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Heterogeneous linearly piezoelectric patches bonded on a linearly elastic body

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ABSTRACT

In [1], we studied the response of a thin homogeneous piezoelectric patch perfectly bonded to an elastic body. Here we extend this study to the case of a very thin heterogeneous patch made of a periodic distribution of piezoelectric inclusions embedded in a linearly elastic matrix and perfectly bonded to an elastic body. Through a rigorous mathematical analysis, we show that various asymptotic models arise, depending on the electromechanical loading together with the relative behavior between the thickness of the patch and the size of the piezoelectric inclusions.

1. Introduction

As the essential technological interest of piezoelectric patches is the monitoring of a deformable body they are bonded to, here, in the same spirit as [1], we intend to propose various asymptotic models for the behavior of the body through the study of the system constituted by a very thin patch made of a periodic distribution of piezoelectric inclusions embedded in a linearly elastic matrix (studied in [2,3]) perfectly bonded to a linearly elastic three-dimensional body.

A reference configuration for the linearly elastic body is a domain Ω of $\mathbb{R}^3$ assimilated to the Euclidean physical space with basis $\{e_1, e_2, e_3\}$. The set Ω lies in $\{x_3 < 0\}$ and a part of its Lipschitz-continuous boundary $\partial\Omega$ is a nonempty domain S of $\{x_3 = 0\}$ such that $S \times (-L, 0)$ is included in Ω for some positive real number L. The patch perfectly bonded to the body occupies $B^h = S \times (0, h)$ and is made of a periodic distribution of linearly piezoelectric inclusions perfectly bonded to a linearly elastic and electrically insulated matrix. More precisely, let $Y = (0,1)^2$ and Y^* be a subdomain strongly included in Y and $I_\varepsilon = \left\{ i \in \mathbb{Z}^2; \varepsilon(i + Y) \subset S \right\}$ then if $s = (\varepsilon, h)$ is the pertinent couple of geometrical parameters of the patch, the piezoelectric inclusions occupy $B_I^s = S_I^\varepsilon \times (0, h)$, $S_I^\varepsilon = \cup_{i \in I_\varepsilon} \varepsilon(i + Y^*)$, while the elastic matrix occupies the remaining part of B^h. Let $\mathcal{O}^h := \Omega \cup S \cup B^h$. The body is clamped on a part Γ_0 of $\partial\Omega \setminus S$ with a positive two-dimensional Hausdorff measure $\mathcal{H}_2(\Gamma_0)$ and subjected to body forces and surface forces on $\Gamma_1 := \partial\Omega \setminus (S \cup \Gamma_0)$ of densities f and $\mathcal{F}$. The whole patch is free of mechanical loading and no electric charges lay in the inclusions. We define $S^\delta := S + \delta e_3, \forall \delta \in \mathbb{R}$ and, if u^s, $e(u^s)$ and σ^s denote the fields of displacement, strain and stress in $\mathcal{O}^h$ and φ^s, D^s stand for the electric potential and the electric displacement, then part of the equations describing the electromechanical equilibriums read as:

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$$\begin{cases} \operatorname{div} \sigma^s = \tilde{f} \text{ in } \mathcal{O}^h \quad u^s = 0 \text{ on } \Gamma_0 \quad \sigma^s n = \mathcal{F} \text{ on } \Gamma_1 \quad \sigma^s n = 0 \text{ on } S^h \cup \partial S \times (0,h) \\ \operatorname{div} D^s = 0 \text{ in } B_I^s \quad D^s n = 0 \text{ on } \partial S_I^\varepsilon \times (0,h) \\ \sigma^s = a e(u^s) \text{ in } \Omega \quad (\sigma^s, D^s) = \frac{1}{h} M^\varepsilon (e(u^s), \nabla \varphi^s) \text{ in } B^h \end{cases} \tag{1}$$

$\tilde{f}$ is the extension of f to B^h by 0, n is the unit outward normal, a denotes the elasticity tensor, which satisfies:

$$a \in L^\infty(\Omega; \operatorname{Lin}(\mathbb{S}^3)),\ \exists c > 0;\ c|e|^2 \le a(x)e \cdot e \quad \forall e \in \mathbb{S}^3 \text{ a.e. } x \in \Omega \tag{2}$$

where $\operatorname{Lin}(\mathbb{S}^N)$ is the space of linear operators on the space $\mathbb{S}^N$ of $N \times N$ symmetric matrices where the inner product and the norm are noted $\cdot$ and $|\cdot|$ as for $\mathbb{R}^3$. If $\mathbb{K} := \mathbb{S}^3 \times \mathbb{R}^3$ is equipped with an inner product and a norm also denoted as previously and $\mathcal{X}_{Y^*}$ and $L^\infty_\#(Y; \operatorname{Lin}(\mathbb{K}))$ denoting the characteristic function of Y^* and the space of Y periodic elements of $L^\infty(\mathbb{R}^2; \operatorname{Lin}(\mathbb{K}))$, respectively, then M^ε is defined by:

$$M^\varepsilon(x) = M(\hat{x}/\varepsilon)\ \forall x \in B^h \quad M \in L^\infty_\#(Y; \operatorname{Lin}(\mathbb{K})) \text{ s.t. } M = \begin{bmatrix} \alpha & -\mathcal{X}_{Y^*}\beta \\ \mathcal{X}_{Y^*}\beta^T & \mathcal{X}_{Y^*}\gamma \end{bmatrix} \tag{3}$$

with

$$\exists \kappa > 0 \text{ s.t. } \kappa |k|^2 \le M(y)k \cdot k \text{ a.e. } y \in Y,\ \forall k \in \mathbb{K} \tag{4}$$

and, in all the sequel, for any function w of $L^2(Y^*; \mathbb{R}^N)$, $\mathcal{X}_{Y^*}w$ is the function defined by $\mathcal{X}_{Y^*}w = 0$ in $Y \setminus Y^*$, $\mathcal{X}_{Y^*}w = w$ in Y^*.

