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Bridging meso- and microscopic anisotropic unilateral damage formulations for microcracked solids

Qi-Zhi Zhu [a,*], Shuang-Shuang Yuan [a], Jian-fu Shao [a,b]

[a] CKey Laboratory of Ministry of Education for Geomechanics and Embankment Engineering, Hohai University, Nanjing, China
[b] Laboratory of Mechanics of Lille, UMR8107 CNRS, University of Lille, Villeneuve-d'Ascq, France

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ABSTRACT

A mathematically consistent and unified description of induced anisotropy and unilateral effects constitutes one of the central tasks in the continuum damage theories developed so far. This paper aims at bridging constitutive damage formulations on meso- and micro-scales with an emphasis on a complete mesoscopic determination of material effective properties for microcracked solids. The key is to introduce a new set of invariants in terms of strain tensor and fabric tensor by making use of the Walpole's tensorial base. This invariant set proves to be equivalent to the classical one, while the new one provides great conveniences to high-order orientation-dependent tensor manipulations. When limited to the case of parallel microcracks, potential relations between ten combination coefficients are established by applying continuity conditions. It is found that the dilute approximation with penny-shaped microcracks is a particular case of the present one. By originally introducing effective strain effect, interactions between microcracks are taken into account with comparison to the Mori–Tanaka method as well as the Ponte-Castaneda and Willis scheme. For completeness, discussions are also addressed on macroscopic formulations with high-order damage variables.

1. Introduction

Microcracking is commonly viewed as the main dissipative mechanism governing the nonlinear mechanical behaviors of quasi-brittle materials like concrete and quasi-brittle rocks [1–3]. In constitutive modeling, the concept of continuum damage has been widely used within the framework of irreversible thermodynamics [4–6]. During the development of damage models, in view of the specific features of microcracks, two essential theoretical issues have attracted much attention and still remain largely open up to now: the first one is the induced anisotropy due to crack growth, nonuniform but in some preferred orientations; the second one, known as unilateral effect, is related to the unilateral contact between the lips of microcracks, which can be open or closed without material interpenetration. Concerning the mechanical phenomena related to the two mechanisms, we can mention the asymmetry of material strength in tension and in compression, the non-linearity of the material response, as well as the effect of confining pressure, volumetric dilatancy, occurrence of unloading–reloading hysteresis in the context of damage–friction coupling at closed cracks.

* Corresponding author.
E-mail address: qizhi.zhu@gmail.com (Q.-Z. Zhu).

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Mathematically consistent description of cracks-related material anisotropy and unilateral effects is an essential and important theoretical issue in Continuum Damage Mechanics. At the early stage, four main methods were explored based on i) two scalar variables, ii) two second-order damage variables, iii) a fourth-order damage variable, and iv) vectorial damage variables, respectively [6,7]. Chaboche [7] discussed and compared the predictive capability of these theories by using a 2D damage problem and setting up four assessment rules. He found that there exist problems of some sort for all the four candidates. On the same topic, by relating macroscopic formulations to mesostructural mechanisms, Halm and Dragon [8] then developed a mesoscopic damage model by employing simultaneously a second-order variable and a fourth order variable; the latter one proved to be mandatory for describing the unilateral effects. However, in the Halm and Dragon's model, both the second- and fourth-rank damage variables have been formulated in the principal directions determined by spectral decomposition. Although such a choice avoided some theoretical difficulties, it brought about some inconsistencies in constitutive damage formulations and the non-objectivity of the mechanical response, as commented by Cormery and Welemane [9]. Welemane and Cormery [10] also made their own contribution to this issue with the help of the tensor representation theory [11] and the piecewise linearity theory [12]. In their work, some basic relations between the combination coefficients have been established by applying the continuity conditions for multilinear functions and the recovery condition of the elongation modulus in the normal direction of cracks. In spite of these significant advances, in-depth developments along this line are still required. On one hand, even in the case of dilute microcracks, the effective stiffness tensor reported by Welemane and Cormery [10] contains some terms whose physical meanings are not clear. On the other hand, there is still a need to establish some trans-scale relationships between the constitutive equations.

Concerning induced anisotropy and unilateral effects, we should mention the school of investigators whom account for unilateral effects by means of the spectral decomposition upon either the stress tensor or the strain tensor. To the authors' knowledge, the choice to separate the stress/strain tensors into positive and negative parts in the principal space can be traced back to the pioneering work by Ortiz [13], mainly based on the hypothesis that damage in quasi-brittle materials is induced by local tensile stress. In that context, the constitutive damage formulations were naturally closely related to principal stress/strain components. It is noted that the Oritz's theory has been followed by numerous authors, particularly those in concrete researches. We mention, among others, the works with stress decomposition [5,14–20] and those using strain decomposition [21–24]. With the spectral decomposition technique, when limited to tensile damage, modeling of nonlinear behaviors of quasi-brittle materials like concrete has gained great success in view of their practical applications. However, this method exhibits obvious disadvantages on both theoretical and numerical aspects: firstly, because of the neglect of the intrinsic fabric by microcracks, the theoretical foundation of the use of stress/strain splitting is not firm and the piecewise linearity for general loading paths has not been proved. To a large extent, the models framed by the spectral decomposition technique should be classed into the category of phenomenological models; secondly, for nonlinear problems, the frequent numerical resolution of the eigenvalue problem implies high computational costs, as commented by Curnier et al. [12], and it is thus not in favor of parallel programming and computing.

The present work is motivated partially by the current situation of the development of the continuum damage mechanics. During more than two decades, the phenomenological damage models and the micromechanical damage models have been investigated individually. The former describes cracking-induced damage by using tensorial internal variables [6]. It is noticed in the literature that the isotropic damage models are still playing a predominant role in practical applications although they exhibit obvious disadvantages in accounting for the mechanical mechanisms behind the main phenomena observed at laboratory. Starting from the microstructure and local behaviors of cracked media, the micromechanical damage models aims at formulating constitutive equations by means of some upscaling methods, such as the homogenization technique. In recent years, these approaches have experienced a great success and have paved a promising way to deliver a mechanisms-based modeling of quasi-brittle materials. However, when compared to the phenomenological models, the mathematical tools used and the constitutive formulations presented in the micromechanical models are somewhat complex, especially for researchers who are mainly concerned with engineering applications. Up to now, few efforts have been devoted to building some links between these two categories of damage models. This work attempts to get out of this trouble and to fill in this gap by dealing with the two critical theoretical issues in continuum damage mechanics.

Along the line of the work by Welemane and Cormery [10], this paper delivers new contributions to the research issues discussed above. Attempts were first made to determine the explicit form of the effective properties for a matrix-cracks system. We start with the classical set of strain-damage invariants largely used in the macroscopic and mesoscopic damage models. For convenience, a new set of invariants is constructed based on the Walpole tensorial base and its equivalence to the classical one is proved. Induced anisotropies and unilateral damage are taken into account in a unified and consistent way. Potential relations between the coefficients companying with the invariants for both the cases of open cracks and closed cracks are explored step by step in Section 3. Efforts in Sections 4 and 5 are addressed to two aspects: i) compare the mesoscopic formulations with the micromechanical results obtained by applying some popular homogenization schemes [25–29], ii) address the theoretical issues with high-order damage variables. The difficulties are revealed and a possible solution is discussed. Finally, in Section 6 are given some concluding remarks.

