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On Hashin–Shtrikman-type bounds for nonlinear conductors

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ABSTRACT

For linear composite conductors, it is known that the celebrated Hashin–Shtrikman bounds can be recovered by the translation method. We investigate whether the same conclusion extends to nonlinear composites in two dimensions. To that purpose, we consider two-phase composites with perfectly conducting inclusions. In that case, explicit expressions of the various bounds considered can be obtained. The bounds provided by the translation method are compared with the nonlinear Hashin–Shtrikman-type bounds delivered by the Talbot–Willis (1985) [2] and the Ponte Castañeda (1991) [3] procedures.

1. Introduction

In a seminal work [1], Hashin and Shtrikman have obtained optimal bounds on the effective conductivity of linear composite conductors with statistically isotropic microstructures. Those bounds are explicit functions of the volume fractions and conductivities of each constitutive phase. Several methods have been proposed to extend the results of Hashin and Shtrikman to nonlinear composites. A first method, proposed by Talbot and Willis [2], makes uses of a *homogeneous* linear comparison medium and generalizes the variational approach introduced by Hashin and Shtrikman. A second method, due to Ponte Castañeda [3], employs a *heterogeneous* linear comparison medium, i.e. a linear comparison composite. Using that last method, any bound on the effective conductivity of the linear comparison composite can be used to generate a corresponding bound for the nonlinear composite. In particular, when the linear Hashin–Shtrikman bound is used, nonlinear Hashin–Shtrikman-type bounds are obtained. A third method, known has the translation method [4], has been introduced independently by Lurie and Cherkaev [5], Murat and Tartar [6]. Originally introduced in the linear context, that method has proved to be very fruitful in a lot of nonlinear homogenization problems [7–12]. For nonlinear isotropic conductors, the translation method has been used to obtain explicit bounds for composites governed by threshold-type energy functions [13,14].

The three methods mentioned above can generate nonlinear bounds of the Hashin–Shtrikman type, i.e. bounds that hold for the whole class of isotropic composites with prescribed volume fractions and conduction properties of the individual phases. As those methods are not mathematically equivalent, it is important to understand the relations between them. The relation between the Talbot–Willis and the Ponte Castañeda methods has been investigated in [15]: for two-phase composites with perfectly conducting inclusions, the two methods have been proved to give the same results if the energy function of the matrix satisfies a certain strong convexity condition. If that condition is violated, the Talbot–Willis method leads to stronger bounds than the Ponte Castañeda method. The relations with the translation method have been studied in [13]: Bounds obtained from the translation method have been proved to be always at least as good as those provided by

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the Ponte Castañeda method. For two-dimensional composites governed by threshold-type energy functions, the translation method actually gives the same bounds as the Ponte Castañeda method. Numerical results suggest that the same conclusion extends to power-law composites, although a rigorous proof is still lacking.

The objective of this paper is to fill some of the gaps in the relations between the translation method and the methods of Ponte Castañeda, Talbot and Willis by investigating conditions under which the translation method may bring a genuine improvement. To that purpose, we consider two-phase composites with perfectly conducting inclusions, in two dimensions. The formulation of the translation method in that context is presented in Sect. 2. If the energy function of the matrix satisfies the strong convexity assumption introduced in [15], we prove in Sect. 3 that the translation method gives the same bounds as the Talbot–Willis and Ponte Castañeda methods. This confirms the numerical observations made in [13] for the special case of power-law composites. For the translation method to bring a genuine improvement, it is therefore necessary that the energy in the matrix is not strongly convex. Building on that observation, in Sect. 4 we provide an example for which bounds obtained from the translation method are indeed stronger than the Talbot–Willis and Ponte Castañeda bounds.

2. Bounding the effective energy via the translation method

Consider a two-dimensional inhomogeneous electric conductor occupying a domain Ω of unit volume. The electric field $\boldsymbol{e}$ and the current density $\boldsymbol{j}$ are related by the local constitutive law

$$\boldsymbol{j} = \frac{\partial w}{\partial \boldsymbol{e}}(\boldsymbol{e}, \boldsymbol{x}) \tag{1}$$

where the convex energy-density function w depends on the location $\boldsymbol{x}$. Denoting by $\bar{\boldsymbol{e}}$ (resp. $\bar{\boldsymbol{j}}$) the spatial average of $\boldsymbol{e}$ (resp. $\boldsymbol{j}$), the effective constitutive law reads as [16,17]

$$\bar{\boldsymbol{j}} = \frac{\mathrm{d}w_{\text{eff}}}{\mathrm{d}\bar{\boldsymbol{e}}}(\bar{\boldsymbol{e}}) \tag{2}$$

where w_{eff} is the effective energy function of the composite material, defined by

$$w_{\text{eff}}(\bar{\boldsymbol{e}}) = \inf_{\boldsymbol{e} \in K(\bar{\boldsymbol{e}})} \int_{\Omega} w(\boldsymbol{e}, \boldsymbol{x}) \, \mathrm{d}\omega \tag{3}$$

In (3), $K(\bar{\boldsymbol{e}})$ is the set of admissible electric fields, as defined by

$$K(\bar{\boldsymbol{e}}) = \{\boldsymbol{e} : \Omega \mapsto \mathbb{R}^2 | \boldsymbol{e} = \nabla V \text{ for some } V : \Omega \mapsto \mathbb{R} \text{ verifying } V(\boldsymbol{x}) = \bar{\boldsymbol{e}} \cdot \boldsymbol{x} \text{ on } \partial\Omega\}$$