The models indexed by $p = (\hat{p}, p_3)$, $\hat{p} = (p_1, p_2) \in \{1, 2\}^2$, $p_3 \in \{1, 2, 3\}$ will be distinguished according to the additional necessary boundary conditions on S_I^ε and $S_I^s := S_I^\varepsilon + he_3$. The case $p_1 = 1$ corresponds to a condition for the electric displacement on S_I^s,

$$D^s n = q^s \quad \text{on } S_I^s, \tag{5$_1$}$$

q^s being a density of charges, while $p_1 = 2$ corresponds to a condition of a given electrical potential:

$$\varphi^s = \varphi_0^s \quad \text{on } S_I^s \tag{5$_2$}$$

roughly speaking $p_1 = 1$ deals with patches used as sensors, whereas $p_1 = 2$ concerns actuators (see [4,5]). The index p_2 accounts for the states of the interface between the piezoelectric inclusions and the body, $p_2 = 1$ corresponds to an electrically impermeable interface, $p_2 = 2$ corresponds to a grounded interface:

$$D^s n = 0 \quad \text{on } S_I^\varepsilon \tag{6$_1$}$$

$$\varphi^s = 0 \quad \text{on } S_I^\varepsilon. \tag{6$_2$}$$

Finally, we assume that s takes a value in a countable set with a sole cluster point such that $p_3 = 1, 2, 3$ if and only if $\lim_{s \to 0}(h/\varepsilon) = 0, 1, +\infty$, respectively.

It will be convenient to use the following notations

$$\begin{cases} \hat{k} = (\hat{e}, \hat{g}) \quad \hat{e} = (e_{\alpha\beta})_{\alpha,\beta \in \{1,2\}} \quad \hat{g} = (g_1, g_2) \quad \forall k = (e, g) \in \mathbb{K} \\ k(r) = k(v, \psi) := (e(v), \nabla \psi) \quad \forall r = (v, \psi) \in H^1(B_I^s; \mathbb{R}^3) \times H^1(B_I^s) \end{cases} \tag{7}$$

An electromechanical state will be an element $r = (v, \psi)$ of

$$V_{\hat{p}} := H^1_{\Gamma_0}(\mathcal{O}^h; \mathbb{R}^3) \times \Psi_{\hat{p}}, \quad \Psi_{(1,1)} = H^1_{m_s}(B_I^s),\ \Psi_{(1,2)} = H^1_{S_I^\varepsilon}(B_I^s),\ \Psi_{(2,1)} = H^1_{S_I^s}(B_I^s),\ \Psi_{(2,2)} = H^1_{S_I^\varepsilon \cup S_I^s}(B_I^s) \tag{8}$$

where for any domain $\mathcal{G}$ of $\mathbb{R}^3$, $H^1_\Gamma(\mathcal{G}; \mathbb{R}^3)$ and $H^1_\Gamma(\mathcal{G})$ respectively denote the subspaces of $H^1(\mathcal{G}; \mathbb{R}^3)$ and $H^1(\mathcal{G})$ of all elements with vanishing traces on a part Γ of $\partial\mathcal{G}$, while $H^1_{m_s}(B_I^s) := \left\{ \varphi \in H^1(B_I^s);\ \int_{\varepsilon(i+Y^*)} \varphi(x)\, \mathrm{d}\hat{x} = 0\ \forall i \in I_\varepsilon \right\}$.

One makes the following assumptions on the data:

$$\begin{cases} \varphi_0^\varepsilon \text{ denotes the restriction to } S \text{ of an element of } H^{1/2}(\{x_3 = 0\}) \text{ still denoted by } \varphi_0^\varepsilon \\ (f, \mathcal{F}, q^\varepsilon) \in L^2(\Omega; \mathbb{R}^3) \times L^2(\Gamma_1; \mathbb{R}^3) \times L^2(S),\ \int_{\varepsilon(i+Y^*)} q^\varepsilon(\hat{x})\, \mathrm{d}\hat{x} = 0\ \forall i \in I_\varepsilon \text{ when } \hat{p} = (1,1) \\ q^s(x + he_3) = q^\varepsilon(x),\ \varphi_0^s(x + he_3) = h\varphi_0^\varepsilon(\hat{x}) \text{ a.e. } x \in S_I^\varepsilon \end{cases} \tag{H$_1$}$$

It is well known that for all φ_0^ε in $H^{1/2}(\{x_3 = 0\})$, there exists an element of $H^1(S \times (-L, 0))$ when $p_2 = 1$, of $H^1_{S-L}(S \times (-L, 0))$ when $p_2 = 2$, still denoted by φ_0^ε, whose trace on S is φ_0^ε. Hence the element φ_{0p}^s of $\Psi_{\hat{p}}$ defined by $\varphi_{0p}^s(x) = h\varphi_0^\varepsilon(\hat{x}, (x_3 - h)L/h)$ satisfies

$$\varphi_{0p}^s = \varphi_0^s \text{ on } S_I^s, \quad \frac{1}{h} \int_{B_I^s} |\nabla \varphi_{0p}^s|^2\, \mathrm{d}x \le C \tag{9}$$

