C. R. Mecanique 345 (2017) 329–336

Contents lists available at ScienceDirect

Comptes Rendus Mecanique

www.sciencedirect.com

Dynamic contact problem with adhesion and damage between thermo–electro–elasto-viscoplastic bodies

Problème dynamique de contact entre deux corps thermo–électro–élasto–viscoplastique avec endommagement et adhésion

Tedjani Hadj ammar [a,*], Abdelkader Saïdi [b], Abdelaziz Azeb Ahmed [a]

[a] *Département de mathématique, Université d'El-Oued, El-Oued, Algeria*
[b] *Institut de Recherche Mathématique Avancée, Université de Strasbourg, 7 Rue René Descartes, 67084 Strasbourg, France*

ARTICLE INFO

Article history:
Received 4 March 2016
Accepted 15 March 2017
Available online 31 March 2017

Keywords:
Thermo-electro-elasto-viscoplastic materials
Normal compliance
Damage
Adhesion

Mots-clés :
Matériaux thermo–électro–élasto–viscoplastique
Réponse normale
Endommagement
Adhésion

ABSTRACT

We study of a dynamic contact problem between two thermo–electro–elasto-viscoplastic bodies with damage and adhesion. The contact is frictionless and is modeled with normal compliance condition. We derive variational formulation for the model and prove an existence and uniqueness result of the weak solution. The proof is based on arguments of evolutionary variational inequalities, parabolic inequalities, differential equations, and fixed point theorem.

RÉSUMÉ

On étudie un problème dynamique de contact entre deux corps thermo–électro–élasto-viscoplastiques avec endommagement et adhésion. Le contact sans frottement est modélisé par une réponse normale. Nous dérivons la formulation variationnelle pour le modèle et prouvons un résultat d'existence et d'unicité de la solution faible. Les démonstrations sont basées sur des techniques d'inéquations variationnelles d'évolution, d'inéquations paraboliques, d'équations différentielles et sur le théorème du point fixe.

Version française abrégée

Dans cette Note, on étudie un problème dynamique de contact sans frottement avec réponse normale, adhésion et endommagement entre deux corps thermo–électro–élasto-viscoplastiques. On suppose que les deux corps occupent deux domaines bornés $\Omega^\ell \subset \mathbb{R}^d$, $\ell = 1, 2$ $(d = 2, 3)$, avec une surface frontière régulière $\Gamma^\ell = \partial\Omega^\ell$, subdivisée en trois parties mesurables Γ_1^ℓ, Γ_2^ℓ et Γ_3^ℓ, d'une part, et de deux parties mesurables Γ_a^ℓ et Γ_b^ℓ, d'autre part, telles que $\mathrm{mes}(\Gamma_1^\ell) > 0$ et

* Corresponding author.
E-mail addresses: hat_olsz@yahoo.com (T. Hadj ammar), abdelkader.saidi@math.unistra.fr (A. Saïdi), aziz-azebahmed@univ-eloued.dz (A. Azeb Ahmed).

http://dx.doi.org/10.1016/j.crme.2017.03.002

$\text{mes}(\Gamma_a^\ell) > 0$. Nous notons par ν^ℓ la normale unitaire sortante à Γ^ℓ, le corps Ω^ℓ est encastré sur Γ_1^ℓ dans une structure fixe. Sur Γ_2^ℓ agissent des tractions surfaciques de densité $\mathbf{f}_2^\ell$ et dans Ω^ℓ agissent des forces volumiques de densité $\mathbf{f}_0^\ell$. Nous supposons également que le potentiel électrique s'annule sur Γ_a^ℓ et qu'une charge électrique de surface de densité q_2^ℓ est imposée sur Γ_b^ℓ. Nous supposons que les matériaux peuvent être endommagés. Le contact entre les deux corps est modélisé par une variable de surface appelée champ d'adhésion, dont l'évolution est décrite par une équation différentielle ordinaire du premier ordre. Les inconnues, dans ce cas, sont les champs des déplacements $\mathbf{u}^\ell$, les champs des contraintes $\boldsymbol{\sigma}^\ell$, les champs des températures θ^ℓ, les champs des endommagements ζ^ℓ, un champ d'adhésion β, les potentiels électriques φ^ℓ et les champs des déplacements électriques $\mathbf{D}^\ell$ avec la loi de comportement électro-élasto-viscoplastique non linéaire. L'évolution de champ de température θ^ℓ est régie par l'équation de chaleur, obtenue à partir de la conservation de l'énergie, et définie par l'équation différentielle (2). L'endommagement ζ^ℓ est une variable interne causé par des déformations, et est donné par l'inclusion différentielle (3). Sous ces hypothèses, le problème thermo–électro-mécanique que nous considérons peut être formulé de la façon suivante.

Problème P. Pour $\ell = 1, 2$, trouver un champ de déplacement $\mathbf{u}^\ell : \Omega^\ell \text{mes}(0,T) \longrightarrow \mathbb{R}^d$, un champ de contrainte $\boldsymbol{\sigma}^\ell : \Omega^\ell \text{mes}(0,T) \longrightarrow \mathbb{S}^d$, un champ de température $\theta^\ell : \Omega^\ell \text{mes}(0,T) \longrightarrow \mathbb{R}$, un champ d'endommagement $\zeta^\ell : \Omega^\ell \text{mes}(0,T) \longrightarrow \mathbb{R}$, un potentiel électrique $\varphi^\ell : \Omega^\ell \text{mes}(0,T) \longrightarrow \mathbb{R}$, un champ d'adhésion $\beta : \Gamma_3 \text{mes}(0,T) \longrightarrow \mathbb{R}$ et un champ de déplacement électrique $\mathbf{D}^\ell : \Omega^\ell \text{mes}(0,T) \longrightarrow \mathbb{R}^d$, satisfaisant aux relations (1)–(11).

Ici et partout dans ce travail, le point au-dessus des variables représente la dérivée par rapport au temps, $\|v\|$ est la norme du vecteur v et les indices ν et τ indiquent les composantes normales et tangentielles des tenseurs et des vecteurs. Les équations (1) représentent la loi constitutive thermo–électro–élasto-viscoplastique dans laquelle $\varepsilon(\mathbf{u}^\ell)$ représente le tenseur des déformations linéarisées, $E(\varphi^\ell) = -\nabla\varphi^\ell$ est le champ électrique, $\mathcal{A}^\ell$ et $\mathcal{G}^\ell$ sont respectivement les opérateurs de viscosité et d'élasticité non linéaire, respectivement, $\mathcal{F}^\ell$ représente le tenseur de viscoplasticité, $\mathcal{E}^\ell$ représente le tenseur piézoélectrique du troisième ordre, $(\mathcal{E}^\ell)^*$ est son transposé et $\mathcal{B}^\ell$ représente le tenseur de permittivité électrique. L'équation (2) représente la conservation de l'énergie, où Θ^ℓ est une fonction constitutive non linéaire. La relation (3) décrit l'évolution du champ d'endommagement où ϕ^ℓ est la fonction source d'endommagement, $\partial\psi_{K^\ell}$ est le sous-différentiel de la fonction indicatrice de l'ensemble des fonctions d'endommagement admissibles K^ℓ. Ensuite, les équations (4) sont respectivement les équations de mouvement du champ de contrainte et d'équilibre du champ de déplacement électrique. Les conditions (5) sont les conditions aux limites classiques de déplacement–traction, tandis que les conditions (10) représentent les conditions aux limites pour les variables électriques. L'équation (6) représente la condition de compliance normale avec adhésion sur la surface de contact Γ_3, dans laquelle γ_ν est le coefficient d'adhésion et R_ν est un opérateur de troncation défini par (12). L'équation (7) représente la condition de contact avec adhésion sur le plan tangentiel, dans lequel p_τ est une fonction donnée et $\mathbf{R}_\tau$ l'opérateur de troncation donnée par (12). L'équation (8) décrit l'évolution du champ d'adhésion avec les paramètres physiques positifs donnés γ_ν, γ_τ et ε_a, avec, $r_+ = \max\{r, 0\}$. Enfin, (11) représente les conditions initiales.