Following [5,6], a lower bound on w_{eff} can be obtained by embedding the original problem in a problem of dimension 4. In more detail, we introduce extended fields $\boldsymbol{E}(\boldsymbol{x}) = (\boldsymbol{e}_1(\boldsymbol{x}), \boldsymbol{e}_2(\boldsymbol{x}))$ obtained by considering 2 electric fields $\boldsymbol{e}_1(\boldsymbol{x})$ and $\boldsymbol{e}_2(\boldsymbol{x})$, written side by side. Introducing the *extended energy*

$$W(\boldsymbol{E}, \boldsymbol{x}) = w(\boldsymbol{e}_1, \boldsymbol{x}) + w(\boldsymbol{e}_2, \boldsymbol{x})$$

as well as the *extended effective energy*

$$W_{\text{eff}}(\bar{\boldsymbol{E}}) = w_{\text{eff}}(\bar{\boldsymbol{e}}_1) + w_{\text{eff}}(\bar{\boldsymbol{e}}_2) \tag{4}$$

it can be readily seen from (3) that

$$W_{\text{eff}}(\bar{\boldsymbol{E}}) = \inf_{\boldsymbol{E} \in \mathcal{K}(\bar{\boldsymbol{E}})} \int_{\Omega} W(\boldsymbol{E}, \boldsymbol{x}) \, \mathrm{d}\omega \tag{5}$$

where $\bar{\boldsymbol{E}} = (\bar{\boldsymbol{e}}_1, \bar{\boldsymbol{e}}_2)$ and $\mathcal{K}(\bar{\boldsymbol{E}}) = \{(\boldsymbol{e}_1, \boldsymbol{e}_2) : \boldsymbol{e}_i \in K(\bar{\boldsymbol{e}}_i)\}$. We now proceed to bound W_{eff} from below. To that purpose, it is convenient to represent extended fields $\boldsymbol{E} = (\boldsymbol{e}_1, \boldsymbol{e}_2)$ by 2×2 matrices, i.e.

$$\boldsymbol{E} = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}$$

where u_i and v_i are the components of the electric field $\boldsymbol{e}_i$ in a reference basis of $\mathbb{R}^2$. For any scalar α and any $\boldsymbol{T}$ in $\mathbb{R}^{2\times 2}$, consider the Legendre transform

$$(W - \alpha \det)^*(\boldsymbol{T}, \boldsymbol{x}) = \sup_{\boldsymbol{E}} \boldsymbol{E} : \boldsymbol{T} - W(\boldsymbol{E}, \boldsymbol{x}) + \alpha \det \boldsymbol{E} \tag{6}$$

where $\boldsymbol{E} : \boldsymbol{T} = \operatorname{tr} \boldsymbol{E}\boldsymbol{T}$ and det is the determinant in $\mathbb{R}^2$, i.e. $\det \boldsymbol{E} = u_1 v_2 - u_2 v_1$. For any $\boldsymbol{E} \in \mathcal{K}(\bar{\boldsymbol{E}})$, it follows from (6) that

$$W(\boldsymbol{E}, \boldsymbol{x}) \geq \boldsymbol{E}(\boldsymbol{x}) : \boldsymbol{T} + \alpha \det \boldsymbol{E}(\boldsymbol{x}) - (W - \alpha \det)^*(\boldsymbol{T}, \boldsymbol{x}) \tag{7}$$

Any $\boldsymbol{E} \in \mathcal{K}(\bar{\boldsymbol{E}})$ satisfies the well-known identities $\int_\Omega \boldsymbol{E}(\boldsymbol{x})\,\mathrm{d}\omega = \bar{\boldsymbol{E}}$ and $\int_\Omega \det \boldsymbol{E}(\boldsymbol{x})\,\mathrm{d}\omega = \det \bar{\boldsymbol{E}}$. Hence, integrating (7) over the domain Ω yields

$$\int_\Omega W(\boldsymbol{E},\boldsymbol{x})\,\mathrm{d}\omega \geq \bar{\boldsymbol{E}} : \boldsymbol{T} + \alpha \det \bar{\boldsymbol{E}} - \int_\Omega (W - \alpha \det)^*(\boldsymbol{T},\boldsymbol{x})\,\mathrm{d}\omega \tag{8}$$

for any $\boldsymbol{E} \in \mathcal{K}(\bar{\boldsymbol{E}})$. Taking the infimum over fields $\boldsymbol{E}$ in $\mathcal{K}(\bar{\boldsymbol{E}})$ gives

$$W_{\text{eff}}(\bar{\boldsymbol{E}}) \geq \bar{\boldsymbol{E}} : \boldsymbol{T} + \alpha \det \bar{\boldsymbol{E}} - \int_\Omega (W - \alpha \det)^*(\boldsymbol{T},\boldsymbol{x})\,\mathrm{d}\omega \tag{9}$$