Thus, classically, seeking an equilibrium state $\mathfrak{s}^s_p$ leads to the problem

$$(\mathcal{P}^s_p): \quad \text{find } \mathfrak{s}^s_p = (u^s_p, \varphi^s_p) \text{ in } V_{\hat{p}} + (p_1 - 1)(0, \varphi^s_{0p}) \text{ such that } \mathcal{M}^s_p(\mathfrak{s}^s_p, r) = \mathcal{L}^s_p(r) \quad \forall r \in V_{\hat{p}}$$

with

$$\mathcal{M}^s_p(\mathfrak{s}, r) := \int_{\Omega} ae(u) \cdot e(v)\,\mathrm{d}x + \frac{1}{h}\int_{B^h} M^\varepsilon k(\mathfrak{s}) \cdot k(r)\,\mathrm{d}x \quad \forall \mathfrak{s} = (u, \varphi),\ \forall r = (v, \psi) \in H^1(\mathcal{O}^h; \mathbb{R}^3) \times H^1(B^s_I) \tag{10}$$

$$\mathcal{L}^s_p(r) := \int_{\Omega} f \cdot v\,\mathrm{d}x + \int_{\Gamma_1} \mathcal{F} \cdot v\,\mathrm{d}\mathcal{H}_2 + (2 - p_1)\int_{S^s_I} q^s \cdot \psi\,\mathrm{d}\hat{x} \quad \forall r = (v, \psi) \in V_{\hat{p}} \tag{11}$$

which, by Stampacchia's theorem, has a unique solution.

2. The asymptotic models

We make the following additional assumption (H$_2$) and proceed in the spirit of [1], but with two-scale convergence techniques instead of weak-convergence ones. Let us recall (see [6,7]) that a sequence of function w_ε in $L^2(S)$ is said to two-scale converge to a limit w_0 belonging to $L^2(S \times Y)$ if, for any ψ in $C^\infty_0(S; C^\infty_\#(Y))$, we have

$$\lim_{\varepsilon \to 0} \int_S w_\varepsilon(\hat{x})\psi(\hat{x}, \hat{x}/\varepsilon)\,\mathrm{d}\hat{x} = \int_S \int_Y w_0(\hat{x}, y)\psi(\hat{x}, y)\,\mathrm{d}\hat{x}\,\mathrm{d}y, \tag{12}$$

$C^\infty_\#(Y)$ being the set of infinitely differentiable functions in $\mathbb{R}^2$ that are periodic of period Y, while $H^1_\#(Y)$ and $H^1_\#(Y; \mathbb{R}^2)$ are the subsets of $H^1(Y)$ and $H^1(Y; \mathbb{R}^2)$ made of all elements whose traces on the opposite sides of Y are equal. So, (H$_2$) reads as:

$$\mathcal{X}_{S^\varepsilon_I}\varphi^\varepsilon_0 \text{ two-scale converges toward } \mathcal{X}_{Y*}\varphi_0 \text{ in } L^2(S \times Y) \text{ with } \varphi_0 \text{ independent of the "fast" variable } y \text{ running in } Y \tag{H$_2$}$$

Note that this can be obtained by the strong convergence in $L^2(S)$ of φ^ε_0 or by a physically realistic assumption of metallization for the upper face of the piezoelectric inclusions (see [2,3]).

In the following, for any w in $L^2(B^s_I; \mathbb{R}^N)$ we denote its extension by zero in $B^h \setminus B^s_I$ by $\tilde{w}$.

Step 1: Convergences

By arguing as in [1], we have:

$$\exists c > 0; \int_\Omega |e(u^s_p)|^2\,\mathrm{d}x + \frac{1}{h}\int_{B^h} (|e(u^s_p)|^2 + |\widetilde{\nabla\varphi^s_p}|^2)\,\mathrm{d}x \le C \tag{13}$$

which implies that

$$k(h, s^s_p) := (e(h, u^s_p), g(h, \varphi^s_p)) := \frac{1}{h}\int_0^h (e(u^s_p), \widetilde{\nabla\varphi^s_p})(\cdot, x_3)\,\mathrm{d}x_3 \tag{14}$$

satisfies

$$|k(h, s^s_p)|_{L^2(S;\mathbb{K})} \le C \tag{15}$$

thus (cf. [6,7]) there exists a not relabeled subsequence such that $k(h, s^s_p)$ will two-scale converge toward some $\bar{k}_p = (\bar{e}_p, \bar{g}_p)$ in $L^2(S \times Y; \mathbb{K})$ with $(\overline{g_p})_3 = \mathcal{X}_{Y*}\varphi_0$ when $\hat{p} = (2, 2)$. In the case $p_3 = 3$, all the components of $\bar{k}_p$ may be identified. When $p_3 = 1$ only some of them are clearly identified, hence, in this case, we introduce the following decomposition of $\mathbb{K}$:

$$\mathbb{K} = {}^1\mathbb{K}_p \oplus {}^2\mathbb{K}_p \oplus {}^3\mathbb{K}_p \tag{16}$$

- when $\hat{p} = (1, 1)$:

$${}^1\mathbb{K}_p = \{k = (e, g) \in \mathbb{K}; e_{i3} = 0, g_3 = 0\},\ {}^2\mathbb{K}_p = \left\{k = (e, g) \in \mathbb{K}; \hat{e} = 0, g_3 = 0\right\},\ {}^3\mathbb{K}_p = \{0\}$$

- when $\hat{p} = (1, 2)$ or $(2, 1)$:

$${}^1\mathbb{K}_p = \{k = (e, g) \in \mathbb{K}; e_{i3} = 0, g = 0\},\ {}^2\mathbb{K}_p = \left\{k = (e, g) \in \mathbb{K}; \hat{e} = 0, \hat{g} = 0\right\},$$
$${}^3\mathbb{K}_p = \{k = (e, g) \in \mathbb{K}; e = 0, g_3 = 0\}$$