2. Introduction of an alternative set of invariants

We are first concerned with the case of a solid matrix weakened by one unique family of parallel non-interacting microcracks. The extension to the general case of multiple crack families will be treated in Section 4. Cracked solids are viewed as

a matrix-cracks system. Geometrically, a set of parallel planar microcracks can be sufficiently described by its normal $\boldsymbol{n}$ and a dimensionless density parameter $d(\boldsymbol{n})$, the latter involving the mean crack size and the crack number per unit volume.

2.1. The classical set of invariants

Assume that there exists for the matrix-cracks system a thermodynamic potential w, which is function of the state variables (macroscopic strain tensor $\boldsymbol{\varepsilon}$ and damage variable $d(\boldsymbol{n})$) and satisfies the following essential conditions [10]: i) $w(\boldsymbol{\varepsilon}, d)$ is objective and of class $\mathcal{C}^1$, implying that the macroscopic stress $\boldsymbol{\sigma}$ is continuous under any loading path; ii) $w(\boldsymbol{\varepsilon}, d)$ is positively definite and of degree two with respect to the macroscopic strain $\boldsymbol{\varepsilon}$ and hence, $\boldsymbol{\sigma}$ defined by $\boldsymbol{\sigma} = \partial w/\partial \boldsymbol{\varepsilon}$ is of degree one with respect to $\boldsymbol{\varepsilon}$. According to the above conditions imposed to the free energy w and by applying the representation theorem for isotropic scalar-valued functions, the free energy w should be a linear combination of the following strain-fabric invariants [11]

$$\mathrm{tr}(\boldsymbol{\varepsilon}.\boldsymbol{\varepsilon}), \quad \mathrm{tr}\boldsymbol{\varepsilon}\mathrm{tr}\boldsymbol{\varepsilon}, \quad \mathrm{tr}\boldsymbol{\varepsilon}\mathrm{tr}(\boldsymbol{\varepsilon}.\boldsymbol{N}), \quad \mathrm{tr}(\boldsymbol{\varepsilon}.\boldsymbol{\varepsilon}.\boldsymbol{N}), \quad \mathrm{tr}^2(\boldsymbol{\varepsilon}.\boldsymbol{N}) \tag{1}$$

where $\boldsymbol{N} = \boldsymbol{n} \otimes \boldsymbol{n}$ defines a second order fabric tensor. The effective stiffness tensor is then the linear combination of the second-order derivative of the above elements with respect to $\boldsymbol{\varepsilon}$

$$\boldsymbol{\delta} \otimes^s \boldsymbol{\delta}, \quad \boldsymbol{\delta} \otimes \boldsymbol{\delta}, \quad \boldsymbol{\delta} \otimes \boldsymbol{N} + \boldsymbol{N} \otimes \boldsymbol{\delta}, \quad \boldsymbol{N} \otimes^s \boldsymbol{\delta} + \boldsymbol{\delta} \otimes^s \boldsymbol{N}, \quad \boldsymbol{N} \otimes \boldsymbol{N} \tag{2}$$

where $\boldsymbol{\delta}$ is the second-order identity tensor. With the notation $\otimes^s$, we define the operation $\left(\boldsymbol{A} \otimes^s \boldsymbol{B}\right)_{ijkl} = \frac{1}{2}\left(A_{ik}B_{jl} + A_{il}B_{jk}\right)$ between any two second-order tensors $\boldsymbol{A}$ and $\boldsymbol{B}$. For any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, a similar definition can be made such that $\left(\boldsymbol{a} \otimes^s \boldsymbol{b}\right)_{ij} = \frac{1}{2}\left(a_i b_j + a_j b_i\right)$.

2.2. A new equivalent set of invariants

The introduction of a new set of invariants is mainly motivated by the requirement of frequent manipulations of inner products between fourth-order orientation-dependent tensors. In the literature, direct use of the set (1) or equivalently (2) has been made in damage formulations. However, this common choice gives rise to the following inconveniences: i) the physical meaning of the combination coefficients associated with the components in Eq. (1) is not clear. As a consequence, some important conditions that are specific to crack problems may have been neglected; ii) it is quite difficult to manipulate inner products and especially inverse operations of four order tensors function of the fourth order elements in Eq. (2).

By defining $\boldsymbol{N} = \boldsymbol{n} \otimes \boldsymbol{n}$ together with $\boldsymbol{T} = \boldsymbol{\delta} - \boldsymbol{n} \otimes \boldsymbol{n}$, the six elements of the Walpole's base are given as follows [30]:

$$\begin{aligned} &\mathbb{E}^1 = \tfrac{1}{2}\boldsymbol{T} \otimes \boldsymbol{T}, \quad \mathbb{E}^2 = \boldsymbol{N} \otimes \boldsymbol{N}, \quad \mathbb{E}^3 = \boldsymbol{T} \otimes^s \boldsymbol{T} - \tfrac{1}{2}\boldsymbol{T} \otimes \boldsymbol{T} \\ &\mathbb{E}^4 = \boldsymbol{N} \otimes^s \boldsymbol{T} + \boldsymbol{T} \otimes^s \boldsymbol{N}, \quad \mathbb{E}^5 = \boldsymbol{N} \otimes \boldsymbol{T}, \quad \mathbb{E}^6 = \boldsymbol{T} \otimes \boldsymbol{N} \end{aligned} \tag{3}$$

The table for inner products between $\mathbb{E}^i, i = 1, ..., 6$ can be found in [30].

It has been proved that any fourth-rank tensor $\mathbb{U}$ presenting the transverse isotropy with the axis of rotation $\boldsymbol{n}$ can be expressed as a linear combination of $\mathbb{E}^i, i = 1 \ldots 6$

$$\mathbb{U} = c\mathbb{E}^1 + d\mathbb{E}^2 + e\mathbb{E}^3 + f\mathbb{E}^4 + g\mathbb{E}^5 + h\mathbb{E}^6 \tag{4}$$

Eq. (4) can also be represented in a compact symbolic form by placing in order the combination coefficients associated with the elements [30]:

$$\mathbb{U} = (c, d, e, f, g, h) \tag{5}$$

The inverse of $\mathbb{U}$ is then calculated simply by following the rule:

$$\mathbb{U}^{-1} = \left(\frac{d}{\ell}, \frac{c}{\ell}, \frac{1}{e}, \frac{1}{f}, -\frac{g}{\ell}, -\frac{h}{\ell}\right) \tag{6}$$

with $\ell = cd - 2gh$. For later use, we also obtain the relation $c/\ell = 1/E$.

The Walpole's base allows one to construct a set of invariants with elements $\boldsymbol{\varepsilon} : \mathbb{E}^i : \boldsymbol{\varepsilon}, i = 1, \ldots, 6$. Further, when there is the diagonal symmetry, one has $g = h$ and the last two elements $\boldsymbol{\varepsilon} : \mathbb{E}^5 : \boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon} : \mathbb{E}^6 : \boldsymbol{\varepsilon}$ can be merged into $\boldsymbol{\varepsilon} : (\mathbb{E}^5 + \mathbb{E}^6) : \boldsymbol{\varepsilon}$. With five independent elastic moduli fully describing the transverse isotropy, namely with the plane-strain bulk modulus k, the transverse shear modulus m, the axial shear modulus p, the axial Young's modulus E and the Poisson's ratio ν, one can establish the following connections for the combination coefficient c, d, e, f, g and h [30]

$$c = 2k, \quad d - 2g^2/c = E, \quad g/c = \nu, \quad e = 2m, \quad f = 2p \tag{7}$$

Therefore, the Walpole's base delivers clear physical meaning to each combination coefficient.