The right-hand side of (9) is thus a lower bound on the extended energy W_{eff}. A lower bound on w_{eff} can be deduced from (9) if the composite is isotropic. In such case, w_{eff} indeed depends on the electric field $\bar{\boldsymbol{e}}$ only through its norm $\bar{e}$, i.e. there exists a function ϕ_{eff} such that $w_{\text{eff}}(\bar{\boldsymbol{e}}) = \phi_{\text{eff}}(\bar{e})$. Consequently, we have $W_{\text{eff}}(\bar{\boldsymbol{e}}_1, \bar{\boldsymbol{e}}_2) = \phi_{\text{eff}}(\bar{e}_1) + \phi_{\text{eff}}(\bar{e}_2)$ where $\bar{e}_i = \|\bar{\boldsymbol{e}}_i\|$. Specializing (9) to $\bar{\boldsymbol{E}}$ of the form $\bar{e}\boldsymbol{N}$ with $\bar{e} \geq 0$ and

$$\boldsymbol{N} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

we thus obtain

$$w_{\text{eff}}(\bar{\boldsymbol{e}}) \geq \frac{1}{2}\left(\bar{e}\boldsymbol{N} : \boldsymbol{T} - \alpha \bar{e}^2 - \int_\Omega (W - \alpha \det)^*(\boldsymbol{T},\boldsymbol{x})\,\mathrm{d}\omega\right) \tag{10}$$

The best bound $w_{\mathsf{T}}(\bar{\boldsymbol{e}})$ delivered by the presented procedure (known as the translation method) is obtained by maximizing the right-hand side of (10) with respect to $(\boldsymbol{T}, \alpha)$, i.e.

$$w_{\mathsf{T}}(\bar{\boldsymbol{e}}) = \sup_{\boldsymbol{T},\alpha} \frac{1}{2}\left(\bar{e}\boldsymbol{N} : \boldsymbol{T} - \alpha \bar{e}^2 - \int_\Omega (W - \alpha \det)^*(\boldsymbol{T},\boldsymbol{x})\,\mathrm{d}\omega\right) \tag{11}$$

The bound w_{T} is relevant only if $w(\boldsymbol{e},\boldsymbol{x})$ has faster than quadratic growth in $\boldsymbol{e}$ (so as to ensure that $(W - \alpha \det)^*(\boldsymbol{T},\boldsymbol{x}) < \infty$). For an energy function $w(\boldsymbol{e},\boldsymbol{x})$ that is quadratic in $\boldsymbol{e}$, the bound (11) is known to coincide with the Hashin–Shtrikman bound [5,6].

In the rest of the paper, we consider two-phase composites constituted of perfectly conducting inclusions in a nonlinear matrix. At a point $\boldsymbol{x}$ in the perfectly conducting phase, the energy $w(\boldsymbol{e},\boldsymbol{x})$ vanishes if $\boldsymbol{e} = 0$ and is equal to $+\infty$ if $\boldsymbol{e} \neq 0$, so that $(W - \alpha \det)^*(\boldsymbol{T},\boldsymbol{x}) = 0$. To alleviate the notations, from now on we drop the $\boldsymbol{x}$ dependence of the energy function: from here on, w (resp. W) simply denotes the energy function (resp. extended energy function) of the matrix. The bound (11) becomes

$$w_{\mathsf{T}}(\bar{\boldsymbol{e}}) = \sup_{\boldsymbol{T},\alpha} w_{\mathsf{T}}(\bar{\boldsymbol{e}}; \boldsymbol{T}, \alpha) \tag{12}$$

where

$$w_{\mathsf{T}}(\bar{\boldsymbol{e}}; \boldsymbol{T}, \alpha) = \frac{1}{2}\left(\bar{e}\boldsymbol{N} : \boldsymbol{T} - \alpha \bar{e}^2 - c(W - \alpha \det)^*(\boldsymbol{T})\right) \tag{13}$$

In (13), c is the volume fraction of the matrix.

3. Two-phase composite with a strongly convex energy function

In this section, we assume that w is isotropic and *strongly convex*, in the sense that w is convex in e^2. Moreover, we assume that w is differentiable and increasing in e with a larger than quadratic growth. An example of energy functions satisfying those assumptions is provided by power-law functions σe^{n+1} with $n > 1$ and $\sigma > 0$.

As a consequence of the stated assumptions on w, we can write the derivative $w'(\boldsymbol{e})$ as

$$w'(\boldsymbol{e}) = h(e)\boldsymbol{e}$$

where h is a positive, unbounded, monotonically increasing function of the norm e. For later reference, we record the expression of the Talbot–Willis bound:

$$w_{\text{TW}}(\bar{\boldsymbol{e}}) = c\, w\left(\frac{\bar{\boldsymbol{e}}\sqrt{2-c}}{c}\right) \tag{14}$$

The result (14) can also be obtained by using the Ponte Castañeda method in conjunction with the Hashin–Shtrikman bound [1] for the linear comparison composite [18].

The goal of this section is to show that the translated bound (13) also coincides with (14). This is accomplished by finding values $(\tilde{\tau}, \tilde{\alpha})$ such that $w_{\mathsf{T}}(\bar{\boldsymbol{e}}; \tilde{\tau}\boldsymbol{N}, \tilde{\alpha}) = w_{\mathrm{TW}}(\bar{\boldsymbol{e}})$ and subsequently showing that the obtained values $(\tilde{\tau}\boldsymbol{N}, \tilde{\alpha})$ maximize the function $(\boldsymbol{T}, \alpha) \mapsto w_{\mathsf{T}}(\bar{\boldsymbol{e}}; \boldsymbol{T}, \alpha)$.