- when $\hat{p} = (2,2)$:

$$^1\mathbb{K}_p = \left\{ k = (e,g) \in \mathbb{K};\, e_{i3} = 0,\, \hat{g} = 0 \right\},\ ^2\mathbb{K}_p = \left\{ k = (e,g) \in \mathbb{K};\, \hat{e} = 0,\, g = 0 \right\},$$
$$^3\mathbb{K}_p = \{ k = (e,g) \in \mathbb{K};\, e = 0,\, g_3 = 0 \}$$

Considering these averages in the transverse direction will be enough to deal with the cases $p_3 \neq 2$. When $p_3 = 2$, it suffices to proceed to a change of unknown (a so-called scaling in the mathematical theory of Kirchhoff–Love plates, see [8,3]) for the electromechanical state in the patch. Let Π^h be the mapping

$$x = (\hat{x}, x_3) \in \mathbb{R}^3 \mapsto \Pi^h x = (\hat{x}, h x_3) \in \mathbb{R}^3 \tag{17}$$

which maps $B^1 = S \times (0,1)$ onto B^h, then the scaling operator is defined by:

$$(S_h r)((\Pi^h)^{-1} x) = (\Pi^h v(x), \varphi(x)) \quad \forall r \in H^1(B^h; \mathbb{R}^3 \times \mathbb{R})\ \forall x \in B^h \tag{18}$$

Let $\mathfrak{s}_{sp} = (u_{sp}, \varphi_{sp}) := S_h \mathfrak{s}^s_p$, we define

$$k(h,\mathfrak{s}_{sp}) = (e(h,\mathfrak{s}_{sp}), g(h,\mathfrak{s}_{sp})),\ e(h,\mathfrak{s}_{sp}) = I_h e(\mathfrak{s}_{sp}) I_h,\ I_h = e_1 \otimes e_1 + e_2 \otimes e_2 + \frac{1}{h} e_3 \otimes e_3,\ g(h,\mathfrak{s}_{sp}) = I_h \nabla \mathfrak{s}_{sp} \tag{19}$$

Then $k(h,\mathfrak{s}_{sp})$ is bounded in $L^2(B^1;\mathbb{K})$ and, up to a subsequence, two-scale converges in $L^2(B^1 \times S)$ to some $\underset{\sim}{k}_p = (\underset{\sim}{e}_p, \underset{\sim}{g}_p)$ in $L^2(B^1 \times Z; \mathbb{K})$. In that case, one will use the additional notations:

$$Z = Y \times (0,1), \quad Z^* = Y^* \times (0,1), \quad z = (y, x_3), \quad k_z(v,\psi) := (e_z(v), \nabla_z \psi), \quad 2(e_z)_{ij}(v) := \frac{\partial v^i}{\partial z_j} + \frac{\partial v^j}{\partial z_i} \tag{20}$$

$$V_{KL} = \left\{ v \in H^1(B^1;\mathbb{R}^3);\, \exists (v^M, v^F) \in H^1(S;\mathbb{R}^2) \times H^2(S) \text{ s.t. } \hat{v}(x) = v^M(\hat{x}) - x_3 \nabla v^F(\hat{x}),\, v_3(x) = v^F(\hat{x}) \right\} \tag{21}$$

By arguing as in [2,3], we have Proposition 2.1.

Proposition 2.1. *Let* $\overline{V} := \left\{ v \in H^1_{\Gamma_0}(\Omega;\mathbb{R}^3);\, \widehat{\gamma_S(v)} \in H^1(S;\mathbb{R}^2) \right\}$, *with* γ_S *the trace operator on* S

i) *the restriction to* Ω *of* u^s_p *converges weakly in* $H^1_{\Gamma_0}(\Omega;\mathbb{R}^3)$ *toward* $\bar{u}_p$ *which belongs to* $\overline{V}$*;*

ii) *when* $p_3 \neq 2$, *if* $(U^s_p, \Phi^s_p) := \frac{1}{h}\int_0^h (u^s_p, \widetilde{\varphi^s_p})\,\mathrm{d}x_3$ *then* $\widehat{U^s_p}$ *converges weakly in* $H^1(S;\mathbb{R}^2)$ *toward* $\widehat{\gamma_S(\bar{u}_p)}$, $(U^s_p)_3$ *converges strongly in* $L^2(S)$ *toward* $(\bar{u}_p)_3$, Φ^s_p *converges strongly to zero in* $L^2(S)$*;*
when $p_3 = 2$, *there exists* $\underset{\sim}{u}_p$ *in* V_{KL} *such that* u_{sp} *weakly converges toward* $\underset{\sim}{u}_p$ *in* $H^1(B^1;\mathbb{R}^3)$ *and* $\underset{\sim}{u}^M_p = \widehat{\gamma_S(\underset{\sim}{u}_p)}$*;*

iii) *when* $p_3 = 3$,
there exists (E, G) *in* $L^2(S;\mathbb{R}^3) \times L^2(S)$ *and* (U^1, w^1, Φ^1) *in* $L^2(S; H^1_\#(Y;\mathbb{R}^2) \times H^1_\#(Y) \times H^1(Y^*))$ *such that*