La formulation variationnelle du problème **P**, est la suivante.

Problème PV. Trouver $\mathbf{u} : (0,T) \to \mathbf{V}$, $\boldsymbol{\sigma} : (0,T) \to \mathcal{H}$, $\theta : (0,T) \to E_1$, $\zeta : (0,T) \to E_1$, $\varphi : (0,T) \to W$, $\beta : (0,T) \to L^\infty(\Gamma_3)$ et $\mathbf{D} : (0,T) \to \mathcal{W}$ satisfaisant les relations (1), (8), (11) et (28)–(30).

Théorème. *Sous les hypothèses (13)–(24), le problème variationnel **PV** admet une solution unique* $\{\mathbf{u}, \boldsymbol{\sigma}, \theta, \zeta, \varphi, \beta, \mathbf{D}\}$ *ayant la régularité (33)–(39).*

1. Introduction

Constitutive laws with internal variables have been used in various publications in order to model the effect of internal variables on the behavior of real bodies like metals, rocks polymers, and so on, for which the rate of deformation depends on the internal variables. Some of the internal state variables considered by many authors are the spatial display of dislocation, the work-hardening of materials, the absolute temperature and the damage field, see for example [1] and references therein for the case of temperature and other internal state variables and the references [2,3] for the case of damage field. The damage is an extremely important topic in engineering, since it affects directly the useful life of the designed structure or component. There exists a very large engineering literature on it. Models taking into account the influence of the internal damage of the material on the contact process have been investigated mathematically. In all these papers, the damage of the material is described with a damage function ζ^ℓ, restricted to have values between zero and one. When $\zeta^\ell = 1$, there is no damage in the material, when $\zeta^\ell = 0$, the material is completely damaged, when $0 < \zeta^\ell < 1$ there is partial damage and the system has a reduced load carrying capacity. Contact problems with damage have been investigated in [1,3]. The novelty with respect to all the above papers is the introduction of an absolute temperature for piezoelectric materials. In this paper, we study a dynamic frictionless contact problem between two thermo–electro–elasto-viscoplastic bodies with damage. The contact is modeled with normal compliance where the adhesion of the contact surfaces is taken into account and is modeled with a surface variable, the bonding field. We derive a variational formulation of the problem and prove the existence of a unique weak solution.

2. Problem statement and variational formulation

The physical setting is as follows. Let us consider two thermo–electro–elasto-viscoplastics bodies, occupying two bounded domains Ω^1, Ω^2 of the space $\mathbb{R}^d (d = 2, 3)$. For each domain Ω^ℓ, the boundary Γ^ℓ is assumed to be Lipschitz continuous, and is partitioned into three disjoint measurable parts Γ_1^ℓ, Γ_2^ℓ and Γ_3^ℓ, on one hand, and on two measurable parts Γ_a^ℓ and Γ_b^ℓ, on the other hand, such that $meas\Gamma_1^\ell > 0$, $meas\Gamma_a^\ell > 0$. Let $T > 0$ and let $[0, T]$ be the time interval of interest. The Ω^ℓ body is submitted to $\mathbf{f}_0^\ell$ forces and volume electric charges of density q_0^ℓ. The bodies are assumed to be clamped on $\Gamma_1^\ell \times (0, T)$. The surface tractions $\mathbf{f}_2^\ell$ act on $\Gamma_2^\ell \times (0, T)$. We also assume that the electrical potential vanishes on $\Gamma_a^\ell \times (0, T)$ and a surface electric charge of density q_2^ℓ is prescribed on $\Gamma_b^\ell \times (0, T)$. The two bodies can enter in contact along the common part $\Gamma_3^1 = \Gamma_3^2 = \Gamma_3$. The classical formulation of the frictionless contact problem with normal compliance, adhesion and damage between two thermo–electro–elasto-viscoplastics bodies is given by:

Problem P. For $\ell = 1, 2$, find a displacement field $\mathbf{u}^\ell : \Omega^\ell \text{mes}(0, T) \longrightarrow \mathbb{R}^d$, a stress field $\boldsymbol{\sigma}^\ell : \Omega^\ell \text{mes}(0, T) \longrightarrow \mathbb{S}^d$, an electric potential field $\varphi^\ell : \Omega^\ell \text{mes}(0, T) \longrightarrow \mathbb{R}$, a temperature $\theta^\ell : \Omega^\ell \times (0, T) \longrightarrow \mathbb{R}$, a damage field $\zeta^\ell : \Omega^\ell \text{mes}(0, T) \longrightarrow \mathbb{R}$, a bonding field $\beta : \Gamma_3 \text{mes}(0, T) \longrightarrow \mathbb{R}$ and a electric displacement field $\mathbf{D}^\ell : \Omega^\ell \text{mes}(0, T) \longrightarrow \mathbb{R}^d$ such that

$$\begin{aligned} &\mathbf{D}^\ell = \mathcal{E}^\ell \varepsilon(\mathbf{u}^\ell) - \mathcal{B}^\ell \nabla\varphi^\ell, \qquad \boldsymbol{\sigma}^\ell = \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\mathbf{u}}^\ell) + \mathcal{G}^\ell \boldsymbol{\varepsilon}(\mathbf{u}^\ell) + (\mathcal{E}^\ell)^* \nabla\varphi^\ell \\ &\quad + \int_0^t \mathcal{F}^\ell\Big(\boldsymbol{\sigma}^\ell(s) - \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\mathbf{u}}^\ell(s)) - (\mathcal{E}^\ell)^* \nabla\varphi^\ell(s), \boldsymbol{\varepsilon}(\mathbf{u}^\ell(s)), \theta^\ell(s), \zeta^\ell(s)\Big)\,\mathrm{d}s, \qquad \text{in } \Omega^\ell \text{mes}(0, T) \end{aligned} \tag{1}$$