We propose here the second set of invariants with the five elements

$$\boldsymbol{\varepsilon} : \mathbb{E}^1 : \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon} : \mathbb{E}^2 : \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon} : \mathbb{E}^3 : \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon} : \mathbb{E}^4 : \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon} : (\mathbb{E}^5 + \mathbb{E}^6) : \boldsymbol{\varepsilon} \tag{8}$$

It is shown that the classical set (1) can be expressed in terms of $\mathbb{E}^i$ in the new set (8)

$$\begin{cases} \mathrm{tr}(\boldsymbol{\varepsilon}.\boldsymbol{\varepsilon}) = \boldsymbol{\varepsilon} : \mathbb{E}^1 : \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon} : \mathbb{E}^2 : \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon} : \mathbb{E}^3 : \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon} : \mathbb{E}^4 : \boldsymbol{\varepsilon} \\ \mathrm{tr}^2\boldsymbol{\varepsilon} = 2\boldsymbol{\varepsilon} : \mathbb{E}^1 : \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon} : \mathbb{E}^2 : \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon} : (\mathbb{E}^5 + \mathbb{E}^6) : \boldsymbol{\varepsilon} \\ \mathrm{tr}\boldsymbol{\varepsilon}\,\mathrm{tr}(\boldsymbol{\varepsilon}.\boldsymbol{N}) = \boldsymbol{\varepsilon} : \mathbb{E}^2 : \boldsymbol{\varepsilon} + \frac{1}{2}\boldsymbol{\varepsilon} : (\mathbb{E}^5 + \mathbb{E}^6) : \boldsymbol{\varepsilon} \\ \mathrm{tr}(\boldsymbol{\varepsilon}.\boldsymbol{\varepsilon}.\boldsymbol{N}) = \boldsymbol{\varepsilon} : \mathbb{E}^2 : \boldsymbol{\varepsilon} + \frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{E}^4 : \boldsymbol{\varepsilon} \\ \mathrm{tr}^2(\boldsymbol{\varepsilon}.\boldsymbol{N}) = \boldsymbol{\varepsilon} : \mathbb{E}^2 : \boldsymbol{\varepsilon} \end{cases} \tag{9}$$

or equivalently in the following matrix form

$$\begin{pmatrix} \mathrm{tr}(\boldsymbol{\varepsilon}.\boldsymbol{\varepsilon}) \\ \mathrm{tr}^2\boldsymbol{\varepsilon} \\ \mathrm{tr}\boldsymbol{\varepsilon}\,\mathrm{tr}(\boldsymbol{\varepsilon}.\boldsymbol{N}) \\ \mathrm{tr}(\boldsymbol{\varepsilon}.\boldsymbol{\varepsilon}.\boldsymbol{N}) \\ \mathrm{tr}^2(\boldsymbol{\varepsilon}.\boldsymbol{N}) \end{pmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}}_{\boldsymbol{A}} \begin{pmatrix} \boldsymbol{\varepsilon} : \mathbb{E}^1 : \boldsymbol{\varepsilon} \\ \boldsymbol{\varepsilon} : \mathbb{E}^2 : \boldsymbol{\varepsilon} \\ \boldsymbol{\varepsilon} : \mathbb{E}^3 : \boldsymbol{\varepsilon} \\ \boldsymbol{\varepsilon} : \mathbb{E}^4 : \boldsymbol{\varepsilon} \\ \boldsymbol{\varepsilon} : (\mathbb{E}^5 + \mathbb{E}^6) : \boldsymbol{\varepsilon} \end{pmatrix} \tag{10}$$

The determinant of the coefficient matrix $\boldsymbol{A}$ is non-zero ($\det\boldsymbol{A} = -1/2$), implying that the two sets (1) and (8) are equivalent. In the following parts, the new invariant set (8) will be used in lieu of the classical set (1) in view that the former can facilitate significantly inner products between and inverse operation upon orientation-dependent tensors.

3. Determination of the effective elasticity tensors under unilateral consideration

The matrix phase with stiffness tensor $\mathbb{C}$ is assumed to be isotropic and linearly elastic. This section focuses on the case of a unique family of microcracks that are highly numerous, randomly distributed in the matrix, and self-similarly propagating. In addition, all microcracks are assumed here to be planar, but not limited to be penny-shaped, unlike the hypothesis adopted in multiscale studies that make direct use of the Eshelby's solution to the ellipsoidal inclusion (crack) problem [27–29,31]. Cracked solids satisfying the above conditions exhibit approximately a transverse isotropy.

3.1. Effective elasticity tensor and free energy

According to the basic results from a linear homogenization procedure [31,32] and by following the previous work [10], the effective elasticity tensor of the solid matrix-cracks system takes the general form

$$\mathbb{C}^{\mathrm{eff},r} = \mathbb{C} - \mathbb{C}^{d,r}, \quad r = +, - \tag{11}$$

Above, the superscript r is used to distinguish the two crack states: open cracks denoted by $r = +$ and closed cracks by $r = -$. The fourth order tensor $\mathbb{C}^d$ is physically the modification brought by microcracks to the elasticity tensor $\mathbb{C}$. $\mathbb{C}$ being linearly isotropic, it is therefore the term $\mathbb{C}^d$ that reflects the transversely isotropic property of $\mathbb{C}^{\mathrm{eff}}$. Thus, $\mathbb{C}^{d,r}$ can be formally expressed as the combination of the elements $\mathbb{E}^i$

$$\mathbb{C}^{d,r} = \left(\varsigma_1^r, \varsigma_2^r, \varsigma_3^r, \varsigma_4^r, \varsigma_5^r, \varsigma_5^r\right), \; r = +, - \tag{12}$$

where ς_i^+ and ς_i^- are the coefficients associated with element $\mathbb{E}^i$, corresponding to the cases of open and closed cracks, respectively. Moreover, both ς_i^+ and ς_i^- are function of the damage variable $d(\boldsymbol{n})$ and satisfy the conditions in the undamaged state

$$\varsigma_i^+(d=0) = 0, \quad \varsigma_i^-(d=0) = 0, \quad i = 1, ..., 5 \tag{13}$$

In other words, $\mathbb{C}^{\mathrm{eff},r}(d=0) = \mathbb{C}$, which corresponds to the undamaged state of material.