$$\bar{k}_p = \overline{k^0_p} + \overline{k^1_p}$$
$$\overline{k^0_p} = (\overline{e^0_p}, \overline{g^0_p}) \quad \widehat{\overline{e^0_p}} = e(\widehat{\gamma_S(\bar{u}_p)}),\, (\overline{e^0_p})_{i3} = E_i,\ 1 \le i \le 3, \quad \widehat{\overline{g^0_p}} = 0,\ (\overline{g^0_p})_3 = G,\ \textit{when } \hat{p} = (2,2)\ G = \mathcal{X}_{Y^*}\varphi_0$$
$$\overline{k^1_p} = (\overline{e^1_p}, \overline{g^1_p}) \quad \widehat{\overline{e^1_p}} = e_y(U^1),\, (\overline{e^1_p})_{\alpha 3} = \partial_{y_\alpha} w^1,\ (\overline{e^1_p})_{33} = 0, \quad \widehat{\overline{g^1_p}} = \mathcal{X}_{Y^*}\nabla_y \Phi^1,\ (\overline{g^1_p})_3 = 0$$

when $p_3 = 1$,
there exists (U^1, Φ^1) *in* $L^2(S; H^1_\#(Y;\mathbb{R}^2) \times H^1(Y^*))$ *with* $\Phi^1 = 0$ *when* $\hat{p} \neq (1,1)$ *such that*

$$^1(\bar{k}_p) = \begin{cases} (e(\widehat{\gamma_S(\bar{u}_p)}) + e_y(U^1), \mathcal{X}_{Y^*}\nabla_y \Phi^1) & \textit{when } \hat{p} = (1,1) \\ e(\widehat{\gamma_S(\bar{u}_p)}) + e_y(U^1) & \textit{when } \hat{p} = (1,2) \textit{ or } (2,1) \\ (e(\widehat{\gamma_S(\bar{u}_p)}) + e_y(U^1), \mathcal{X}_{Y^*}\varphi_0) & \textit{when } \hat{p} = (2,2) \end{cases}$$

when $p_3 = 2$,
there exists (u^1_p, φ^1_p) *in* $\left(L^2(B^1; H^1_\#(Y;\mathbb{R}^3) \cap L^2(S; H^1(Z;\mathbb{R}^3)) \right) \times L^2(S; H^1(Z^*))$ *such that*

$$\hat{\underset{\sim}{e}}_p = \hat{e}(\underset{\sim}{u}_p) + \hat{e}_z(\underset{\sim}{u}^1_p), \quad (\underset{\sim}{e}_p)_{i3} = (e_z)_{i3}(\hat{\underset{\sim}{u}}_p)$$
$$\underset{\sim}{g}_p = \mathcal{X}_{Z^*}\nabla_z \varphi^1_p$$

φ^1_p *vanishes on* S *when* $p_2 = 2$, $\varphi^1_p = \varphi_0$ *on* $S + e_3$ *when* $p_1 = 2$.

Step 2: Identification of $\bar{u}_p$ and $\underline{u}_p$
The process of identification of $\bar{u}_p$ and $\underline{u}_p$ as the unique solutions to variational problems (which implies the convergence of the whole sequences involved in Proposition 2.1) depends strongly on p_3.

Case $p_3 = 3$ (*i.e.* $h/\varepsilon \to +\infty$): going to the limit in the variational formulation of $(\mathcal{P}_p^s)$ with the test function $r^* = (v^*, \psi^*)$ where

$$v^*(x) = \begin{cases} v(x) \text{ a.e. } x \text{ in } \Omega \\ v(x) + \min\{x_3, \varepsilon, h - x_3\}\,(v^1(\hat{x}, \hat{x}/\varepsilon) + 2(x_3/\varepsilon)\widehat{F}(\hat{x}), 2w^1(\hat{x}, \hat{x}/\varepsilon) + (x_3/\varepsilon)F_3(\hat{x})) \text{ a.e. } x \text{ in } B^h \end{cases} \tag{22}$$

with (v^1, w^1) in $C_0^\infty(S; C_\#^\infty(Y; \mathbb{R}^2 \times \mathbb{R})$, F in $H^1(S; \mathbb{R}^3)$ and for all v in $\overline{V}$ the same symbol v denotes the extension into $H^1_{\Gamma_0}(\mathcal{O}^h; \mathbb{R}^3)$ by a Kirchhoff–Love displacement in B^h;

$$\psi^*(x) = \min\{x_3, \varepsilon, h - x_3\}\,(\psi^1(\hat{x}, \hat{x}/\varepsilon) + (x_3/\varepsilon)H(\hat{x})) \text{ a.e. } x \text{ in } B_I^s \text{ with } \psi^1 \text{ in } C_0^\infty(S; C_\#^\infty(\overline{Y^*})), H \text{ in } H^1(S) \tag{23}$$

yields

$$\int_\Omega ae(\bar{u}_p) \cdot e(v)\,\mathrm{d}x + \int_{S\times Y} M(y)\bar{k}_p(\hat{x}, y) \cdot (k_p^0(\widehat{\gamma_S(v)}, F, H) + k_p^1(v^1, w^1, \psi^1))\,\mathrm{d}\hat{x}\,\mathrm{d}y = \int_\Omega f \cdot v\,\mathrm{d}x + \int_{\Gamma_1} \mathcal{F} \cdot v\,\mathrm{d}\mathcal{H}_2 \tag{24}$$

where

$$\begin{aligned}
&k_p^0(v, F, H) = (e_p^0, g_p^0)(v, F, H) \in \mathbb{K} \text{ s.t.}\\
&\widehat{e_p^0} = e(v),\ (e_p^0)_{i3} = F_i,\ 1 \le i \le 3,\ \widehat{g_p^0} = 0,\ (g_p^0)_3 = \mathcal{X}_{Y^*}H \quad \forall (v, F, H) \text{ in } H^1(S; \mathbb{R}^2) \times L^2(S; \mathbb{R}^3) \times L^2(S)\\
&k_p^1(v^1, z^1, \psi^1) = (e_p^1, g_p^1)(v^1, z^1, \psi^1) \in \mathbb{K} \text{ s.t.}\\
&\widehat{e_p^1} = e_y(v^1),\ (e_p^1)_{\alpha 3} = \partial_{y_\alpha} z^1,\ (e_p^1)_{33} = 0,\ \widehat{g_p^1} = \mathcal{X}_{Y^*}\nabla_y \psi^1,\ (g_p^1)_3 = 0 \quad \forall (v^1, z^1, \psi^1)\\
&\quad \text{in } \Sigma := H^1_\#(Y; \mathbb{R}^2) \times H^1_\#(Y) \times H^1(Y^*)
\end{aligned}$$