$$\dot{\theta}^\ell - \kappa_0^\ell \Delta\theta^\ell = \Theta^\ell\big(\boldsymbol{\sigma}^\ell - \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\mathbf{u}}^\ell), \boldsymbol{\varepsilon}(\mathbf{u}^\ell), \theta^\ell, \zeta^\ell\big) + p^\ell \quad \text{in } \Omega^\ell \text{mes}(0, T) \tag{2}$$

$$\dot{\zeta}^\ell - \kappa^\ell \Delta\zeta^\ell + \partial\psi_{K^\ell}(\zeta^\ell) \ni \phi^\ell\big(\boldsymbol{\sigma}^\ell - \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\mathbf{u}}^\ell) - (\mathcal{E}^\ell)^* \nabla\varphi^\ell(s), \boldsymbol{\varepsilon}(\mathbf{u}^\ell), \theta^\ell(s), \zeta^\ell\big) \quad \text{in } \Omega^\ell \text{mes}(0, T) \tag{3}$$

$$\rho^\ell \ddot{\mathbf{u}}^\ell = \mathrm{Div}\,\boldsymbol{\sigma}^\ell + \mathbf{f}_0^\ell, \quad \mathrm{div}\,\mathbf{D}^\ell - q_0^\ell = 0 \qquad \text{in } \Omega^\ell \text{mes}(0, T) \tag{4}$$

$$\mathbf{u}^\ell = 0 \quad \text{on } \Gamma_1^\ell \text{mes}(0, T), \quad \text{and} \quad \boldsymbol{\sigma}^\ell \nu^\ell = \mathbf{f}_2^\ell \quad \text{on } \Gamma_2^\ell \text{mes}(0, T) \tag{5}$$

$$\sigma_\nu^1 = \sigma_\nu^2 \equiv \sigma_\nu, \text{ where } \sigma_\nu = -p_\nu([u_\nu]) + \gamma_\nu \beta^2 R_\nu([u_\nu]) \quad \text{on } \Gamma_3 \text{mes}(0, T) \tag{6}$$

$$\boldsymbol{\sigma}_\tau^1 = -\boldsymbol{\sigma}_\tau^2 \equiv \boldsymbol{\sigma}_\tau, \text{ where } \boldsymbol{\sigma}_\tau = p_\tau(\beta)\mathbf{R}_\tau([\mathbf{u}_\tau]) \quad \text{on } \Gamma_3 \text{mes}(0, T) \tag{7}$$

$$\dot{\beta} = -\Big(\beta\big(\gamma_\nu (R_\nu([u_\nu]))^2 + \gamma_\tau |\mathbf{R}_\tau([\mathbf{u}_\tau])|^2\big) - \varepsilon_a\Big)_+ \quad \text{on } \Gamma_3 \text{mes}(0, T) \tag{8}$$

$$\kappa_0^\ell \frac{\partial^\ell \theta^\ell}{\partial \nu^\ell} + \alpha^\ell \theta^\ell = 0 \quad \frac{\partial \zeta^\ell}{\partial \nu^\ell} = 0 \quad \text{on } \Gamma^\ell \text{mes}(0, T) \tag{9}$$

$$\varphi^\ell = 0 \text{ on } \Gamma_a^\ell \text{mes}(0, T), \quad \text{and} \quad \mathbf{D}^\ell.\nu^\ell = q_2^\ell \text{ on } \Gamma_b^\ell \text{mes}(0, T) \tag{10}$$

$$\mathbf{u}^\ell(0) = \mathbf{u}_0^\ell, \ \dot{\mathbf{u}}^\ell(0) = \mathbf{v}_0^\ell, \ \theta^\ell(0) = \theta_0^\ell, \quad \zeta^\ell(0) = \zeta_0^\ell \quad \text{in } \Omega^\ell, \quad \text{and} \quad \beta(0) = \beta_0 \quad \text{on } \Gamma_3 \tag{11}$$

First, equations (1) represent the thermo–electro–elasto-viscoplastic constitutive law with damage. Equation (2) represents the energy conservation where Θ^ℓ is a nonlinear constitutive function that represents the heat generated by the work of internal forces and p^ℓ is a given volume heat source. Inclusion (3) describes the evolution of the damage field, governed by the source damage function ϕ^ℓ, where $\partial\psi_{K^\ell}$ is the subdifferential of indicator function of the set K of admissible damage functions given by $K^\ell = \{\xi \in H^1(\Omega^\ell);\ 0 \le \xi \le 1, \quad \text{a.e. in } \Omega^\ell\}$. Equations (4) are the equilibrium equations for the stress and electric-displacement fields, respectively. The equations (5) represent the displacement-traction boundary condition. Condition (6) represents the normal compliance conditions with adhesion, where γ_ν is a given adhesion coefficient and $[u_\nu] = u_\nu^1 + u_\nu^2$ stands for the displacements in normal direction. The contribution of the adhesive to the normal traction is represented by the term $\gamma_\nu \beta^2 R_\nu([u_\nu])$, the adhesive traction is tensile and is proportional, with proportionality coefficient γ_ν, to the square of the intensity of adhesion and to the normal displacement, but as long as it does not exceed the bond length L. The maximal tensile traction is $\gamma_\nu \beta^2 L$. R_ν is the truncation operator defined by

$$R_\nu(s) = \begin{cases} L & \text{if} \quad s < -L \\ -s & \text{if} \quad -L \le s \le 0 \\ 0 & \text{if} \quad s > 0 \end{cases} \qquad \mathbf{R}_\tau(\mathbf{v}) = \begin{cases} \mathbf{v} & \text{if} \quad |\mathbf{v}| \le L \\ -L\dfrac{\mathbf{v}}{|\mathbf{v}|} & \text{if} \quad |\mathbf{v}| > L \end{cases} \tag{12}$$

Here $L > 0$ is the characteristic length of the bond, beyond which it does not offer any additional traction. The introduction of the operator R_ν, together with the operator $\mathbf{R}_\tau$ defined below, is motivated by mathematical arguments but it is not restrictive from a physical point of view, since no restriction on the size of the parameter L is made in what follows. Condition (7) represents the adhesive contact condition on the tangential plane, where $[\mathbf{u}_\tau] = \mathbf{u}_\tau^1 - \mathbf{u}_\tau^2$ stands for the jump of the displacements in tangential direction. This condition shows that the shear on the contact surface depends on the bonding field and on the tangential displacement, but as long as it does not exceed the bond length L. Next, equation (8) represents the ordinary differential equation that describes the evolution of the bonding field and it was already used in [4], see also [3] for more details. Here, besides γ_ν, two new adhesion coefficients are involved, γ_τ and ε_a. Notice that in this model once debonding occurs bonding cannot be reestablished since, as it follows from (8), $\dot{\beta} \leq 0$. Boundary condition (9) represents a Fourier boundary condition for the temperature and homogeneous Neumann boundary condition for the damage field. (10) represent the electric boundary conditions. Finally, (11) represents the initial conditions.