In addition, the isotropic elasticity tensor $\mathbb{C}$ can also be expressed in terms of $\mathbb{E}^i$

$$\mathbb{C} = (2\lambda + 2\mu, \lambda + 2\mu, 2\mu, 2\mu, \lambda, \lambda) \tag{14}$$

with λ and the shear modulus μ being the two Lamé's constants of the matrix phase. By combining Eqs. (12) and (14), $\mathbb{C}^{\mathrm{eff},r}$ is finally written in the Walpole's base:

$$\mathbb{C}^{\mathrm{eff},r} = \left(2\lambda + 2\mu - \varsigma_1^r, \lambda + 2\mu - \varsigma_2^r, 2\mu - \varsigma_3^r, 2\mu - \varsigma_4^r, \lambda - \varsigma_5^r, \lambda - \varsigma_5^r\right), \; r = +, - \tag{15}$$

By further defining the differences $\Delta\mathbb{C} = \mathbb{C}^{\mathrm{eff},+} - \mathbb{C}^{\mathrm{eff},-}$ and $\Delta\varsigma_i = \varsigma_i^- - \varsigma_i^+$, one has

$$\Delta\mathbb{C}^{\mathrm{eff}} = (\Delta\varsigma_1, \Delta\varsigma_2, \Delta\varsigma_3, \Delta\varsigma_4, \Delta\varsigma_5, \Delta\varsigma_5) \tag{16}$$

The unilateral contact implies that $\mathbb{C}^{\mathrm{eff}}$ will suffer a jump $\Delta\mathbb{C}^{\mathrm{eff}}$ at an opening/closure transition of microcracks, but that both the stress and free energy must be continuous under any loading path.

In summary, microcracks can be open or closed, leading to unilateral effects. For each state, there are five coefficients involved in the effective elasticity tensor. Thus, we have in total ten parameters ς_i^r to be determined for the complete description of $\mathbb{C}^{\text{eff}}$. The rest of this section is devoted to establishing the basic relations between these parameters.

3.2. Application of continuously differential property

The continuity in stress implies that

$$\Delta\boldsymbol{\sigma} = \boldsymbol{\sigma}^+ - \boldsymbol{\sigma}^- = \Delta\mathbb{C}^{\text{eff}} : \boldsymbol{\varepsilon} = 0 \tag{17}$$

which should be valid for any strain state. Thus, the continuously differential property of w means that $\Delta\mathbb{C}^{\text{eff}}$ is singular. By making use of the inverse rule (6), one can establish the conditions

$$\begin{cases} \Delta\varsigma_1\Delta\varsigma_2 - 2(\Delta\varsigma_5)^2 = 0 \\ \Delta\varsigma_3 = 0 \\ \Delta\varsigma_4 = 0 \end{cases} \tag{18}$$

which leads to the expression of the jump $\Delta\mathbb{C}^{\text{eff}}$

$$\Delta\mathbb{C}^{\text{eff}} = (\Delta\varsigma_1, \Delta\varsigma_2, 0, 0, \Delta\varsigma_5, \Delta\varsigma_5) \tag{19}$$

$$= \frac{1}{2}\Delta\varsigma_1\boldsymbol{T}\otimes\boldsymbol{T} + \Delta\varsigma_2\boldsymbol{N}\otimes\boldsymbol{N} + \sqrt{\frac{1}{2}\Delta\varsigma_1\Delta\varsigma_2}\,(\boldsymbol{N}\otimes\boldsymbol{T} + \boldsymbol{T}\otimes\boldsymbol{N}) \tag{20}$$

or equivalently in the form

$$\Delta\mathbb{C}^{\text{eff}} = \frac{\partial\hbar(\boldsymbol{n},\boldsymbol{\varepsilon})}{\partial\boldsymbol{\varepsilon}} \otimes \frac{\partial\hbar(\boldsymbol{n},\boldsymbol{\varepsilon})}{\partial\boldsymbol{\varepsilon}} \tag{21}$$

with

$$\hbar(\boldsymbol{n},\boldsymbol{\varepsilon}) = \left(\sqrt{\frac{\Delta\varsigma_1}{2}}\boldsymbol{T} + \sqrt{\Delta\varsigma_2}\boldsymbol{N}\right) : \boldsymbol{\varepsilon} \tag{22}$$

According to Curnier et al. [12] and Welemane and Cormery [10], the salient feature shown by Eq. (21) is necessary to ensure the continuity of the multilinear function $\hbar(\boldsymbol{n},\boldsymbol{\varepsilon})$. In addition, $\hbar(\boldsymbol{n},\boldsymbol{\varepsilon}) = 0$ describes the opening/closure transition condition. More precisely, microcracks with normal $\boldsymbol{n}$ are referred to as open if $\hbar(\boldsymbol{n},\boldsymbol{\varepsilon}) > 0$ and, on the contrary, referred to as closed if $\hbar(\boldsymbol{n},\boldsymbol{\varepsilon}) < 0$. The explicit form of $\hbar(\boldsymbol{n},\boldsymbol{\varepsilon})$ will be given later.

It is worth noting that the conditions (18) ensure simultaneously stress continuity and energy continuity across the hypersurface that separates the two sub-domains describing the open state and the closed state of cracks, respectively.

3.3. Recovery condition on the elasticity modulus

Under the hypothesis that all cracks are planar and frictionless, the axial Young modulus, denoted by E, will be completely recovered when cracks become closed. To make appear the term related to the Young's modulus E, we first determine the effective compliance tensor $\mathbb{S}^{\text{eff},-}$ as the inverse of $\mathbb{C}^{\text{eff},-}$. According to the operation rule, one has

$$\mathbb{S}^{\text{eff},-} = \left(\frac{\lambda+2\mu-\varsigma_2^-}{\ell^-}, \frac{2\lambda+2\mu-\varsigma_1^-}{\ell^-}, \frac{1}{2\mu-\varsigma_3^-}, \frac{1}{2\mu-\varsigma_4^-}, \frac{\lambda-\varsigma_5^-}{\ell^c}, \frac{\lambda-\varsigma_5^-}{\ell^c}\right) \tag{23}$$

with $\ell^- = (2\lambda+2\mu-\varsigma_1^-)(\lambda+2\mu-\varsigma_2^-) - 2(\lambda-\varsigma_5^-)^2$.

As aforementioned, the coefficient associated with $\mathbb{E}^2$ is related to the axial Young's modulus as follows [30]:

$$\frac{2\lambda+2\mu-\varsigma_1^-}{\ell^-} = \frac{1}{E} \tag{24}$$

Further rearrangement of Eq. (24) gives

$$[\varsigma_1^-\varsigma_2^- - 2(\varsigma_5^-)^2] + [(E-\lambda-2\mu)\varsigma_1^- - 2(\lambda+\mu)\varsigma_2^- + 4\lambda\varsigma_5^-] = 0 \tag{25}$$

This equality must hold for any combination of the damage (implicitly involved in ς_i^-) and the elastic constants of the matrix, implying the following conditions

$$\begin{cases} \varsigma_1^-\varsigma_2^- - 2(\varsigma_5^-)^2 = 0 \\ (E-\lambda-2\mu)\varsigma_1^- - 2(\lambda+\mu)\varsigma_2^- + 4\lambda\varsigma_5^- = 0 \end{cases} \tag{26}$$

It follows

$$\varsigma_5^- = \nu\varsigma_1^-, \quad \varsigma_2^- = 2\nu^2\varsigma_1^- \tag{27}$$

where ν denotes the Poisson's ratio of the matrix phase.