Let $\{{}^ik, 1 \le i \le 7\}$ be a basis of $\mathbb{K}^{\mathrm{eff}}_{\hat{p},3} := \mathbb{S}^3 \times \{0\} \times \{0\} \times \mathbb{R}$, and ${}^ik^* = {}^ik$ for $1 \le i \le 6$ and ${}^7k^* = \mathcal{X}_{Y^*}{}^7k$, each cell problem

$$({}^i\mathcal{P}): \quad \text{find } {}^i\sigma \text{ in } \Sigma \text{ such that} \int_Y M(y)(k_p^1({}^i\sigma) + {}^ik^*) \cdot k_p^1(\tau)\,\mathrm{d}y = 0 \quad \forall \tau \in \Sigma$$

has a unique solution (up to a constant field) and let M_p^{eff} in $\mathrm{Lin}(\mathbb{K}_p^{\mathrm{eff}})$ be defined by

$$(M_p^{\mathrm{eff}})_{ij} = \int_Y M(y)k_p^1({}^i\sigma + {}^ik^*) \cdot (k_p^1({}^j\sigma) + {}^jk^*)\,\mathrm{d}y \quad 1 \le i, j \le 7 \tag{25}$$

then we have:

$$\begin{aligned}
&\int_\Omega ae(\bar{u}_p) \cdot e(v)\,\mathrm{d}x + \int_S M_p^{\mathrm{eff}} k_p^0(\widehat{\gamma_S(\bar{u}_p)}, E, G) \cdot k_p^0(\hat{v}, F, H)\,\mathrm{d}\hat{x}\\
&\qquad = \int_\Omega f \cdot v\,\mathrm{d}x + \int_{\Gamma_1} \mathcal{F} \cdot v\,\mathrm{d}\mathcal{H}_2 \quad \forall (v, F, H) \in H^1(S; \mathbb{R}^2) \times L^2(S; \mathbb{R}^3) \times L^2(S)
\end{aligned} \tag{26}$$

The additional global variables E, G may be easily eliminated through the decomposition

$$\mathbb{K}^{\mathrm{eff}}_{\hat{p},3} = {}^1\mathbb{K}^{\mathrm{eff}}_{\hat{p},3} \oplus {}^2\mathbb{K}^{\mathrm{eff}}_{\hat{p},3} \tag{27}$$

with

$$\text{if } \hat{p} \ne (2,2),\ {}^1\mathbb{K}^{\mathrm{eff}}_{\hat{p},3} = \left\{k = (e, g) \in \mathbb{K}^{\mathrm{eff}}_{\hat{p},3};\ e_{i3} = 0,\ g_3 = 0\right\},\ {}^2\mathbb{K}^{\mathrm{eff}}_{\hat{p},3} = \left\{k = (e, g) \in \mathbb{K}^{\mathrm{eff}}_{\hat{p},3};\ \hat{e} = 0\right\} \tag{28}$$

$$\text{if } \hat{p} = (2,2),\ {}^1\mathbb{K}^{\mathrm{eff}}_{\hat{p},3} = \left\{k = (e, g) \in \mathbb{K}^{\mathrm{eff}}_{\hat{p},3};\ e_{i3} = 0\right\},\ {}^2\mathbb{K}^{\mathrm{eff}}_{\hat{p},3} = \left\{k = (e, g) \in \mathbb{K}^{\mathrm{eff}}_{\hat{p},3};\ \hat{e} = 0,\ g_3 = 0\right\} \tag{29}$$

which induces a decomposition of M_p^{eff} on linear operators ${}^{ij}M_p^{\mathrm{eff}}$ mapping ${}^i\mathbb{K}_p^{\mathrm{eff}}$ into ${}^j\mathbb{K}_p^{\mathrm{eff}}$, and if

$$\widetilde{M_p^{\mathrm{eff}}} := {}^{11}M_p^{\mathrm{eff}} - {}^{12}M_p^{\mathrm{eff}}\,({}^{22}M_p^{\mathrm{eff}})^{-1}\,{}^{21}M_p^{\mathrm{eff}} \tag{30}$$

then $\bar{u}_p$ is the unique solution to

$$(\bar{\mathcal{P}}_{\hat{p},3})\begin{cases}\text{find } \bar{u}_p \text{ in } \overline{V} \text{ such that for all } v \text{ in } \overline{V}\\ \int_\Omega ae(\bar{u}_p)\cdot e(v)\,\mathrm{d}x+\int_S \widetilde{M_p^{\text{eff}}}(e(\widehat{\gamma_S(\bar{u}_p)}))\cdot e(\widehat{\gamma_S(v)})\,\mathrm{d}\hat{x}=\int_\Omega f\cdot v\,\mathrm{d}x+\int_{\Gamma_1}\mathcal{F}\cdot v\,\mathrm{d}\mathcal{H}_2, \text{ if } \hat{p}\neq(2,2)\\ \int_\Omega ae(\bar{u}_p)\cdot e(v)\,\mathrm{d}x+\int_S \widetilde{M_p^{\text{eff}}}(e(\widehat{\gamma_S(\bar{u}_p)}),\varphi_0 e_3)\cdot(e(\widehat{\gamma_S(v)}),0)\,\mathrm{d}\hat{x}=\int_\Omega f\cdot v\,\mathrm{d}x+\int_{\Gamma_1}\mathcal{F}\cdot v\,\mathrm{d}\mathcal{H}_2, \text{ if } \hat{p}=(2,2)\end{cases}$$