We now proceed to obtain a variational formulation of problem **P**. For this purpose, we introduce additional notation and assumptions on the problem data. Here and in what follows the indices i and j run between 1 and d, the summation convention over repeated indices is adopted and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable. Let $E_0 = L^2(\Omega^1) \times L^2(\Omega^2)$, $H^\ell = [L^2(\Omega^\ell)]^d$, $\mathcal{H}^\ell = [L^2(\Omega^\ell)]_s^{d\times d}$, and define the following spaces: $\mathcal{Z} = \{\theta \in L^\infty(0,T;L^2(\Gamma_3));\ 0 \leq \theta \leq 1,\ \text{a.e. on } \Gamma_3\}$, $V^\ell = \{v^\ell \in [H^1(\Omega^\ell)]^d;\ v^\ell|_{\Gamma_1^\ell} = 0\}$, $W^\ell = \{\psi^\ell \in H^1(\Omega^\ell);\ \psi^\ell|_{\Gamma_a^\ell} = 0\}$, $\mathcal{W}^\ell = \{\mathbf{D}^\ell \in H^\ell;\ \operatorname{div}\mathbf{D}^\ell \in L^2(\Omega^\ell)\}$, $\mathbf{V} = V^1 \times V^2$, $E_1 = H^1(\Omega^1) \times H^1(\Omega^2)$, $H = H^1 \times H^2$, $\mathcal{H} = \mathcal{H}^1 \times \mathcal{H}^2$, $W = W^1 \times W^2$, $\mathcal{W} = \mathcal{W}^1 \times \mathcal{W}^2$. We now list assumptions on the data. Assume that the operators $\mathcal{A}^\ell, \mathcal{G}^\ell, \mathcal{F}^\ell, \Theta^\ell, \phi^\ell, \mathcal{E}^\ell$ and $\mathcal{B}^\ell$ satisfy the following conditions ($C^1_{\mathcal{A}^\ell}, C^2_{\mathcal{A}^\ell}, m_{\mathcal{A}^\ell}, L_{\mathcal{G}^\ell}, L_{\mathcal{F}^\ell}, L_{\Theta^\ell}, L_{\phi^\ell}$ and $m_{\mathcal{B}^\ell}$ being positive constants).

$\mathcal{A} : \Omega \times \mathbb{S}_n \to \mathbb{S}_n$ satisfies the following properties:

$$\left.\begin{array}{l}
\text{(a) } \mathcal{A}^\ell : \Omega^\ell \times \mathbb{S}^d \to \mathbb{S}^d \\
\text{(b) } |\mathcal{A}^\ell(\mathbf{x},\boldsymbol{\xi})| \leq C^1_{\mathcal{A}^\ell}|\boldsymbol{\xi}| + C^2_{\mathcal{A}^\ell} \quad \forall \boldsymbol{\xi} \in \mathbb{S}^d,\ \text{a.e. } \mathbf{x} \in \Omega^\ell \\
\text{(c) } (\mathcal{A}^\ell(\mathbf{x},\boldsymbol{\xi}_1) - \mathcal{A}^\ell(\mathbf{x},\boldsymbol{\xi}_2)) \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \geq m_{\mathcal{A}^\ell}|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|^2\ \forall \boldsymbol{\xi}_1,\boldsymbol{\xi}_2 \in \mathbb{S}^d,\ \text{a.e. } \mathbf{x} \in \Omega^\ell \\
\text{(d) for any } \boldsymbol{\xi} \in \mathbb{S}^d, \mathbf{x} \mapsto \mathcal{A}^\ell(\mathbf{x},\boldsymbol{\xi}) \text{ is measurable on } \Omega^\ell \\
\text{(e) the mapping } \boldsymbol{\xi} \mapsto \mathcal{A}^\ell(\mathbf{x},\boldsymbol{\xi}) \text{ is continuous on } \mathbb{S}^d,\ \text{a.e. } \mathbf{x} \in \Omega^\ell
\end{array}\right\} \tag{13}$$

$$\left.\begin{array}{l}
\text{(a) } \mathcal{G}^\ell : \Omega^\ell \times \mathbb{S}^d \to \mathbb{S}^d \\
\text{(b) } |\mathcal{G}^\ell(\mathbf{x},\boldsymbol{\xi}_1) - \mathcal{G}^\ell(\mathbf{x},\boldsymbol{\xi}_2)| \leq L_{\mathcal{G}^\ell}|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|,\ \ \forall \boldsymbol{\xi}_1,\boldsymbol{\xi}_2 \in \mathbb{S}^d,\ \text{a.e. } \mathbf{x} \in \Omega^\ell \\
\text{(c) For any } \boldsymbol{\xi} \in \mathbb{S}^d, \mathbf{x} \mapsto \mathcal{G}^\ell(\mathbf{x},\boldsymbol{\xi}) \text{ is measurable on } \Omega^\ell \\
\text{(d) The mapping } \mathbf{x} \mapsto \mathcal{G}^\ell(\mathbf{x},\mathbf{0}) \text{ belongs to } \mathcal{H}^\ell
\end{array}\right\} \tag{14}$$

$$\left.\begin{array}{l}
\text{(a) } \mathcal{F}^\ell : \Omega^\ell \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \times \mathbb{R} \to \mathbb{S}^d \\
\text{(b) } |\mathcal{F}^\ell(\mathbf{x},\boldsymbol{\eta}_1,\boldsymbol{\xi}_1,d_1,r_1) - \mathcal{F}^\ell(\mathbf{x},\boldsymbol{\eta}_2,\boldsymbol{\xi}_2,d_2,r_2)| \leq L_{\mathcal{F}^\ell}\big(|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2| + |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2| + \\
\qquad |d_1 - d_2| + |r_1 - r_2|\big),\ \ \forall \boldsymbol{\eta}_1,\boldsymbol{\eta}_2,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2 \in \mathbb{S}^d,\ \forall d_1,d_2,r_1,r_2 \in \mathbb{R},\ \text{a.e. } \mathbf{x} \in \Omega^\ell \\
\text{(c) for any } \boldsymbol{\eta},\boldsymbol{\xi} \in \mathbb{S}^d,\ \ d,r \in \mathbb{R}, \mathbf{x} \mapsto \mathcal{F}^\ell(\mathbf{x},\boldsymbol{\eta},\boldsymbol{\xi},d,r) \text{ is measurable in } \Omega^\ell \\
\text{(d) the mapping } \mathbf{x} \mapsto \mathcal{F}^\ell(\mathbf{x},\mathbf{0},\mathbf{0},0,0) \text{ belongs to } \mathcal{H}^\ell
\end{array}\right\} \tag{15}$$