3.4. Complementary conditions on local inelastic strain

Cracks are discontinuities in the solid matrix. Macroscopic strain of cracked solids can be decomposed into two additive parts, respectively attributed to the deformation of the matrix phase and the existence of microcracks. Denoting by $[\![\boldsymbol{u}]\!]$ the displacement jump vector between two crack lips, the contribution of a family of planar microcracks to the total strain takes the form [26,27]:

$$\boldsymbol{\varepsilon}^c = \mathcal{N}\int_{\omega^+} (\boldsymbol{n} \otimes^s [\![\boldsymbol{u}]\!])\mathrm{d}S = \beta\boldsymbol{N} + \boldsymbol{\gamma} \otimes^s \boldsymbol{n} \tag{28}$$

where the integration is performed over the upper surface ω^+ of the penny-shaped cracks; $\mathcal{N}$ is the crack number per unit volume; β and $\boldsymbol{\gamma}$ are the opening degree of cracks and the relative sliding between the surfaces in the crack plane, respectively. It is then possible to define

$$\beta = \mathcal{N}\int_{\omega^+} [\![u_n]\!]\mathrm{d}S; \quad \boldsymbol{\gamma} = \mathcal{N}\int_{\omega^+} [\![\boldsymbol{u}_t]\!]\mathrm{d}S, \quad \text{with } [\![\boldsymbol{u}_t]\!] = [\![\boldsymbol{u}]\!] - [\![u_n]\!]\boldsymbol{n} \tag{29}$$

The inelastic strain (28) is general. For open cracks or closed frictionless cracks, the macroscopic stress–strain relation reads

$$\boldsymbol{\sigma} = \mathbb{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^c) \tag{30}$$

or equivalently

$$\boldsymbol{\sigma} = (\mathbb{C} - \mathbb{C}^{d,r}) : \boldsymbol{\varepsilon}, \ r = +, - \tag{31}$$

$\boldsymbol{\varepsilon}^c$ is then related linearly to the total strain $\boldsymbol{\varepsilon}$

$$\boldsymbol{\varepsilon}^c = \mathbb{S} : \mathbb{C}^{d,r} : \boldsymbol{\varepsilon}, \ r = +, - \tag{32}$$

where $\mathbb{S} = \mathbb{C}^{-1}$ is the compliance tensor of the matrix phase and can be expressed in the Walpole's base as follows:

$$\mathbb{S} = \frac{1}{E}(1-\nu, 1, 1+\nu, 1+\nu, -\nu, -\nu) \tag{33}$$

When noticing the properties

$$\boldsymbol{\varepsilon}^c : \boldsymbol{N} = \boldsymbol{\varepsilon}^c : \boldsymbol{\delta} = \beta \tag{34}$$

we obtain by multiplying the two sides of Eq. (32) with $\boldsymbol{N}$

$$\beta = \frac{1}{E}\left[(\varsigma_2^r - 2\nu\varsigma_5^r)\boldsymbol{\varepsilon} : \boldsymbol{N} + (\varsigma_5^r - \nu\varsigma_1^r)\boldsymbol{\varepsilon} : \boldsymbol{T}\right] \tag{35}$$

and then with $\boldsymbol{\delta}$

$$\begin{aligned}\beta = {} & \frac{1}{E}\left[(\varsigma_2^r - 2\nu\varsigma_5^r)\boldsymbol{\varepsilon} : \boldsymbol{N} + (\varsigma_5^r - \nu\varsigma_1^r)\boldsymbol{\varepsilon} : \boldsymbol{T}\right] \\ & + \frac{1}{E}\left\{2[(1-\nu)\varsigma_5^r - \nu\varsigma_2^r]\boldsymbol{\varepsilon} : \boldsymbol{N} + [(1-\nu)\varsigma_1^r - 2\nu\varsigma_5^r]\boldsymbol{\varepsilon} : \boldsymbol{T}\right\}\end{aligned} \tag{36}$$

It follows by comparing Eqs. (35) and (36)

$$2[(1-\nu)\varsigma_5^r - \nu\varsigma_2^r]\boldsymbol{\varepsilon} : \boldsymbol{N} + [(1-\nu)\varsigma_1^r - 2\nu\varsigma_5^r]\boldsymbol{\varepsilon} : \boldsymbol{T} = 0 \tag{37}$$

which must hold for any strain state, implying that

$$\begin{cases}(1-\nu)\varsigma_5^r - \nu\varsigma_2^r = 0 \\ (1-\nu)\varsigma_1^r - 2\nu\varsigma_5^r = 0\end{cases} \tag{38}$$

For closed cracks, when $\varsigma_1^- \neq 0$, combination of Eqs. (27) and (38) gives

$$(1+\nu)(1-2\nu) = 0 \tag{39}$$

The solutions to function (39) are $\nu = -1$ and $\nu = 0.5$, which are, however, generally impossible for quasi-brittle materials. Therefore, we must have

$$\varsigma_1^- = \varsigma_2^- = \varsigma_5^- = 0 \tag{40}$$

From Eq. (38), one also derives the following relation between ς_i^+ for open microcracks

$$\varsigma_1^+ \varsigma_2^+ - 2(\varsigma_5^+)^2 = 0 \tag{41}$$

which can be verified by introduction of Eq. (40) into Eq. (18).

3.5. Condition on the strain energy

According to Eq. (11), in the case of open cracks, the strain free energy reads:

$$w^+ = \frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon} - \frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{C}^{d,+} : \boldsymbol{\varepsilon} \tag{42}$$

For the matrix-cracks system, the total energy should be the sum of two parts: the first one represents the elastic energy related to the solid matrix and the second one, named as the blocked (stored) energy and denoted by $w^b = w^b(d, \boldsymbol{\varepsilon}^c)$, is quadratic with respect to the strain $\boldsymbol{\varepsilon}^c$. The general form of the total free energy is then given by:

$$w^+ = \frac{1}{2}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^c) : \mathbb{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^c) + \frac{1}{2}\boldsymbol{\varepsilon}^c : \mathbb{C}^c : \boldsymbol{\varepsilon}^c \tag{43}$$

Above, the fourth-rank tensor $\mathbb{C}^c$ is function of the damage variable and the elastic properties of the matrix phase. By using Eq. (32), the equivalence between Eq. (42) and Eq. (43) leads to the equation:

$$\mathbb{C}^c = \mathbb{C} : [(\mathbb{S} : \mathbb{C}^{d,+})^{-1} - \mathbb{I}] \tag{44}$$

On the other hand, by making use of Eq. (38), one has

$$\mathbb{S} : \mathbb{C}^{d,+} = \frac{1}{E}\left(0, \varsigma_2^+ - 2\nu\varsigma_5^+, (1+\nu)\varsigma_3^+, (1+\nu)\varsigma_4^+, \varsigma_5^+ - \nu\varsigma_1^+, 0\right) \tag{45}$$

The inverse $(\mathbb{S} : \mathbb{C}^{d,+})^{-1}$ has to be performed indirectly because $\ell^+ = 0$. For this reason, Eq. (44) is reformulated in the form

$$\mathbb{C} = \left(\mathbb{C}^c + \mathbb{C}\right) : \left(\mathbb{S} : \mathbb{C}^{d,+}\right) \tag{46}$$

Multiplying the two sides of Eq. (46) with the fourth-order tensor $\left(\frac{E}{\varsigma_2^+ - 2\nu\varsigma_5^+}\mathbb{E}^2 + \frac{E}{(1+\nu)\varsigma_4^+}\mathbb{E}^4\right)$ gives

$$\frac{E^2}{(1+\nu)(1-2\nu)}\left(\frac{1-\nu}{\varsigma_2^+ - 2\nu\varsigma_5^+}\mathbb{E}^2 + \frac{1-2\nu}{(1+\nu)\varsigma_4^o}\mathbb{E}^4 + \frac{\nu}{\varsigma_2^+ - 2\nu\varsigma_5^+}\mathbb{E}^5\right) = (\mathbb{C}^c + \mathbb{C}) : (\mathbb{E}^2 + \mathbb{E}^4) \tag{47}$$