3.1. Bounds obtained for $\boldsymbol{T}$ parallel to $\boldsymbol{N}$

For a matrix $\boldsymbol{T}$ of the form $\boldsymbol{T} = \tau\boldsymbol{N}$, we have

$$(W - \alpha \det)^*(\tau\boldsymbol{N}) = \sup_{\boldsymbol{E}=\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}} f(\boldsymbol{E}; \tau, \alpha) \tag{15}$$

where

$$f(\boldsymbol{E}; \tau, \alpha) = \tau(v_1 + u_2) - w(\boldsymbol{e}_1) - w(\boldsymbol{e}_2) + \alpha(u_1 v_2 - u_2 v_1) \tag{16}$$

Setting $h_i = h(e_i)$, the stationarity conditions in (16) read as

$$\begin{aligned} -h_1 u_1 + \alpha v_2 = 0, \quad & \tau - h_1 v_1 - \alpha u_2 = 0 \\ -h_2 v_2 + \alpha u_1 = 0, \quad & \tau - h_2 u_2 - \alpha v_1 = 0 \end{aligned} \tag{17}$$

As a result of the nonconvexity of the function $f(\cdot; \tau, \alpha)$, the system (17) generally admits multiple solutions. Limiting the discussion to the case $\tau, \alpha \geq 0$, a close inspection shows that there are three branches of solutions to (17):

- branch 1 corresponds to solutions of the form

$$\boldsymbol{E} = \begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix} \text{ with } \tau = h(v)v + \alpha v \tag{18}$$

  Since the function $v \mapsto h(v)v + \alpha v$ is strictly increasing from 0 to ∞ (in the domain $v \geq 0$), there is a unique $v \geq 0$ that satisfies the relation $\tau = h(v)v + \alpha v$ in (18). The value taken by $f(\cdot; \tau, \alpha)$ for the matrix $\boldsymbol{E}$ in (18) is denoted by $f_1(\tau, \alpha)$;
- branch 2 corresponds to solutions of the form

$$\boldsymbol{E} = \begin{pmatrix} 0 & u_2 \\ v_1 & 0 \end{pmatrix} \text{ with } u_2 \neq v_1, \tau = h_1 v_1 + \alpha u_2 = h_2 u_2 + \alpha v_1 \tag{19}$$

  We denote by $f_2(\tau, \alpha)$ the maximum value taken by $f(\cdot; \tau, \alpha)$ over $\boldsymbol{E}$ of the form (19);
- branch 3 corresponds to solutions of the form

$$\boldsymbol{E}_{\pm} = \begin{pmatrix} \pm u_1 & \frac{\tau}{2\alpha} \\ \frac{\tau}{2\alpha} & \pm u_1 \end{pmatrix} \text{ with } h\left(\sqrt{u_1^2 + \frac{\tau^2}{4\alpha^2}}\right) = \alpha \tag{20}$$

  Such $\boldsymbol{E}_{\pm}$ exists as long as $h(0) \leq \alpha$ and $\tau \leq \tau_c(\alpha)$. Here $\tau_c(\alpha) \geq 0$ is the scalar defined implicitly by

$$h\left(\frac{\tau_c(\alpha)}{2\alpha}\right) = \alpha \tag{21}$$

  It can easily be verified that $f(\boldsymbol{E}_+; \tau, \alpha) = f(\boldsymbol{E}_-; \tau, \alpha)$. That common value, denoted by $f_3(\tau, \alpha)$, can be expressed in terms of $\tau_c(\alpha)$ as

$$f_3(\tau, \alpha) = \frac{\tau^2}{2\alpha} + \frac{\tau_c^2(\alpha)}{4\alpha} - 2w\left(\frac{\tau_c(\alpha)}{2\alpha}\frac{\bar{\boldsymbol{e}}}{\bar{e}}\right)$$

The function $(W - \alpha \det)^*(\tau\boldsymbol{N})$ is the maximum of the three branches detailed above, i.e.

$$(W - \alpha \det)^*(\tau\boldsymbol{N}) = \max_{1 \leq i \leq 3} f_i(\tau, \alpha) \tag{22}$$

A general explicit expression of $(W - \alpha \det)^*(\tau\boldsymbol{N})$ – holding for arbitrary values of (τ, α) and any form of the energy w – remains out of reach. What can be proved, however, is that

$$(W - \alpha \det)^*(\tau\boldsymbol{N}) = f_3(\tau, \alpha) \text{ for } 0 \leq \tau \leq \tau_c(\alpha) \text{ and } h(0) \leq \alpha \tag{23}$$

Let us justify the property (23). We consider $\alpha \geq h(0)$ as fixed and denote by $v(\tau) \geq 0$ the solution to the equation

$$\tau = h(v(\tau))v(\tau) + \alpha v(\tau) \tag{24}$$

so that

$$f_1(\tau,\alpha) = 2\tau v(\tau) - 2w\left(v(\tau)\frac{\bar{\boldsymbol{e}}}{\bar{e}}\right) - \alpha v(\tau)^2 \tag{25}$$

It can easily be verified from (21) and (24) that

$$v(\tau_c(\alpha)) = \frac{\tau_c(\alpha)}{2\alpha}$$

For $v \geq 0$, the function $v \mapsto h(v)v + \alpha v$ is strictly increasing from 0 to ∞. Hence $\tau \mapsto v(\tau)$ is increasing with τ, so that

$$v(\tau) \leq v(\tau_c(\alpha)) = \frac{\tau_c(\alpha)}{2\alpha} \text{ for } 0 \leq \tau \leq \tau_c(\alpha) \tag{26}$$

Using (24) and (25), we obtain by differentiation

$$\frac{\partial}{\partial\tau}(f_1 - f_3)(\tau,\alpha) = 2v(\tau) - \frac{\tau}{\alpha} = \frac{v(\tau)}{\alpha}\Big(\alpha - h(v(\tau))\Big) \tag{27}$$