$$\left.\begin{array}{l}
\text{(a) } \Theta^\ell : \Omega^\ell \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \\
\text{(b) } |\Theta^\ell(\mathbf{x},\boldsymbol{\eta}_1,\boldsymbol{\xi}_1,\alpha_1,d_1) - \Theta^\ell(\mathbf{x},\boldsymbol{\eta}_2,\boldsymbol{\xi}_2,\alpha_2,d_2)| \leq L_{\Theta^\ell}\big(|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2| + |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2| \\
\quad + |\alpha_1 - \alpha_2| + |d_1 - d_2|\big),\quad \forall \boldsymbol{\eta}_1,\boldsymbol{\eta}_2,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2 \in \mathbb{S}^d, \forall \alpha_1,\alpha_2,d_1,d_2 \in \mathbb{R} \text{ a.e. } \mathbf{x} \in \Omega^\ell \\
\text{(c) for any } \boldsymbol{\eta},\boldsymbol{\xi} \in \mathbb{S}^d, \alpha,d \in \mathbb{R},\quad \mathbf{x} \mapsto \Theta^\ell(\mathbf{x},\boldsymbol{\eta},\boldsymbol{\xi},\alpha,d) \text{ is measurable on } \Omega^\ell \\
\text{(d) the mapping } \mathbf{x} \mapsto \Theta^\ell(\mathbf{x},\mathbf{0},\mathbf{0},0,0) \text{ belongs to } L^2(\Omega^\ell) \\
\text{(e) } \Theta^\ell(\mathbf{x},\boldsymbol{\eta},\boldsymbol{\xi},\alpha,d) \text{ is bounded for all } \boldsymbol{\eta},\boldsymbol{\xi} \in \mathbb{S}^d, \alpha,d \in \mathbb{R} \text{ a.e. } \mathbf{x} \in \Omega^\ell
\end{array}\right\} \tag{16}$$

$$\left.\begin{array}{l}
\text{(a) } \phi^\ell : \Omega^\ell \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \\
\text{(b) } |\phi^\ell(\mathbf{x},\boldsymbol{\eta}_1,\boldsymbol{\xi}_1,\alpha_1,d_1) - \phi^\ell(\mathbf{x},\boldsymbol{\eta}_2,\boldsymbol{\xi}_2,\alpha_2,d_2)| \leq L_{\phi^\ell}\big(|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2| + |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2| \\
\quad + |\alpha_1 - \alpha_2| + |d_1 - d_2|\big),\ \ \forall \boldsymbol{\eta}_1,\boldsymbol{\eta}_2,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2 \in \mathbb{S}^d,\ \forall \alpha_1,\alpha_2,d_1,d_2 \in \mathbb{R} \text{ a.e. } \mathbf{x} \in \Omega^\ell \\
\text{(c) for any } \boldsymbol{\eta},\boldsymbol{\xi} \in \mathbb{S}^d, \alpha,d \in \mathbb{R},\ \mathbf{x} \mapsto \phi^\ell(\mathbf{x},\boldsymbol{\eta},\boldsymbol{\xi},\alpha,d) \text{ is measurable on } \Omega^\ell \\
\text{(d) the mapping } \mathbf{x} \mapsto \phi^\ell(\mathbf{x},\mathbf{0},\mathbf{0},0,0) \text{ belongs to } L^2(\Omega^\ell) \\
\text{(e) } \phi^\ell(\mathbf{x},\boldsymbol{\eta},\boldsymbol{\xi},\alpha,d) \text{ is bounded for all } \boldsymbol{\eta},\boldsymbol{\xi} \in \mathbb{S}^d, \alpha,d \in \mathbb{R} \text{ a.e. } \mathbf{x} \in \Omega^\ell
\end{array}\right\} \tag{17}$$

$$\left.\begin{array}{l}
\text{(a) } \mathcal{E}^\ell : \Omega^\ell \times \mathbb{S}^d \to \mathbb{R}^d \\
\text{(b) } \mathcal{E}^\ell(\mathbf{x},\tau) = (e^\ell_{ijk}(\mathbf{x})\tau_{jk}),\ \ \forall \tau = (\tau_{ij}) \in \mathbb{S}^d \text{ a.e. } \mathbf{x} \in \Omega^\ell,\ \text{and } e^\ell_{ijk} = e^\ell_{ikj} \in L^\infty(\Omega^\ell)
\end{array}\right\} \tag{18}$$

$$\left.\begin{array}{l}\text{(a) } \mathcal{B}^\ell : \Omega^\ell \times \mathbb{R}^d \to \mathbb{R}^d \\ \text{(b) } \mathcal{B}^\ell(\mathbf{x}, \mathbf{E}) = (b^\ell_{ij}(\mathbf{x}) E_j) \quad \forall \mathbf{E} = (E_i) \in \mathbb{R}^d, \ \text{a.e. } \mathbf{x} \in \Omega^\ell \\ \text{(c) } b^\ell_{ij} = b^\ell_{ji}, \ b^\ell_{ij} \in L^\infty(\Omega^\ell), \quad 1 \le i, j \le d \\ \text{(d) } \mathcal{B}^\ell \mathbf{E}.\mathbf{E} \ge m_{\mathcal{B}^\ell} |\mathbf{E}|^2, \ \ \forall \mathbf{E} = (E_i) \in \mathbb{R}^d, \ \text{a.e. } \mathbf{x} \in \Omega^\ell \end{array}\right\} \tag{19}$$

The *normal compliance functions* p_ν and the *tangential compliance functions* p_τ satisfy the assumptions (L_ν, L_τ and M_τ being positive constants):

$$\left.\begin{array}{l}\text{(a) } \exists L_\nu > 0 \text{ such that } |p_\nu(\mathbf{x}, r_1) - p_\nu(\mathbf{x}, r_2)| \le L_\nu |r_1 - r_2|, \ \ \forall r_1, r_2 \in \mathbb{R}, \ \text{a.e. } \mathbf{x} \in \Gamma_3 \\ \text{(b) the mapping } \mathbf{x} \mapsto p_\nu(\mathbf{x}, r) \text{ is measurable on } \Gamma_3, \ \forall r \in \mathbb{R} \\ \text{(c) } p_\nu(\mathbf{x}, r) = 0, \text{ for all } r \le 0, \ \text{a.e. } \mathbf{x} \in \Gamma_3 \end{array}\right\} \tag{20}$$

$$\left.\begin{array}{l}\text{(a) } \exists L_\tau > 0 \text{ such that } |p_\tau(\mathbf{x}, d_1) - p_\tau(\mathbf{x}, d_2)| \le L_\tau |d_1 - d_2|, \ \ \forall d_1, d_2 \in \mathbb{R}, \ \text{a.e. } \mathbf{x} \in \Gamma_3 \\ \text{(b) } \exists M_\tau > 0 \text{ such that } |p_\tau(\mathbf{x}, d)| \le M_\tau, \ \ \forall d \in \mathbb{R}, \ \text{a.e. } \mathbf{x} \in \Gamma_3 \\ \text{(c) the mapping } \mathbf{x} \mapsto p_\tau(\mathbf{x}, d) \text{ is measurable on } \Gamma_3, \ \forall d \in \mathbb{R} \\ \text{(d) the mapping } \mathbf{x} \mapsto p_\tau(\mathbf{x}, 0) \in L^2(\Gamma_3) \end{array}\right\} \tag{21}$$

The mass density satisfies

$$\rho^\ell \in L^\infty(\Omega^\ell), \ \ \text{there exists } \rho_0 > 0 \text{ such that } \rho^\ell(x) \ge \rho_0 \text{ a.e. } x \in \Omega^\ell, \ \ell = 1, 2 \tag{22}$$