Next, using the properties between $\boldsymbol{\varepsilon}^c$ and $\mathbb{E}^i$

$$\mathbb{E}^1 : \boldsymbol{\varepsilon}^c = \mathbb{E}^3 : \boldsymbol{\varepsilon}^c = \mathbb{E}^5 : \boldsymbol{\varepsilon}^c = 0 \tag{48}$$

as well as the relation

$$\mathbb{E}^1 + \mathbb{E}^2 + \mathbb{E}^3 + \mathbb{E}^4 = \mathbb{I} \tag{49}$$

we have

$$\boldsymbol{\varepsilon}^c = \mathbb{I} : \boldsymbol{\varepsilon}^c = (\mathbb{E}^2 + \mathbb{E}^4) : \boldsymbol{\varepsilon}^c \tag{50}$$

The total strain energy (43) is finally expressed in the form

$$w^+ = \frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^c : \mathbb{C} : \boldsymbol{\varepsilon} + \frac{1}{2}\frac{E^2}{(1+\nu)(1-2\nu)}\boldsymbol{\varepsilon}^c : \left(\frac{1-\nu}{\varsigma_2^+ - 2\nu\varsigma_5^+}\mathbb{E}^2 + \frac{1-2\nu}{(1+\nu)\varsigma_4^+}\mathbb{E}^4\right) : \boldsymbol{\varepsilon}^c \tag{51}$$

or equivalently by introducing the relations (38)

$$w^+ = \frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^c : \mathbb{C} : \boldsymbol{\varepsilon} + \frac{1}{2}\boldsymbol{\varepsilon}^c : \left(\frac{2\lambda^2}{\varsigma_1^+}\mathbb{E}^2 + \frac{4\mu^2}{\varsigma_4^+}\mathbb{E}^4\right) : \boldsymbol{\varepsilon}^c \tag{52}$$

The thermodynamical forces associated with the variable $\boldsymbol{\varepsilon}^c$ can be determined by means of a standard differentiation

$$\boldsymbol{F}_\varepsilon = -\frac{\partial w^c}{\partial \boldsymbol{\varepsilon}^c} = \mathbb{C} : \boldsymbol{\varepsilon} - \left(\frac{2\lambda^2}{\varsigma_1^+} \mathbb{E}^2 + \frac{4\mu^2}{\varsigma_4^+} \mathbb{E}^4 \right) : \boldsymbol{\varepsilon}^c \tag{53}$$

$\boldsymbol{\varepsilon}^c$ being the inelastic strain by microcracks, $\boldsymbol{F}_\varepsilon$ can be viewed as the local stresses applied onto microcracks. Thus, at the crack opening-closure transition, we should have the cancellation of $\boldsymbol{F}_\varepsilon$, that is $\boldsymbol{F}_\varepsilon = 0$:

$$\mathbb{C} : \boldsymbol{\varepsilon} - \left(\frac{2\lambda^2}{\varsigma_1^+} \mathbb{E}^2 + \frac{4\mu^2}{\varsigma_4^+} \mathbb{E}^4 \right) : \boldsymbol{\varepsilon}^c = 0 \tag{54}$$

Back insertion into Eq. (52) gives

$$w^+ = \frac{1}{2} \boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon} - \frac{1}{2} \boldsymbol{\varepsilon} : \mathbb{C} : \left(\frac{\varsigma_1^+}{2\lambda^2} \mathbb{E}^2 + \frac{\varsigma_4^+}{4\mu^2} \mathbb{E}^4 \right) : \mathbb{C} : \boldsymbol{\varepsilon} \tag{55}$$

The effective stiffness tensor is then obtained

$$\mathbb{C}^{\text{eff},+} = \left(2\lambda + 2\mu - \varsigma_1^+, \lambda + 2\mu - \varsigma_2^+, 2\mu, 2\mu - \varsigma_4^+, \lambda - \varsigma_5^+, \lambda - \varsigma_5^+ \right) \tag{56}$$

Its comparison with the general form (15) allows us to determine the coefficients ς_3

$$\varsigma_3^+ = \varsigma_3^- = 0 \tag{57}$$

In summary, when the cracks are closed, the effective stiffness tensor takes the form

$$\mathbb{C}^{\text{eff},-} = \left(2\lambda + 2\mu, \lambda + 2\mu, 2\mu, 2\mu - \varsigma_4^-, \lambda, \lambda \right) \tag{58}$$

Recall that $\varsigma_4^- = \varsigma_4^+$ according to the condition (18).

3.6. Summary on the coefficients ς_i^r

The relationships between the combination coefficients $\varsigma_i^r, r = +, -, i = 1, ..., 5$ are finally summarized as follows

$$\begin{cases} (1-\nu)\varsigma_5^+ = \nu\varsigma_2^+ \\ (1-\nu)\varsigma_1^+ = 2\nu\varsigma_5^+ \\ \varsigma_1^- = \varsigma_2^- = \varsigma_5^- = 0 \\ \varsigma_3^+ = \varsigma_3^- = 0 \\ \varsigma_4^+ = \varsigma_4^- \neq 0 \end{cases} \tag{59}$$

Accordingly, the jump of the elasticity tensor is simplified into the form

$$\Delta\mathbb{C}^{\text{eff}} = \varsigma_1^+ \mathbb{E}^1 + \varsigma_2^+ \mathbb{E}^2 + \varsigma_5^+ (\mathbb{E}^5 + \mathbb{E}^6) \tag{60}$$

When accounting for Eq. (38), $\Delta\mathbb{C}$ also takes the form

$$\Delta\mathbb{C}^{\text{eff}} = \frac{\varsigma_1^+}{2\lambda^2} (\lambda \boldsymbol{\delta} + 2\mu \boldsymbol{N}) \otimes (\lambda \boldsymbol{\delta} + 2\mu \boldsymbol{N}) \tag{61}$$

which is consistent with the requirement in Eq. (21). The crack opening-closure transition condition finally reads

$$\hbar(\boldsymbol{n}, \boldsymbol{\varepsilon}) = (\lambda \boldsymbol{\delta} + 2\mu \boldsymbol{N}) : \boldsymbol{\varepsilon} = 0 \tag{62}$$

or equivalently

$$\hbar(\boldsymbol{n}, \boldsymbol{\varepsilon}) = \boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{N} = \boldsymbol{0} \tag{63}$$

3.7. Comparisons with the existing results for dilute microcracks

By applying the standard linear homogenization procedure to a system composed of a matrix and a family of penny-shaped microcracks, the total free energy predicted by the dilute scheme reads [33]

$$w^{\text{dilute}} = \frac{1}{2} \boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon} - \frac{1}{2} d\boldsymbol{\varepsilon} : \mathbb{C} : \left(\frac{1}{h_0} \mathbb{E}^2 + \frac{1}{2h_1} \mathbb{E}^4 \right) : \mathbb{C} : \boldsymbol{\varepsilon} \tag{64}$$

for open microcracks, and

$$w^{\text{dilute}} = \frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon} - \frac{1}{2} d\boldsymbol{\varepsilon} : \mathbb{C} : \left(\frac{1}{2h_1}\mathbb{E}^4\right) : \mathbb{C} : \boldsymbol{\varepsilon} \tag{65}$$

for closed frictionless microcracks, with $h_0 = \frac{3E}{16(1-\nu^2)}$ and $h_1 = h_0(1 - \frac{\nu}{2})$. Comparison of Eq. (64) with Eq. (55) leads to the relations:

$$\varsigma_1^+ = \frac{2d\lambda^2}{h_0}; \quad \varsigma_4^+ = \frac{2d\mu^2}{h_1} \tag{66}$$

Obviously, the same crack opening-transition condition as Eq. (63) can be derived. It is also found that the result by the dilute homogenization scheme is a particular case where microcracks are assumed to be of penny-shaped form.