In view of (26) and recalling that h is increasing, we have $h(v(\tau)) \leq h(\frac{\tau_c(\alpha)}{2\alpha}) = \alpha$ for $\tau \leq \tau_c(\alpha)$. Hence

$$0 \leq \frac{\partial}{\partial\tau}(f_1 - f_3)(\tau,\alpha) \text{ for } 0 \leq \tau \leq \tau_c(\alpha)$$

This shows that the function $\tau \mapsto (f_1 - f_3)(\tau,\alpha)$ is increasing on $[0, \tau_c(\alpha)]$. Hence,

$$(f_1 - f_3)(\tau,\alpha) \leq (f_1 - f_3)(\tau_c(\alpha),\alpha) \text{ for } 0 \leq \tau \leq \tau_c(\alpha)$$

For $\tau = \tau_c(\alpha)$, it can easily be verified that the matrices $\boldsymbol{E}$ in (18) and $\boldsymbol{E}_\pm$ in (20) coincide and are equal to $\frac{\tau_c(\alpha)}{2\alpha}\boldsymbol{N}$. Therefore, $(f_1 - f_3)(\tau_c(\alpha),\alpha) = 0$. We thus obtain that

$$(f_1 - f_3)(\tau,\alpha) \leq 0 \text{ for } 0 \leq \tau \leq \tau_c(\alpha)$$

A similar reasoning can be used to show that

$$(f_2 - f_3)(\tau,\alpha) \leq 0 \text{ for } 0 \leq \tau \leq \tau_c(\alpha)$$

Substituting in (22) gives the desired result (23). □

The property (23) allows us to evaluate the bound $w_\mathsf{T}(\bar{\boldsymbol{e}}; \tau\boldsymbol{N}, \alpha)$ for any (τ,α) that satisfy the conditions $0 \leq \tau \leq \tau_c(\alpha)$ and $h(0) \leq \alpha$. In particular, consider the special values defined by

$$\tilde{\alpha} = h\left(\frac{\bar{e}\sqrt{2-c}}{c}\right), \quad \frac{\tilde{\tau}}{2\tilde{\alpha}} = \frac{\bar{e}}{c} \tag{28}$$

Since h is increasing, we necessarily have $h(0) \leq \tilde{\alpha}$. We also note from (28) and (21) that $\tau_c(\tilde{\alpha})/2\tilde{\alpha} = \bar{e}\sqrt{2-c}/c$, hence $\tau_c(\tilde{\alpha}) \geq \tilde{\tau}$. Use of (23) yields

$$(W - \tilde{\alpha}\det)^*(\tilde{\tau}\boldsymbol{N}) = f_3(\tilde{\tau},\tilde{\alpha}) = (4-c)\tilde{\alpha}\left(\frac{\bar{e}}{c}\right)^2 - 2w\left(\frac{\bar{\boldsymbol{e}}\sqrt{2-c}}{c}\right)$$

Substituting in (13) gives

$$w_\mathsf{T}(\bar{\boldsymbol{e}}; \tilde{\tau}\boldsymbol{N}, \tilde{\alpha}) = \frac{1}{2}\Big(2\tilde{\tau}\bar{e} - \tilde{\alpha}\bar{e}^2 - cf_3(\tilde{\tau},\tilde{\alpha})\Big) = c\,w\left(\frac{\bar{\boldsymbol{e}}\sqrt{2-c}}{c}\right) \tag{29}$$

which shows that the translated bound $w_\mathsf{T}(\bar{\boldsymbol{e}})$ in (12) is at least as good as the Talbot–Willis bound $w_{\mathrm{TW}}(\bar{\boldsymbol{e}})$ in (14). In the next section, we show that the values $(\tilde{\tau}\boldsymbol{N}, \tilde{\alpha})$ are in fact optimal in (12), i.e. that the translated bound $w_\mathsf{T}(\bar{\boldsymbol{e}})$ is equal to $w_{\mathrm{TW}}(\bar{\boldsymbol{e}})$.

3.2. Optimized bound

Observe that the bound (12) is defined by a *concave optimization* problem over $(\boldsymbol{T}, \alpha)$. The function $(W - \alpha \det)^*(\boldsymbol{T})$ is indeed convex in $(\boldsymbol{T}, \alpha)$, since it is defined in (6) as the pointwise supremum of a family of linear functions in $(\boldsymbol{T}, \alpha)$. Hence the values of $(\boldsymbol{T}, \alpha)$ reaching the supremum in (12) are fully characterized by the optimality conditions

$$\frac{1}{c}(\bar{e}\boldsymbol{N}, -\bar{e}^2) \in \partial (W - \alpha \det)^*(\boldsymbol{T}) \tag{30}$$

where $\partial (W - \alpha \det)^*(\boldsymbol{T})$ is the subdifferential of the convex function $(\boldsymbol{T}, \alpha) \mapsto (W - \alpha \det)^*(\boldsymbol{T})$. We recall that vectors $(\boldsymbol{A}, a)$ in the subdifferential $\partial (W - \tilde{\alpha} \det)^*(\tilde{\tau}\boldsymbol{N})$ are characterized by the property [19,20]

$$(W - \tilde{\alpha} \det)^*(\tilde{\tau}\boldsymbol{N}) + \boldsymbol{A} : (\boldsymbol{T} - \tilde{\tau}\boldsymbol{N}) + a(\alpha - \tilde{\alpha}) \leq (W - \alpha \det)^*(\boldsymbol{T}) \tag{31}$$

for any $(\boldsymbol{T}, \alpha)$.