The adhesion coefficients γ_ν, γ_τ and ε_a satisfy the conditions

$$\gamma_\nu, \gamma_\tau \in L^\infty(\Gamma_3), \ \ \varepsilon_a \in L^2(\Gamma_3), \ \gamma_\nu, \gamma_\tau, \varepsilon_a \ge 0, \ \text{a.e. on } \Gamma_3 \tag{23}$$

The bodies forces, surface tractions and the volume heat source have the regularity

$$\left.\begin{array}{l}\mathbf{f}^\ell_0 \in L^2(0, T; L^2(\Omega^\ell)^d), \ \mathbf{f}^\ell_2 \in L^2(0, T; L^2(\Gamma^\ell_2)^d), \ q^\ell_0 \in C(0, T; L^2(\Omega^\ell)), \ q^\ell_2 \in C(0, T; L^2(\Gamma^\ell_b)) \\ q^\ell_2\big|_{\Gamma_3} = 0, \ \ \mathbf{u}^\ell_0 \in \mathbf{V}^\ell, \ \mathbf{v}^\ell_0 \in H^\ell, \ \zeta^\ell_0 \in K^\ell, \ \beta_0 \in L^2(\Gamma_3), \ 1 \ge \beta_0 \ge 0, \ \kappa^\ell_0 > 0, \ \kappa^\ell > 0 \end{array}\right\} \tag{24}$$

We define the mappings $a_0 : E_1 \times E_1 \to \mathbb{R}$, $a : E_1 \times E_1 \to \mathbb{R}$, $\mathbf{f} : [0, T] \to \mathbf{V}'$, $q : [0, T] \to W$, $j_{\nu c} : \mathbf{V} \times \mathbf{V} \to \mathbb{R}$ and $j_{\text{ad}} : L^\infty(\Gamma_3) \times \mathbf{V} \times \mathbf{V} \to \mathbb{R}$ respectively, by

$$\left.\begin{array}{l} a_0(\zeta, \xi) = \displaystyle\sum_{\ell=1}^{2} \kappa^\ell_0 \int_{\Omega^\ell} \nabla \zeta^\ell . \nabla \xi^\ell \, \mathrm{d}x + \sum_{\ell=1}^{2} \alpha^\ell \int_{\Gamma^\ell} \zeta^\ell \xi^\ell \mathrm{d}a, \quad a(\zeta, \xi) = \sum_{\ell=1}^{2} \kappa^\ell \int_{\Omega^\ell} \nabla \zeta^\ell . \nabla \xi^\ell \mathrm{d}x \\ (\mathbf{f}(t), \mathbf{v})_{\mathbf{V}' \times \mathbf{V}} = \displaystyle\sum_{\ell=1}^{2} \int_{\Omega^\ell} \mathbf{f}^\ell_0(t) \cdot \mathbf{v}^\ell \, \mathrm{d}x + \sum_{\ell=1}^{2} \int_{\Gamma^\ell_2} \mathbf{f}^\ell_2(t) \cdot \mathbf{v}^\ell \, \mathrm{d}a \quad \forall \mathbf{v} \in \mathbf{V} \\ (q(t), \zeta)_W = \displaystyle\sum_{\ell=1}^{2} \int_{\Omega^\ell} q^\ell_0(t) \zeta^\ell \, \mathrm{d}x - \sum_{\ell=1}^{2} \int_{\Gamma^\ell_b} q^\ell_2(t) \zeta^\ell \, \mathrm{d}a \quad \forall \zeta \in W, \quad j_{\nu c}(\mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} p_\nu([u_\nu])[v_\nu] \, \mathrm{d}a \\ j_{\text{ad}}(\beta, \mathbf{u}, \mathbf{v}) = \displaystyle\int_{\Gamma_3} \Big(-\gamma_\nu \beta^2 R_\nu([u_\nu])[v_\nu] + p_\tau(\beta) \mathbf{R}_\tau([\mathbf{u}_\tau])[\mathbf{v}_\tau] \Big) \, \mathrm{d}a \end{array}\right\} \tag{25}$$

Using standard arguments, we obtain the variational formulation of the mechanical problem **P**.

Problem PV. Find $\mathbf{u} : [0, T] \to \mathbf{V}$, $\boldsymbol{\sigma} : (0, T) \to \mathcal{H}$, $\zeta : (0, T) \to E_1$, $\varphi : (0, T) \to W$, $\beta : (0, T) \to L^\infty(\Gamma_3)$ and $\mathbf{D} : (0, T) \to \mathcal{W}$ such that for all $t \in (0, T)$,

$$\boldsymbol{\sigma}^\ell = \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\mathbf{u}}^\ell) + \mathcal{G}^\ell \boldsymbol{\varepsilon}(\mathbf{u}^\ell) + (\mathcal{E}^\ell)^* \nabla \varphi^\ell + \int_0^t \mathcal{F}^\ell \Big(\boldsymbol{\sigma}^\ell(s) - \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\mathbf{u}}^\ell(s)) - (\mathcal{E}^\ell)^* \nabla \varphi^\ell(s), \boldsymbol{\varepsilon}(\mathbf{u}^\ell(s)), \zeta^\ell(s) \Big) \, \mathrm{d}s \tag{26}$$

$$\mathbf{D}^\ell = \mathcal{E}^\ell \varepsilon(\mathbf{u}^\ell) - \mathcal{B}^\ell \nabla \varphi^\ell \tag{27}$$

$$(\ddot{u}, v)_{\mathbf{V}' \times \mathbf{V}} + \sum_{\ell=1}^{2} (\boldsymbol{\sigma}^\ell, \ \varepsilon(\mathbf{v}^\ell))_{\mathcal{H}^\ell} + j_{\text{ad}}(\beta(t), \mathbf{u}(t), \mathbf{v}) + j_{\nu c}(\mathbf{u}(t), \mathbf{v}) = (\mathbf{f}(t), \mathbf{v})_{\mathbf{V}' \times \mathbf{V}} \quad \forall \mathbf{v} \in \mathbf{V} \tag{28}$$

$$\left.\begin{array}{l}\zeta(t)\in K,\qquad \sum_{\ell=1}^{2}(\dot{\zeta}^{\ell}(t),\xi^{\ell}-\zeta^{\ell}(t))_{L^2(\Omega^\ell)}+a(\zeta(t),\xi-\zeta(t))\geq \\ \sum_{\ell=1}^{2}\Big(\phi^{\ell}\Big(\boldsymbol{\sigma}^{\ell}(t)-\mathcal{A}^{\ell}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}^{\ell}(t))-(\mathcal{E}^{\ell})^{*}\nabla\varphi^{\ell}(s),\boldsymbol{\varepsilon}(\mathbf{u}^{\ell}(t)),\zeta^{\ell}(t)\Big),\xi^{\ell}-\zeta^{\ell}(t)\Big)_{L^2(\Omega^\ell)},\quad \forall\xi\in K\end{array}\right\} \tag{29}$$

$$\sum_{\ell=1}^{2}(\mathcal{B}^{\ell}\nabla\varphi^{\ell}(t),\nabla\phi^{\ell})_{H^\ell}-\sum_{\ell=1}^{2}(\mathcal{E}^{\ell}\varepsilon(\mathbf{u}^{\ell}(t)),\nabla\phi^{\ell})_{H^\ell}=(q(t),\phi)_W,\quad \forall\phi\in W \tag{30}$$

$$\dot{\beta}(t)=-\Big(\beta(t)\big(\gamma_\nu(R_\nu([u_\nu(t)]))^2+\gamma_\tau\,|\mathbf{R}_\tau([\mathbf{u}_\tau(t)])|^2\big)-\varepsilon_a\Big)_+ \tag{31}$$

$$\mathbf{u}(0)=\mathbf{u}_0,\quad \dot{\mathbf{u}}(0)=\mathbf{v}_0,\quad \zeta(0)=\zeta_0,\quad \beta(0)=\beta_0 \tag{32}$$