4. Extension to the case of multiple crack families

We are concerned with the simplest case where the coefficients $\varsigma_i^r(d)$ are linear with respect to d. When accounting for the properties (13), the free energy w takes the general expression

$$w(\boldsymbol{\varepsilon}, d) = \frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon} - \frac{1}{2} d\boldsymbol{\varepsilon} : \mathbb{C} : \left(\kappa_n \mathbb{E}^2 + \kappa_t \mathbb{E}^4\right) : \mathbb{C} : \boldsymbol{\varepsilon} \tag{67}$$

Above, κ_n and κ_t are the model's parameters respectively associated with the normal and tangential operators. In the case of open cracks, the following relations are set down

$$\kappa_n = \frac{\varsigma_1^+}{2d\lambda^2}, \quad \kappa_t = \frac{\varsigma_4^+}{4d\mu^2} \tag{68}$$

For closed frictionless cracks, we should set $\kappa_n = 0$ in order to model the Young modulus' recovery.

From the free energy (67) is derived the effective stiffness tensor $\mathbb{C}^{\text{eff}}$ as follows:

$$\mathbb{C}^{\text{eff}} = \mathbb{C} - d\mathbb{C} : \left(\kappa_n \mathbb{E}^2 + \kappa_t \mathbb{E}^4\right) : \mathbb{C} \tag{69}$$

The macroscopic stress–strain relationship is given by

$$\boldsymbol{\sigma} = \mathbb{C}^{\text{eff}} : \boldsymbol{\varepsilon} \tag{70}$$

or equivalently $\boldsymbol{\sigma} = \mathbb{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^c)$ with

$$\boldsymbol{\varepsilon}^c = d\left(\kappa_n \mathbb{E}^2 + \kappa_t \mathbb{E}^4\right) : \mathbb{C} : \boldsymbol{\varepsilon} \tag{71}$$

4.1. Without interactions between microcracks

When no interactions between microcracks are taken into account, the extension of the basic results to the case of multiple crack families can be carried out by a simple summation. Under this simplification, the free energy is written in the form

$$w(\boldsymbol{\varepsilon}, \{d\}) = \frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon} - \frac{1}{2}\sum_{j=1}^{n} d^j \boldsymbol{\varepsilon} : \mathbb{C} : \left(\kappa_n^j \mathbb{E}^{2,j} + \kappa_t \mathbb{E}^{4,j}\right) : \mathbb{C} : \boldsymbol{\varepsilon} \tag{72}$$

The macroscopic stress–strain relationship is given as

$$\boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon} - \sum_{j=1}^{n} d^j \mathbb{C} : \left(\kappa_n^j \mathbb{E}^{2,j} + \kappa_t \mathbb{E}^{4,j}\right) : \mathbb{C} : \boldsymbol{\varepsilon} \tag{73}$$

or equivalently $\boldsymbol{\sigma} = \mathbb{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^c)$ with $\boldsymbol{\varepsilon}^c = \sum_{j=1}^{n} \epsilon^{c,j}$ and

$$\epsilon^{c,j} = d^j \left(\kappa_n^j \mathbb{E}^{2,j} + \kappa_t \mathbb{E}^{4,j}\right) : \mathbb{C} : \boldsymbol{\varepsilon} \tag{74}$$

For the rth crack family with normal n^r, the thermodynamic force associated with the damage variable d^r is derived from the free energy

$$F_{d^r} = -\frac{\partial w}{\partial d^r} = \frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{C} : \left(\kappa_n^r \mathbb{E}^{2,r} + \kappa_t \mathbb{E}^{4,r}\right) : \mathbb{C} : \boldsymbol{\varepsilon} \tag{75}$$

4.2. Account of crack interactions via effective strain

In Eq. (74), the local inelastic strain $\epsilon^{c,j}$ is linearly dependent on the macroscopic strain $\boldsymbol{\varepsilon}$, but independent on the inelastic strains. The formulation corresponds exactly to the dilute case of crack distributions. In order to improve the prediction of the effective properties, a simple way to take into account the effect of crack interactions is to introduce the concept of effective strain. For this, we assume that the local strain $\epsilon^{c,j}$ is related to the effective strain but not simply to the total strain. To this end, the following formula is constructed on the basis of the results for the dilute case

$$\epsilon^{c,j} = d^j \left(\kappa_n^j \mathbb{E}^{2,j} + \kappa_t \mathbb{E}^{4,j}\right) : \mathbb{C} : \left(\boldsymbol{\varepsilon} - \mathbb{P}^j : \boldsymbol{\varepsilon}^c\right) \tag{76}$$

in which the fourth-order tensor $\mathbb{P}$ is used to account for the impact of the total inelastic strain $\boldsymbol{\varepsilon}^c$ on the determination of local inelastic strain by microcracks. It follows:

$$\boldsymbol{\varepsilon}^c = \sum_{j=1}^{n} \epsilon^{c,j} = \sum_{j=1}^{n} d^j \left(\kappa_n^j \mathbb{E}^{2,j} + \kappa_t \mathbb{E}^{4,j}\right) : \mathbb{C} : \left(\boldsymbol{\varepsilon} - \mathbb{P}^j : \boldsymbol{\varepsilon}^c\right), \tag{77}$$

from which is derived the expression of $\boldsymbol{\varepsilon}^c$

$$\boldsymbol{\varepsilon}^c = \left[\mathbb{I} + \sum_{j=1}^{n} d^j \left(\kappa_n^j \mathbb{E}^{2,j} + \kappa_t \mathbb{E}^{4,j}\right) : \mathbb{C} : \mathbb{P}^j\right]^{-1} : \sum_{j=1}^{n} d^j \left(\kappa_n^j \mathbb{E}^{2,j} + \kappa_t \mathbb{E}^{4,j}\right) : \mathbb{C} : \boldsymbol{\varepsilon} \tag{78}$$

Recall that in the dilute case, the modification by microcracks to the overall stiffness reads:

$$\mathbb{C}^d = \sum_{j=1}^{n} d^j \mathbb{C} : \left(\kappa_n^j \mathbb{E}^{2,j} + \kappa_t \mathbb{E}^{4,j}\right) : \mathbb{C} \tag{79}$$