We now show that $(\tilde{\tau}\boldsymbol{N}, \tilde{\alpha})$ satisfies the optimality condition (30). Let $\tilde{\boldsymbol{E}}_\pm$ be the value taken by the matrix $\boldsymbol{E}_\pm$ in (20) for the case $(\tau, \alpha) = (\tilde{\tau}, \tilde{\alpha})$, i.e.

$$\tilde{\boldsymbol{E}}_\pm = \frac{\bar{e}}{c}\begin{pmatrix} \pm\sqrt{1-c} & 1 \\ 1 & \pm\sqrt{1-c} \end{pmatrix} \tag{32}$$

Since $(W - \tilde{\alpha} \det)^*(\tilde{\tau}\boldsymbol{N}) = f_3(\tilde{\tau}, \tilde{\alpha})$, we know that

$$(W - \tilde{\alpha} \det)^*(\tilde{\tau}\boldsymbol{N}) = \tilde{\tau}\tilde{\boldsymbol{E}}_+ : \boldsymbol{N} + \tilde{\alpha} \det \tilde{\boldsymbol{E}}_+ - W(\tilde{\boldsymbol{E}}_+)$$

For any $\boldsymbol{T}$ and α, we thus have

$$(W - \tilde{\alpha} \det)^*(\tilde{\tau}\boldsymbol{N}) + \tilde{\boldsymbol{E}}_+ : (\boldsymbol{T} - \tilde{\tau}\boldsymbol{N}) + (\alpha - \tilde{\alpha}) \det \tilde{\boldsymbol{E}}_+ = \tilde{\boldsymbol{E}}_+ : \boldsymbol{T} + \alpha \det \tilde{\boldsymbol{E}}_+ - W(\tilde{\boldsymbol{E}}_+) \tag{33}$$

The right-hand side of (33) is bounded from above by $(W - \alpha \det)^*(\boldsymbol{T})$, hence

$$(W - \tilde{\alpha} \det)^*(\tilde{\tau}\boldsymbol{N}) + \tilde{\boldsymbol{E}}_+ : (\boldsymbol{T} - \tilde{\tau}\boldsymbol{N}) + (\alpha - \tilde{\alpha}) \det \tilde{\boldsymbol{E}}_+ \leq (W - \alpha \det)^*(\boldsymbol{T}) \tag{34}$$

The same argument but replacing $\boldsymbol{E}_+$ with $\boldsymbol{E}_-$ gives

$$(W - \tilde{\alpha} \det)^*(\tilde{\tau}\boldsymbol{N}) + \tilde{\boldsymbol{E}}_- : (\boldsymbol{T} - \tilde{\tau}\boldsymbol{N}) + (\alpha - \tilde{\alpha}) \det \tilde{\boldsymbol{E}}_- \leq (W - \alpha \det)^*(\boldsymbol{T}) \tag{35}$$

Summing (34) and (35) and further noting that $\frac{1}{2}(\tilde{\boldsymbol{E}}_+ + \tilde{\boldsymbol{E}}_-) = \frac{\bar{e}}{c}\boldsymbol{N}$ and $\det\ \tilde{\boldsymbol{E}}_\pm = -\bar{e}^2/c$, we obtain

$$(W - \tilde{\alpha} \det)^*(\tilde{\tau}\boldsymbol{N}) + \frac{\bar{e}}{c}\boldsymbol{N} : (\boldsymbol{T} - \tilde{\tau}\boldsymbol{N}) - \frac{\bar{e}^2}{c}(\alpha - \tilde{\alpha}) \leq (W - \alpha \det)^*(\boldsymbol{T})$$

which, in view of (31), can be written as

$$(\frac{\bar{e}}{c}\boldsymbol{N}, -\frac{\bar{e}^2}{c}) \in \partial (W - \tilde{\alpha} \det)^*(\tilde{\tau}\boldsymbol{N})$$

Hence $(\tilde{\tau}\boldsymbol{N}, \tilde{\alpha})$ satisfies the optimality conditions (30), meaning that the optimized bound $w_{\mathsf{T}}(\bar{\boldsymbol{e}})$ is equal to $w_{\mathsf{T}}(\bar{\boldsymbol{e}}; \tilde{\tau}\boldsymbol{N}, \tilde{\alpha})$. Since $w_{\mathsf{T}}(\bar{\boldsymbol{e}}; \tilde{\tau}\boldsymbol{N}, \tilde{\alpha}) = w_{\mathrm{TW}}(\bar{\boldsymbol{e}})$, the conclusion is that the bound w_{T} obtained from the translation method coincides with the Talbot–Willis bound w_{TW}.

4. Example in the not strongly convex case

In this section, we investigate an example in which the energy w in the matrix is not strongly convex, i.e. w is not convex in e^2. More precisely, we take w as

$$w(\boldsymbol{e}) = \begin{cases} \epsilon e & \text{if } e < 1 \\ +\infty & \text{if } e \geq 1 \end{cases} \tag{36}$$

where $\epsilon > 0$ is a fixed parameter. The composite considered is a two-dimensional version of that considered in [15] for comparing the Talbot–Willis and the Ponte Castañeda methods.