The existence of the unique solution to Problem **PV** is stated and proved in the next section.

Remark 1. We note that, in Problem **P** and in Problem **PV**, we do not need to impose explicitly the restriction $0\leq\beta\leq 1$. Indeed, equation (31) guarantees that $\beta(x,t)\leq\beta_0(x)$ and, therefore, the assumption $0\leq\beta_0(x)\leq 1$, a.e. $x\in\Gamma_3$ shows that $\beta(x,t)\leq 1$ for $t\geq 0$, a.e. $x\in\Gamma_3$. On the other hand, if $\beta(x,t_0)=0$ at time t_0, then it follows from (31) that $\dot{\beta}(x,t)=0$ for all $t\geq t_0$ and therefore, $\beta(x,t)=0$ for all $t\geq t_0$, a.e. $x\in\Gamma_3$. We conclude that $0\leq\beta(x,t)\leq 1$ for all $t\in[0,T]$, a.e. $x\in\Gamma_3$.

3. Existence and uniqueness result

Now, we propose our existence and uniqueness result.

Theorem 3.1. *Under the above assumptions and the initial data satisfy* $\mathbf{u}_0\in\mathbf{V}$, $\mathbf{v}_0\in H$, $\zeta_0^{\ell}\in K^{\ell}$, $\beta_0\in L^2(\Gamma_3)$, $0\leq\beta_0\leq 1$, a.e. *on* Γ_3. *Then there exists a unique solution* $\{\mathbf{u},\boldsymbol{\sigma},\zeta,\varphi,\beta,\mathbf{D}\}$ *to problem* ***PV****. Moreover, the solution satisfies*

$$\mathbf{u}\in H^1(0,T;\mathbf{V})\cap C^1(0,T;H),\ \ddot{\mathbf{u}}\in L^2(0,T;\mathbf{V}') \tag{33}$$

$$\varphi\in C(0,T;W) \tag{34}$$

$$\beta\in W^{1,\infty}(0,T;L^2(\Gamma_3))\cap\mathcal{Z} \tag{35}$$

$$\boldsymbol{\sigma}\in L^2(0,T;\mathcal{H}),\ (\mathrm{Div}\,\boldsymbol{\sigma}^1,\mathrm{Div}\,\boldsymbol{\sigma}^2)\in L^2(0,T;\mathbf{V}') \tag{36}$$

$$\zeta\in H^1(0,T;E_0)\cap L^2(0,T;E_1) \tag{37}$$

$$\theta\in H^1(0,T;E_0)\cap L^2(0,T;E_1) \tag{38}$$

$$\mathbf{D}\in C(0,T;\mathcal{W}) \tag{39}$$

The proof of Theorem 3.1 is carried out in several steps that we prove in what follows, everywhere in this section we suppose that assumptions of Theorem 3.1 hold, and let a $\eta\in L^2(0,T;\mathbf{V}')$ be given. In the first step, we consider the following variational problem.

Problem PV$_\eta$. Find a $\mathbf{u}_\eta:[0,T]\to\mathbf{V}$, $\varphi_\eta:[0,T]\to W$ and $\beta_\eta:[0,T]\to L^2(\Gamma_3)$ such that

$$\left.\begin{array}{l}(\ddot{u}_\eta(t),v)_{V'\times V}+\sum_{\ell=1}^{2}(\mathcal{A}^{\ell}\varepsilon(\dot{\mathbf{u}}^{\ell}(t)),\ \varepsilon(\mathbf{v}^{\ell}))_{\mathcal{H}^\ell}=(\mathbf{f}(t)-\eta(t),\mathbf{v})_{V'\times V}\ \ \forall\mathbf{v}\in\mathbf{V},\ \textit{a.e.}\ t\in(0,T)\\ \sum_{\ell=1}^{2}(\mathcal{B}^{\ell}\nabla\varphi_\eta^{\ell}(t),\nabla\phi^{\ell})_{H^\ell}-\sum_{\ell=1}^{2}(\mathcal{E}^{\ell}\varepsilon(\mathbf{u}_\eta^{\ell}(t)),\nabla\phi^{\ell})_{H^\ell}=(q(t),\phi)_W,\quad\forall\phi\in W,\ \ \textit{a.e.}\ t\in(0,T)\\ \dot{\beta}_\eta(t)=-\Big(\beta_\eta(t)\big(\gamma_\nu(R_\nu([u_{\eta\nu}(t)]))^2+\gamma_\tau\,\big|\mathbf{R}_\tau([\mathbf{u}_{\eta\tau}(t)])\big|^2\big)-\varepsilon_a\Big)_+,\ \textit{a.e.}\ t\in(0,T)\\ \mathbf{u}_\eta^{\ell}(0)=\mathbf{u}_0^{\ell},\ \ \dot{\mathbf{u}}_\eta^{\ell}(0)=\mathbf{v}_0^{\ell},\ \ \beta_\eta(0)=\beta_0\quad\text{in }\Omega^{\ell}\end{array}\right\} \tag{40}$$

We have the following result for the problem.

Lemma 3.2. *There exists a unique solution to Problem* ***PV***$_\eta$ *and it has its regularity expressed in (33)–(35).*

For more details about the proof of this lemma, see [4] (lemmas 4.3; 4.5; 4.6).

In the second step, we let $\lambda\in L^2(0,T;E_0)$ be given and consider the following problem.

Problem PV$_\lambda$. *Find a* $\theta_\lambda = (\theta_\lambda^1, \theta_\lambda^2) : [0, T] \to E_0$, such that

$$\theta_\lambda(0) = \theta_0, \quad \sum_{\ell=1}^{2} (\dot{\theta}_\lambda^\ell(t) - \lambda^\ell(t) - \rho^\ell(t), \xi^\ell)_{L^2(\Omega^\ell)} + a_0(\theta_\lambda^\ell(t), \xi) = 0 \tag{41}$$

We have the following result for the problem.