Case $p_3=1$ (*i.e.* $h/\varepsilon\to 0$): By using successively test functions ${}^I r=({}^I v,\psi)$, $1\le I\le 3$ and $r^*=(v^*,\psi^*)$ defined by:

$$\begin{aligned}&({}^I v(x))_i=x_3\delta_{Ii}\theta^1(\hat{x})\theta^2(\hat{x}/\varepsilon)\\ &\psi(x)=\eta(x_3)\theta^1(\hat{x})\theta^2(\hat{x}/\varepsilon)\\ &\eta(x_3)=\begin{cases}x_3 & \text{if } \hat{p}=(1,1) \text{ or } (1,2)\\ x_3-h & \text{if } \hat{p}=(2,1)\\ 0 & \text{if } \hat{p}=(2,2)\end{cases}\\ &(\theta^1,\theta^2)\in C_0^\infty(S)\times C_\#^\infty(Y)\\ &\delta_{Ii} \text{ being the Kronecker symbol}\end{aligned}$$

and

$$\begin{aligned}&v^*=v \text{ in } \Omega\\ &v^*(x)=v(x)+\varepsilon v^1(\hat{x},\hat{x}/\varepsilon) \text{ a.e. in } B^h \text{ where } v \text{ denotes the extension into } H^1_{\Gamma_0}(\mathcal{O}^h;\mathbb{R}^3)\\ &\qquad\text{of any element } v \text{ of } \overline{V} \text{ by a Kirchhoff–Love displacement in } B^h\\ &\psi^*=\varepsilon\psi^1(\hat{x},\hat{x}/\varepsilon)\\ &(v^1,\psi^1)\in C_0^\infty(S;H^1_\#(Y;\mathbb{R}^2))\times C_0^\infty(S;H^1(Y^*))\end{aligned}$$

we obtain

$$\int_\Omega ae(\bar{u}_p)\cdot e(v)\,\mathrm{d}x+\int_{S\times Y}\widetilde{M}_p\,{}^1(\bar{k}_p)\cdot{}^1k(v^*,\psi^*)\,\mathrm{d}\hat{x}\,\mathrm{d}y=\int_\Omega f\cdot v\,\mathrm{d}x+\int_{\Gamma_1}\mathcal{F}\cdot v\,\mathrm{d}\mathcal{H}_2 \tag{31}$$

where ${}^1k(v^*,\psi^*)=({}^1e,{}^1g)$ is the element of ${}^1\mathbb{K}_p$ such that:

$$\begin{aligned}&\widehat{{}^1e}=e(\widehat{\gamma_S(v)})+e_y(v^1)\\ &\widehat{{}^1g}=\nabla_y\psi^1 \quad \text{if } \hat{p}=(1,1)\\ &({}^1g)_3=0 \qquad \text{if } \hat{p}=(2,2)\end{aligned}$$

and

$$\widetilde{M}_p:={}^{11}M_p-{}^{12}M_p({}^{22}M_p)^{-1\,21}M_p \tag{32}$$

${}^{IJ}M_p$ being the elements of the decomposition of M in linear mappings from ${}^I\mathbb{K}_p$ into ${}^J\mathbb{K}_p$, $1\le I,J\le 3$. Let $\{k^i,1\le i\le 3\}$ be a basis of $\mathbb{S}^2$ when $\hat{p}\neq(2,2)$, and $\{k^i,1\le i\le 4\}$ be a basis of $\mathbb{S}^2\times\{0\}\times\{0\}\times\mathbb{R}$ when $\hat{p}=(2,2)$, each cell problem

$$(\mathcal{P}^i):\begin{cases}\text{find } (w^i,\psi^i) \text{ in } H^1_\#(Y;\mathbb{R}^2)\times H^1(Y^*) \text{ such that, for all } (w,\psi) \text{ in } H^1_\#(Y;\mathbb{R}^2)\times H^1(Y^*)\\ \int_Y \widetilde{M}_p({}^1k_p(w^i,\psi^i)+k^i)\cdot{}^1k_p(w,\psi)\,\mathrm{d}y=0\end{cases}$$

has a unique solution (up to a constant field) and let $\widetilde{M}_p^{\text{eff}}$ in $\mathrm{Lin}(\mathbb{S}^2)$ be defined by

$$(\widetilde{M}_p^{\text{eff}})_{ij}:=\int_Y \widetilde{M}_p({}^1k_p(w^i,\psi^i)+k^i)\cdot({}^1k_p(w^j,\psi^j)+k^j)\,\mathrm{d}y \tag{33}$$

so that $\bar{u}_p$ is the unique solution to

$$(\bar{\mathcal{P}}_{\hat{p},1}) \begin{cases} \text{find } \bar{u}_p \text{ in } \overline{V} \text{ such that, for all } v \text{ in } \overline{V} \\ \int_\Omega ae(\bar{u}_p)\cdot e(v)\,\mathrm{d}x + \int_S \widetilde{M}_p^{\mathrm{eff}} e(\widehat{\gamma_s(\bar{u}_p)})\cdot e(\widehat{\gamma_s(v)})\,\mathrm{d}\hat{x} = \int_\Omega f\cdot v\,\mathrm{d}x + \int_{\Gamma_1} \mathcal{F}\cdot v\,\mathrm{d}\mathcal{H}_2, \ \text{if } \hat{p}\neq(2,2) \\ \int_\Omega ae(\bar{u}_p)\cdot e(v)\,\mathrm{d}x + \int_S \widetilde{M}_p^{\mathrm{eff}} (e(\widehat{\gamma_s(\bar{u}_p)}), \varphi_0 e_3)\cdot (e(\widehat{\gamma_s(v)}),0)\,\mathrm{d}\hat{x} = \int_\Omega f\cdot v\,\mathrm{d}x + \int_{\Gamma_1} \mathcal{F}\cdot v\,\mathrm{d}\mathcal{H}_2, \ \text{if } \hat{p}=(2,2) \end{cases}$$