The Talbot–Willis bound w_{TW} is given by

$$w_{\mathrm{TW}}(\bar{\boldsymbol{e}}) = \begin{cases} \epsilon\bar{e} & \text{if } \bar{e} \leq \frac{c}{2-c} \\ \left(\frac{2-c}{c}\right)\epsilon\bar{e}^2 & \text{if } \frac{c}{2-c} \leq \bar{e} \leq \frac{c}{\sqrt{2-c}} \\ +\infty & \text{if } \frac{c}{\sqrt{2-c}} < \bar{e} \end{cases} \tag{37}$$

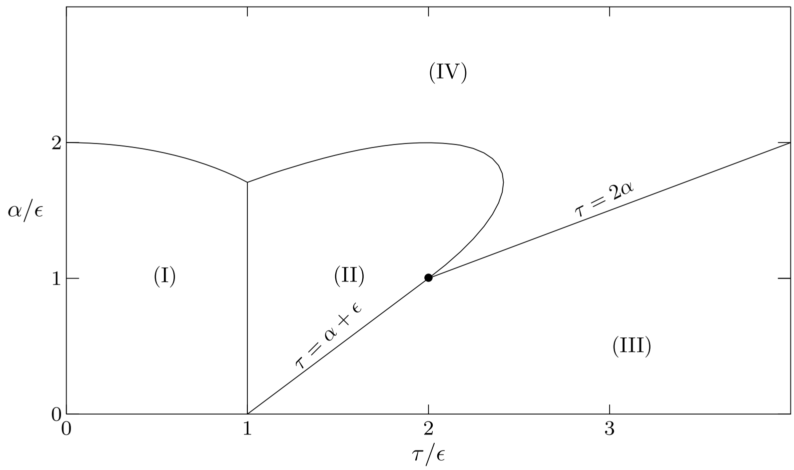

Fig. 1. Chart of the function $(\tau, \alpha) \mapsto (W - \alpha \det)^*(\tau \boldsymbol{N})$.

The Ponte Castañeda bound w_{PC} is given by

$$w_{PC}(\bar{\boldsymbol{e}}) = \begin{cases} \left(\frac{2-c}{c}\right)\epsilon\bar{e}^2 & \text{if } \bar{e} \leq \frac{c}{\sqrt{2-c}} \\ +\infty & \text{if } \frac{c}{\sqrt{2-c}} < \bar{e} \end{cases} \tag{38}$$

The expressions (37) and (38) can be obtained by adapting the calculations presented in [15] to the 2D case. We now proceed to evaluate the bounds provided by (12) and (13). As in Sect. 3, the main difficulty lies in solving the nonconvex maximization problem (15) that defines the Legendre transform $(W - \alpha \det)^*(\boldsymbol{T})$. For the energy w in (36), the maximization problem (15) can be solved in closed form. The resulting expression of $(W - \alpha \det)^*(\tau \boldsymbol{N})$ depends on the value of (τ, α) as represented in Fig. 1. In more detail, we have

$$(W - \alpha \det)^*(\tau \boldsymbol{N}) = \begin{cases} 0 & \text{in (I)} \\ \tau - \epsilon & \text{in (II)} \\ 2\tau - \alpha - 2\epsilon & \text{in (III)} \\ \dfrac{\tau^2}{2\alpha} + \alpha - 2\epsilon & \text{in (IV)} \end{cases} \tag{39}$$

In (39), (I)–(IV) are domains in the (τ, α) plane that are represented in Fig. 1. The common boundary between the domains (II) and (IV) is the elliptical arc with equation $\tau^2 - 2\alpha\tau + 2\alpha^2 - 2\alpha\epsilon = 0$. The boundary between the domains (I) and (IV) is the elliptical arc with equation $\tau^2 + 2\alpha(\alpha - 2\epsilon) = 0$.

The result (39) allows the bounds in (12) and (13) to be evaluated. As a first illustration, consider the special case $\tau = 2\epsilon$, $\alpha = \epsilon$. From (13) and (39) we have

$$w_{\mathsf{T}}(\bar{\boldsymbol{e}}; 2\epsilon \boldsymbol{N}, \epsilon) = \frac{\epsilon}{2}(4\bar{e} - \bar{e}^2 - c) \tag{40}$$

It can easily be verified that $\bar{e} < \frac{1}{2}(4\bar{e} - \bar{e}^2 - c)$ if $1 - \sqrt{1-c} < \bar{e}$. Hence $w_{\mathsf{T}}(\bar{\boldsymbol{e}}; 2\epsilon \boldsymbol{N}, \epsilon)$ improves on the Talbot–Willis bound w_{TW} for any $\bar{e}$ between $1 - \sqrt{1-c}$ and $\frac{c}{2-c}$.

Better bounds than (40) can be obtained by fully optimizing (13) with respect to τ and α. Omitting the detail of the calculations, the optimal values of (τ, α) are found to depend on $\bar{e}$ as follows:

- for $\bar{e} \leq 1 - \sqrt{1-c}$, the optimal values of (τ, α) are equal to $(1, 0)$;
- for $1 - \sqrt{1-c} \leq \bar{e} \leq \frac{c}{\sqrt{2-c}}$, the optimal values of (τ, α) are given by

  $$\frac{\alpha}{\epsilon} = 1 - \frac{\bar{e}^2 - 2\bar{e} + c}{\sqrt{(\bar{e}^2 - 2\bar{e} + c)^2 + (2\bar{e} - c)^2}}, \; \tau = \alpha + \sqrt{\alpha(2\epsilon - \alpha)}$$