It is usually quite difficult to determine the $\mathbb{P}$-tensor theoretically or even numerically. For simplicity, we assume that the $\mathbb{P}$-tensor is identical for all considered crack families. One then obtains the effective stiffness tensor

$$\mathbb{C}^{\text{eff}} = \mathbb{C} - \left[\mathbb{I} + \mathbb{C}^d : \mathbb{P} : \mathbb{S}\right]^{-1} : \mathbb{C}^d \tag{80}$$

where $\mathbb{C}^d$ has been defined in Eq. (79). The prediction (80) is in the same form as that in some previous works [34,35]. Particularly, when $\mathbb{P} = \mathbb{I}$, the effective strain corresponds to the elastic strain. Accordingly, one has

$$\mathbb{C}^{\text{eff}} = \left[\mathbb{I} + \mathbb{C}^d : \mathbb{S}\right]^{-1} : \mathbb{C} \tag{81}$$

which is exactly the prediction by the Mori–Tanaka method. The effective compliance tensor $\mathbb{S}^{\text{eff}} = (\mathbb{C}^{\text{eff}})^{-1}$ takes the form

$$\mathbb{S}^{\text{eff}} = \mathbb{S} + \mathbb{S} : \mathbb{C}^d : \mathbb{S} = \mathbb{S} + \sum_{j=1}^{n} d^j \left(\kappa_n^j \mathbb{E}^{2,j} + \kappa_t \mathbb{E}^{4,j}\right) \tag{82}$$

The Gibbs energy is given by

$$\psi(\boldsymbol{\sigma}, \{d\}) = \frac{1}{2}\boldsymbol{\sigma} : \mathbb{S}^{\text{eff}} : \boldsymbol{\sigma} = \frac{1}{2}\boldsymbol{\sigma} : \mathbb{S} : \boldsymbol{\sigma} + \frac{1}{2}\sum_{j=1}^{n} d^j \boldsymbol{\sigma} : \left(\kappa_n^j \mathbb{E}^{2,j} + \kappa_t \mathbb{E}^{4,j}\right) : \boldsymbol{\sigma} \tag{83}$$

Recall that the constant κ_n^j is zero-valued when microcracks in the jth family are closed. From the difference $\Delta\psi$ is derived the opening/closure transition condition for microcracks with normal $\boldsymbol{n}$

$$\boldsymbol{\sigma} : (\boldsymbol{n} \otimes \boldsymbol{n}) = 0 \tag{84}$$

Differing from Eq. (63) derived under the dilute condition, the above multilinear function is expressed in the stress space.

5. Formulations and difficulties with high-order damage variables

We are now in a position to derive constitutive formulations using high-order damage variables. More precisely, we define a second-order tensor and a fourth-order damage tensor, as classically performed in mesoscopic damage models:

$$\boldsymbol{D} = \sum_{j=1}^{n} d^j \boldsymbol{n}^j \otimes \boldsymbol{n}^j, \quad \mathbb{D} = \sum_{j=1}^{n} d^j \boldsymbol{n}^j \otimes \boldsymbol{n}^j \otimes \boldsymbol{n}^j \otimes \boldsymbol{n}^j \tag{85}$$

With these two damage variables, the strain free energy (72) becomes

$$w^{+}(\boldsymbol{\varepsilon},\boldsymbol{D},\mathbb{D})=\frac{1}{2}\boldsymbol{\varepsilon}:\mathbb{C}:\boldsymbol{\varepsilon}-\frac{1}{2}\boldsymbol{\varepsilon}:\mathbb{C}:\left[\kappa_t\left(\boldsymbol{D}\otimes^{s}\boldsymbol{\delta}+\boldsymbol{\delta}\otimes^{s}\boldsymbol{D}\right)+(k_n-2k_t)\,\mathbb{D}\right]:\mathbb{C}:\boldsymbol{\varepsilon} \tag{86}$$

for the case of open cracks, and

$$w^{-}(\boldsymbol{\varepsilon},\boldsymbol{D},\mathbb{D})=\frac{1}{2}\boldsymbol{\varepsilon}:\mathbb{C}:\boldsymbol{\varepsilon}-\frac{1}{2}\boldsymbol{\varepsilon}:\mathbb{C}:\left[\kappa_t\left(\boldsymbol{D}\otimes^{s}\boldsymbol{\delta}+\boldsymbol{\delta}\otimes^{s}\boldsymbol{D}\right)-2k_t\mathbb{D}\right]:\mathbb{C}:\boldsymbol{\varepsilon} \tag{87}$$

for the case of closed frictionless cracks. The difference is then obtained as follows:

$$\Delta w=-\frac{1}{2}k_n\boldsymbol{\varepsilon}:\mathbb{C}:\mathbb{D}:\mathbb{C}:\boldsymbol{\varepsilon} \tag{88}$$

Now, difficulties arise. We can derive without difficulty the stress–strain relations as well as the effective tensor from both Eqs. (86) and (87). However, we do not know how to set up an opening/closure transition condition from Eq. (88) which should be a multilinear function such as Eq. (22) in order to define a hypersurface separating the cases of open and closed cracks. To handle this troublesome theoretical problem, Halm and Dragon [8] proposed a spectral decomposition upon the second-order tensor $\boldsymbol{D}$ in such a way that

$$\boldsymbol{D}=\sum_{i=1}^{3}D_i\boldsymbol{N}^i\otimes\boldsymbol{N}^i \tag{89}$$

Then, both the eigenvalues D_i and the principal directions $\boldsymbol{N}^i$ are used to discretize the fourth-order damage tensor $\mathbb{D}$, that is

$$\mathbb{D}=\sum_{i=1}^{3}D_i\boldsymbol{N}^i\otimes\boldsymbol{N}^i\otimes\boldsymbol{N}^i\otimes\boldsymbol{N}^i \tag{90}$$

As have been commented by Cormery and Welemane [9], this operation is a very grossly simplifying approximation.

In order to get out of this troublesome situation, a potential routine is to take into account the dissipative mechanism by frictional sliding. For closed cracks, it is reasonable to perform damage–friction coupling analyses. On this aspect, constitutive formulations under isotropic assumptions [36,37] as well as anisotropic damage formulations [27,28,38] have been achieved. As for the development involving higher-order damage variables, one can benefit directly form the procedure established in the above works.

6. Concluding remarks

The mesoscopic thermodynamic damage formulations accounting for both induced material anisotropies and unilateral effects have been achieved in this work. Important links between mesoscopic constitutive equations and microscopic ones are originally set up. Some further remarks are made as follows.

For a family of non-interacting microcracks, the strain free energy formulated as a linear combination of five invariants takes the same form as the dilute approximation obtained by applying the linear homogenization approach. However, the latter strongly relies on the penny-shaped assumption for microcracks in order to make direct use of the Eshelby's solution, while the present results have released this constraint: microcracks should be planar but not limited to be a disc plane (penny-shaped).

For the case of multiple interacting crack families, the present results allows a new insight into the Ponte-Castaneda and Willis's homogenization scheme [34]: to determine the local strain inside inhomogeneities, each inhomogeneity can be treated as an isolated inclusion embedded in the same solid matrix, which, however, is subjected to an effective uniform strain at remote boundaries. In particular, for the Mori–Tanaka scheme, this effective macroscopic uniform strain takes the mean strain measured over the matrix phase (the total elastic strain).

In the *purely elastic* damage context, macroscopic phenomenological approaches even with high-order variables have met more or less difficulties, especially in the simultaneous description of induced material anisotropies and unilateral effects. It is now shown that the key difficulty resides in deriving an opening/closure transition condition in terms of high-order tensors.

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