (14)–(20) (Fig. 1):

$$N = -R\sin\alpha_0 = -P\tan\alpha_0 \tag{7}$$

$$Q = R\cos\alpha_0 = P \tag{8}$$

$$M = Q(x+u) + Nw = P(x + u + w\tan\alpha_0) \tag{9}$$

We noted that in the moment equation ([1], Eq. (23)), the authors tacitly assume $u = 0$ and also omit the negative sign in Eqs. (15) and (18) for N. Using Eqs. (4), (5), and (6), the set of differential equations ([1], Eqs. (21)–(23)) are obtained:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{P}{EA}\tan\alpha_0 \tag{10}$$

$$\frac{\mathrm{d}w}{\mathrm{d}x} - \sin\theta = \frac{P}{GA} \tag{11}$$

$$-\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{P}{EI}(x + u + w\tan\alpha_0) \tag{12}$$

This system, as the authors show, has a closed form solution—[1], Eqs. (26) and (39).

Now let us analyse this derivation. First, we assume the infinitesimal strain tensor. Hence, we must have the following:

$$\varepsilon_{xx} = O(\delta) \quad \text{and} \quad \gamma_{xz} = O(\delta) \tag{13}$$

where δ is small. This can be achieved if in Eqs. (2) and (3), we have the following:

$$\left|\frac{\mathrm{d}u}{\mathrm{d}x}\right| \le \delta \qquad \left|\frac{\mathrm{d}w}{\mathrm{d}x}\right| \le \delta \qquad \left|\frac{\mathrm{d}\theta}{\mathrm{d}x}\right| \le \delta \qquad |\sin\theta| \le \delta \tag{14}$$

Thus, $\sin\theta$ should be small, and, therefore, θ should be small. Hence, the following hold.

$$\sin\theta \approx \theta \quad \text{and} \quad \cos\theta \approx 1 \tag{15}$$

Here, we note that the derivation of equilibrium equations (7), (8), and (9) stipulates that forces N and Q are in fact horizontal and vertical, respectively, to the deformed cross section. They become axial and shear forces only if we assume the infinitesimal strain tensor.

By approximation (15), Eqs. (10), (11), and (12) take the following forms:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{P}{EA}\alpha_0 \tag{16}$$

$$\frac{\mathrm{d}w}{\mathrm{d}x} - \theta = \frac{P}{kGA} \tag{17}$$

$$-\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{P}{EI}(x + u + w\alpha_0) \tag{18}$$

The boundary conditions of the problem are as follows:

$$w(0) = 0 \quad \text{and} \quad M(0) = 0 \tag{19}$$

In addition, we consider two symmetry conditions:

$$u(L) = 0 \tag{20}$$

$$\theta(L) = 0 \tag{21}$$

where L is the beam half span. The condition that reaction force is perpendicular to the deformed base curve is also considered.

$$\frac{\mathrm{d}w}{\mathrm{d}x}(0) = \tan\alpha_0 \approx \alpha_0 \tag{22}$$

The integration of Eq. (16) under condition (20) yields the following expression:

$$u = \frac{P}{EA}\alpha_0(L - x) \tag{23}$$

From (17) we obtain the relation given below:

$$\theta = \frac{\mathrm{d}w}{\mathrm{d}x} - \frac{P}{kGA} \tag{24}$$

Substituting this into (18) and considering (23) yields the following differential equation:

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + \frac{P\alpha_0}{EI} w = -\frac{P}{EI}\left(x + \frac{P}{EA}\alpha_0(L - x)\right) \tag{25}$$

The solution of this equation, subject to boundary conditions $(19)_1$ and (21), is as follows:

$$\frac{w}{L} = \frac{1}{\alpha_0}\left[\left(\frac{1+\omega^2\psi}{\omega\cos\omega}\right)\sin\frac{\omega x}{L} - \frac{x}{L}\right] + \frac{P}{EA}\left[\cos\frac{\omega x}{L} - 1 + x + \sin\frac{\omega x}{L}\left(\tan\omega - \frac{1}{\omega\cos\omega}\right)\right] \tag{26}$$

where

$$\omega^2 \equiv p\alpha_0, \qquad p \equiv \frac{PL^2}{EI} \quad \text{and} \quad \psi \equiv \frac{EI}{kGAL^2} \tag{27}$$

If we assume that $\omega^2 > 0$, then the moment boundary condition $(19)_2$ can be fulfilled only if $1/EA = 0$, that is, if the beam is inextensible. This means that the final solution of the problem is given as follows:

$$u = 0 \tag{28}$$

$$\frac{w}{L} = \frac{1}{\alpha_0}\left[\left(\frac{1+\omega^2\psi}{\omega\cos\omega}\right)\sin\frac{\omega x}{L} - \frac{x}{L}\right] \tag{29}$$

We note that the authors did not discuss the moment boundary condition, since as stated earlier, they start tacitly with $u = 0$, and therefore, the moment condition is automatically satisfied. However, they assume the validity of Eq. (23)—[1], Eq. (26)—without any restriction. Using condition (22) and solution (29) yields the relation between α_0 and p:

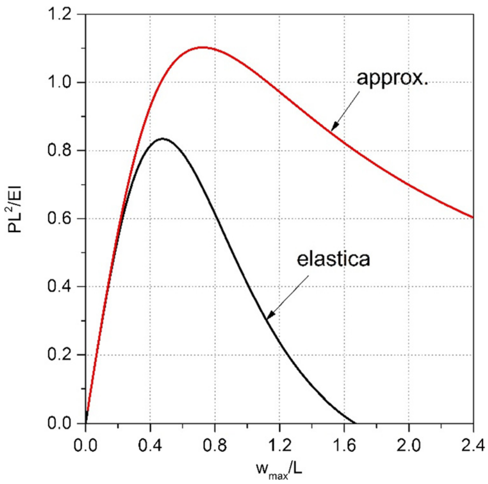

Fig. 2. Comparisons of the present solution with the elastic solution.

$$\cos\omega - \frac{1+\psi\omega^2}{1+\alpha_0^2} = 0 \tag{30}$$

From this relation, we can calculate α_0 or p. At $x = L$, the deflection is maximum. Using (26) and (33), we find that the following relation holds:

$$\frac{w_{\max}}{L} = \frac{1}{\alpha_0}\left[\frac{\tan\omega}{\omega}\left(1+\psi\omega^2\right) - 1\right] \tag{31}$$

Before we compare the present solution with the one given by the authors, we note that their Eq. (41) does not derive from their Eq. (40) and from the condition $\frac{\mathrm{d}w}{\mathrm{d}x}(0) = \tan\alpha_0$, and, therefore, Eq. (44)—and also Eq. (51), but not (52)—is wrong. The correct Eq. (44), which follows from Eq. (40) and the condition $\frac{\mathrm{d}w}{\mathrm{d}x}(0) = \tan\alpha_0$, is as follows:

$$\cos\sqrt{ps} = \left[\frac{1+s^2}{(1-\mu s)(1+\psi ps)}\right]^{-1} \tag{32}$$

where $s \equiv \tan\alpha_0$, and μ is coefficient of the friction.

Now, comparison of solutions (29)–(31) with the author's solution given by their Eqs. (39) and (53) and the present Eq. (32), with $\mu = 0$, shows that the solutions formally coincide. The only difference is the parameterisation: in the present solution, the parameter is α_0, while in the authors' solution, it is s_0. This implies that, for given p and ψ, both solutions—the present, which correctly assumes small θ, and the authors' solution, which assumes unlimited θ—yield the same deflection.

Now we will present some numerical comparisons to determine the limitations of the author's solution. For simplicity, we limit ourselves only to case $\psi = 0$ and compare the present beam model with Euler's elastic, i.e. inextensible shear-less large deflection beam. In this case, the load factor and maximal deflection are obtained as follows by using (30) and (31):

$$p \equiv \frac{PL^2}{EI} = \frac{1}{\alpha_0}\arccos^2\left(\frac{1}{1+\alpha_0^2}\right) \tag{33}$$

$$\frac{w_{\max}}{L} = \frac{\sqrt{2+\alpha_0^2}}{\arccos\left(\frac{1}{1+\alpha_0^2}\right)} - \frac{1}{\alpha_0} \tag{34}$$

This load factor given by (33) has an extreme point where $p_{\max} \approx 1.102532$ and $w_{\max}/L \approx 0.724334$. For the elastic solution, at the extreme point we have $p_{\max}^{(e)} \approx 0.833976$ and $w_{\max}^{(e)}/L \approx 0.476378$ [3,7] (see Fig. 2). Thus, the present solution yields a value of p that is 32% higher than that for the elastica solution. The results of the calculations presented in Tables 1 and 2 show that the present solution (and therefore, also the authors' solution) is valid for deflections up to approximately $0.2 \times L$. We can thus conclude that the authors' solution is valid only for small beam deflections of inextensible beam.

Finally, we would like to note that the authors have mismatched the reference and deformed beam configurations. For example, they defined the curvature of the beam base curve in their Eq. (5). If this is so, then in the formula, $\mathrm{d}s$ is the arc length of the deformed base curve: i.e. $\mathrm{d}s = \sqrt{(\mathrm{d}x + \mathrm{d}u)^2 + \mathrm{d}w^2}$. Further, θ should be the angle of the tangent to the deformed base curve [9]. However, θ is, by assumption, the rotation angle of the cross section, and $\mathrm{d}s = \sqrt{(\mathrm{d}x + \mathrm{d}u)^2 + \mathrm{d}w^2} \neq \mathrm{d}x/\cos\theta$. Hence, their Eq. (5) does not define the curvature of the deformed base curve. As

Table 1
Comparison of the results for the maximum deflection w_{max}/L for the given load factor $p = PL^2/EI$ (diff is the relative difference in %).

$p = PL^2/EI$	w_{max}/L			diff	
	[1]	Present	Elastica	Present	Elastica
0.25	0.0844	0.084402	0.084887	0.00	−0.57
0.5	0.1760	0.175984	0.181147	0.01	−2.85
0.75	0.2874	0.287445	0.324168	−0.02	−11.33
1	0.4653	0.465255	–	0.01	–
1.1025	0.7189	0.718924	–	0.00	–

Table 2
Comparison of results for the load factor $p = PL^2/EI$ for the given maximum deflections w_{max}/L (diff is the relative error in %).

w_{max}/L	[1]	Present	diff	Elastica	diff
0.1	0.296	0.294690	0.44	0.292414	1.23
0.2	0.560	0.559564	0.08	0.542208	3.28
0.3	0.774	0.773701	0.04	0.719593	7.56
0.4	0.929	0.929092	−0.01	0.813989	14.13

we have shown, this mismatching of the configurations was also observed in their derivation of the beam equations. The correct derivation of beam equations for large deflections can be found in [10,11] and in the references therein.

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