  For those values, (τ, α) is on (II) ∩ (IV). The corresponding value of $w_{\mathsf{T}}(\bar{\boldsymbol{e}}; \tau \boldsymbol{N}, \alpha)$ is

  $$w_{\mathsf{T}}(\bar{\boldsymbol{e}}; \tau \boldsymbol{N}, \alpha) = \epsilon \left(\bar{e} - \frac{1}{2}\bar{e}^2 + \frac{1}{2}\sqrt{(\bar{e}^2 - 2\bar{e} + c)^2 + (2\bar{e} - c)^2} \right)$$

- for $\frac{c}{\sqrt{2-c}} < \bar{e} \leq c$, the optimal bound is obtained by taking $\tau = 2\alpha\bar{e}/c$. For α large enough, the point (τ, α) is in the domain (IV) so that

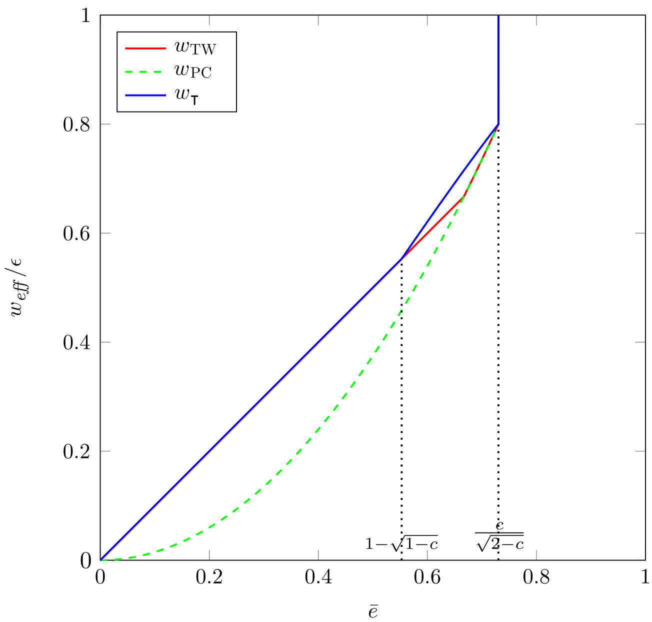

Fig. 2. Comparison of the bounds in the case $c = 0.8$.

$$w_T(\bar{\boldsymbol{e}}; \tau \boldsymbol{N}, \alpha) = \frac{1}{2}\alpha\left(\bar{e}^2\frac{2-c}{c} - c\right) + \epsilon c \tag{41}$$

Taking the limit $\alpha \to \infty$ in (41) gives $w_T(\bar{\boldsymbol{e}}; \tau \boldsymbol{N}, \alpha) \to +\infty$;

– for $c < \bar{e}$, the optimal bound is obtained by taking $\alpha = 0$. For τ large enough, the point (τ, α) is in the domain (III) so that

$$w_T(\bar{\boldsymbol{e}}; \tau \boldsymbol{N}, 0) = \tau(\bar{e} - c) + \epsilon c \tag{42}$$

Taking the limit $\tau \to \infty$ in (42) gives $w_T(\bar{\boldsymbol{e}}; \tau \boldsymbol{N}, 0) \to +\infty$.

In summary, the optimized bound w_T is given by

$$w_T(\bar{e}) = \begin{cases} \epsilon\bar{e} & \text{if } \bar{e} \le 1 - \sqrt{1-c} \\ \epsilon\left(\bar{e} - \frac{1}{2}\bar{e}^2 + \frac{1}{2}\sqrt{(\bar{e}^2 - 2\bar{e} + c)^2 + (2\bar{e} - c)^2}\right) & \text{if } 1 - \sqrt{1-c} \le \bar{e} \le \frac{c}{\sqrt{2-c}} \\ +\infty & \text{if } \frac{c}{\sqrt{2-c}} < \bar{e} \end{cases} \tag{43}$$

The bounds w_{TW}, w_{PC} and w_T are plotted in Fig. 2 as a function of the norm $\bar{e}$ of the effective electric field. The volume fraction c of the matrix is set to 0.8. For $\bar{e} \le 1 - \sqrt{1-c}$, the bounds w_{TW} and w_T coincide with the energy w of the matrix. The most important observation is that the translation bound w_T strictly improves both on w_{TW} and w_{PC} for $1 - \sqrt{1-c} \le \bar{e} \le c/\sqrt{2-c}$. In the limit $\bar{e} \to c/\sqrt{2-c}$, all the three bounds converge towards the same value ϵc. For $\bar{e} > c/\sqrt{2-c}$, the three bounds are equal to $+\infty$.

5. Concluding remarks

For two-phase composites with perfectly conducting inclusions and a strongly convex energy function, the analysis presented in Sect. 3 shows that the translation method gives the same Hashin–Shtrikman-type bounds as the Talbot–Willis and the Ponte Castañeda methods. It should be observed, however, that the calculations involved in the translation method are significantly more complicated than those involved in the Talbot–Willis and the Ponte Castañeda procedures. This is mainly a consequence of the embedding of the original problem in an extended problem of dimension 4, which is an essential ingredient of the translation method when applied to two-dimensional conductors.

Although it is rather special, the example presented in Sect. 4 shows that the translation method has the potential to produce nonlinear Hashin–Shtrikman-type bounds that are strictly stronger than those provided by the Talbot–Willis and the Ponte Castañeda methods. In that regard, it can be noted that the translation method can be used in a more general fashion by embedding the original problem in an extended problem of dimension higher than 4 (by considering three – or more – copies of the original problem). Although the calculations become even more involved, the resulting Hashin–Shtrikman-type bounds may improve on the Talbot–Willis and the Ponte Castañeda bounds even for strongly convex energy functions [21].

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