Case $p_3 = 2$ (*i.e.* $h/\varepsilon \to 1$): as only local unknowns u_p^1, φ_p^1 enter $\underline{k}_p$, it suffices to eliminate the space variable by introducing suitable oscillating test functions in the variational formulation of $(\mathcal{P}_p^s)$ and to go to the limit as in [3]. Let $\mathcal{I} = \{1,2\}^3 \cup \{*\}$ with $\{*\} = \varnothing$ if $\hat{p} \neq (2,2)$ and $k^I \in \mathbb{S}^2 \times \mathbb{R}^3$, $I \in \mathcal{I}$, and W be defined by

$$k^I := ((-x_3)^{I_3-1} e_{I_1} \otimes_S e_{I_2}, 0) \text{ if } I \in \{1,2\}^3,\ k^I := (0, e_3) \text{ if } I \in \{*\} \tag{34}$$

$$W := \left(H^1(Z;\mathbb{R}^3) \cap L^2((-1,1); H^1_\#(Y;\mathbb{R}^3))\right) \times \Psi^1 \tag{35}$$

$$\Psi^1 := \left\{ \psi \in H^1(Z^*);\ \psi = 0 \text{ on } S \text{ if } p_2 = 2,\ \psi = 0 \text{ on } S + e_3 \text{ if } p_1 = 2 \right\} \tag{36}$$

each cell problem

$$(\mathcal{P}^I): \quad \text{find } \sigma^I \text{ in } W \text{ such that } \int_Z M_p(y)(k^I + k_z(\sigma^I))\cdot k_z(\rho)\,\mathrm{d}z = 0, \quad \forall \rho \in W$$

has a unique solution (up to constant fields) and let M_p^{eff} be defined by

$$(M_p^{\mathrm{eff}})_{IJ} := \int_Z M_p(y)(k^I + k_z(\sigma^I))\cdot(k^J + k_z(\sigma^J))\,\mathrm{d}z, \quad 1 \le I, J \le \sharp(\mathcal{I}) \tag{37}$$

then $(\bar{u}_p, \underline{u}_p)$ is the unique solution to

$$(\bar{\mathcal{P}}_{\hat{p},2}) \begin{cases} \text{find } (\bar{u}_p,\underline{u}_p) \text{ in } \mathcal{V} = \left\{ (\bar{v},\underline{v}) \text{ in } H^1_{\Gamma_0}(\Omega;\mathbb{R}^3) \times V_{KL}(B^1) \text{ s.t } \widehat{\gamma_S(\bar{v})} = (\underline{v})^M \right\} \text{ such that, for all } (\bar{v},\underline{v}) \text{ in } \mathcal{V} \\ \int_\Omega ae(\bar{u}_p)\cdot e(\bar{v})\,\mathrm{d}x + \int_{B^1} M^{\mathrm{eff}}(e(\underline{u}_p^M), D^2\underline{u}_p^F)\cdot(e(\underline{v}^M), D^2\underline{v}^F)\,\mathrm{d}x = \int_\Omega f\cdot\bar{v}\,\mathrm{d}x + \int_{\Gamma_1} \mathcal{F}\cdot\bar{v}\,\mathrm{d}\mathcal{H}_2, \ \text{if } \hat{p}\neq(2,2) \\ \int_\Omega ae(\bar{u}_p)\cdot e(\bar{v})\,\mathrm{d}x + \int_{B^1} M^{\mathrm{eff}}(e(\underline{u}_p^M), D^2\underline{u}_p^F, \varphi_0 e_3)\cdot(e(\underline{v}^M), D^2\underline{v}^F, 0)\,\mathrm{d}x \\ \qquad = \int_\Omega f\cdot\bar{v}\,\mathrm{d}x + \int_{\Gamma_1} \mathcal{F}\cdot\bar{v}\,\mathrm{d}\mathcal{H}_2, \ \text{if } \hat{p}=(2,2) \end{cases}$$

Hence, we have the following convergence results

Theorem 2.1. *When s goes to zero, u_p^s weakly converges in $H^1_{\Gamma_0}(\Omega;\mathbb{R}^3)$ toward $\bar{u}_p$, while, when $p_3 = 2$, u_{sp} weakly converges in $H^1(B^1;\mathbb{R}^3)$ toward $\underline{u}_p$, the unique solution to $(\overline{\mathcal{P}}_p)$.*

3. Concluding remarks

Hence our limit models described by $\overline{\mathcal{P}}_p$ correspond to the equilibrium of the genuine body subjected to loading $(f,\mathcal{F})$ and reinforced along S, this reinforcement being nonlocal when $p_3 = 2$. The characteristics of this elastic reinforcement may depend on the dielectric or piezoelectric coefficients inasmuch as these terms appear in the expression of $\widetilde{M}_p^{\mathrm{eff}}$ or M_p^{eff} (see [4], where the systematic influence of crystal symmetries has been carried out). When $\hat{p} = (2,2)$, the applied electric potential creates a source surface term and the patch acts as an actuator. The piezoelectric inclusions being disconnected, the patch cannot act as a sensor.

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