Lemma 3.3. *There exists a unique solution to problem* ***PV***$_\lambda$ *and it has its regularity expressed in (38).*

Proof. By an application of the Poincaré–Friedrichs inequality, we can find a constant $\beta' > 0$ such that

$$\int_{\Omega^\ell} |\nabla \xi^\ell|^2 \mathrm{d}x + \frac{\alpha^\ell}{\kappa_0^\ell} \int_{\Gamma^\ell} |\xi^\ell|^2 \mathrm{d}a \geq \beta' \int_{\Omega^\ell} |\xi^\ell|^2 \mathrm{d}x \quad \forall \xi^\ell \in V^\ell, \quad \ell = 1, 2$$

Thus, we obtain

$$a_0(\xi, \xi) \geq c_1 \|\xi\|_V^2 \quad \forall \xi = (\xi^1, \xi^2) \in V \tag{42}$$

where $c_1 = \min(\kappa_0^1, \kappa_0^2) \cdot \min(1, \beta')/2$, which implies that a_0 is a_0 is V-elliptic. Consequently, based on classical arguments of functional analysis concerning parabolic equations, the parabolic equation (41) has a unique solution θ_λ satisfying (38). □

In the third step, we let $\mu \in L^2(0, T; E_0)$ be given and consider the following problem.

Problem PV$_\mu$. *Find a* $\zeta_\mu = (\zeta_\mu^1, \zeta_\mu^2) : [0, T] \to E_0$, such that

$$\zeta_\mu(t) \in K, \; \sum_{\ell=1}^{2} (\dot{\zeta}_\mu^\ell(t) - \mu^\ell(t), \xi^\ell - \zeta_\mu^\ell(t))_{L^2(\Omega^\ell)} + a(\zeta_\mu(t), \xi - \zeta_\mu(t)) \geq 0, \;\; \forall \xi \in K, \; \text{a.e.}\, t \in (0, T) \tag{43}$$

The following abstract result is for parabolic variational inequalities (see, e.g., [3, p. 47])

Theorem 3.4. *Let $X \subset Y = Y' \subset X'$ be a Gelfand triple. Let F be a nonempty, closed, and convex set of X. Assume that $a(.,.) : X \times X \to \mathbb{R}$ is a continuous and symmetric bilinear form such that for some constants $\alpha > 0$ and c_0,*

$$a(v, v) + c_0 \|v\|_Y^2 \geq \alpha \|v\|_X^2 \quad \forall v \in X \tag{44}$$

Then, for every $u_0 \in F$ and $f \in L^2(0, T; Y)$, there exists a unique function $u \in H^1(0, T; Y) \cap L^2(0, T; X)$ such that $u(0) = u_0$, $u(t) \in F$ $\forall t \in [0, T]$, and

$$(\dot{u}(t), v - u(t))_{X' \times X} + a(u(t), v - u(t)) \geq (f(t), v - u(t))_Y \;\; \forall v \in F \; a.e. \; t \in (0, T) \tag{45}$$

We prove next the unique solvability of problem **PV**$_\mu$.

Lemma 3.5. *There exists a unique solution to problem* ***PV***$_\mu$ *and it has its regularity expressed in (37).*

Proof. The inclusion mapping of $(E_1, \|.\|_{E_1})$ into $(E_0, \|.\|_{E_0})$ is continuous and its range is dense. We denote by E_1' the dual space of E_1 and, identifying the dual of E_0 with itself, we can write the Gelfand triple

$$E_1 \subset E_0 = E_0' \subset E_1'$$

We use the notation $(.,.)_{E_1' \times E_1}$ to represent the duality pairing between E' and E_1. We have

$$(\zeta, \xi)_{E_1' \times E_1} = (\zeta, \xi)_{E_0} \quad \forall \zeta \in E_0, \xi \in E_1$$

and we note that $K = K^1 \times K^2$ is a closed convex set in E_1. Then, using $\kappa_0^\ell > 0$ (see, (24)), and the fact that $\zeta_0 \in K$ in (24), it is easy to see that Lemma 3.5 is a straight consequence of Theorem 3.4. □

In the fourth step, we consider the following fixed point problem.

Problem PV$_{\eta\lambda\mu}$. *Find a* $\boldsymbol{\sigma}_{\eta\lambda\mu} = (\boldsymbol{\sigma}_{\eta\lambda\mu}^1, \boldsymbol{\sigma}_{\eta\lambda\mu}^2) : [0, T] \to \mathcal{H}$ *such that*

$$\boldsymbol{\sigma}_{\eta\lambda\mu}^\ell(t) = \mathcal{G}^\ell \boldsymbol{\varepsilon}(\mathbf{u}_\eta^\ell(t)) + \int_0^t \mathcal{F}^\ell(\boldsymbol{\sigma}_{\eta\lambda\mu}^\ell(s), \boldsymbol{\varepsilon}(\mathbf{u}_\eta^\ell(s)), \theta_\lambda^\ell(s), \zeta_\mu^\ell(s))\, \mathrm{d}s, \quad \text{a.e.}\, t \in (0, T), \quad \ell = 1, 2 \tag{46}$$

By using arguments similar to those in the proof lemma 3.7 in [1], we have the following result.

Lemma 3.6. *There exists a unique solution to problem* $\mathbf{PV}_{\eta\lambda\mu}$ *and it satisfies (36).*

We consider the element: $\Lambda(\eta,\lambda,\mu)(t) = \big(\Lambda^1(\eta,\lambda,\mu)(t), \Lambda^2(\eta,\lambda,\mu)(t), \Lambda^3(\eta,\lambda,\mu)(t)\big) \in \mathbf{V}' \times E_0 \times E_0$, defined by the equations

$$\forall \mathbf{v} \in \mathbf{V}, \quad (\Lambda^1(\eta,\lambda)(t), \mathbf{v})_{\mathbf{V}'\times\mathbf{V}} = \sum_{\ell=1}^{2} \big(\mathcal{G}^\ell \varepsilon(\mathbf{u}_\eta^\ell(t)),\ \varepsilon(\mathbf{v}^\ell)\big)_{\mathcal{H}^\ell} + \sum_{\ell=1}^{2} \big((\mathcal{E}^\ell)^* \nabla \varphi_\eta^\ell,\ \varepsilon(\mathbf{v}^\ell)\big)_{\mathcal{H}^\ell}$$

$$+ \sum_{\ell=1}^{2} \Big(\int_0^t \mathcal{F}^\ell\big(\boldsymbol{\sigma}_{\eta\lambda}^\ell, \varepsilon(\mathbf{u}_\eta^\ell(s)), \theta_\lambda(s), \zeta_\mu(s)\big)\, \mathrm{d}s,\ \varepsilon(\mathbf{v}^\ell)\Big)_{\mathcal{H}^\ell} + j_{\mathrm{ad}}(\beta_\eta(t), \mathbf{u}_\eta(t), \mathbf{v}) + j_{\nu c}(\mathbf{u}_\eta(t), \mathbf{v})$$

$$\Lambda^2(\eta,\theta,\mu)(t) = \Big(\Theta^1\big(\boldsymbol{\sigma}_{\eta\lambda\mu}^1(t), \boldsymbol{\varepsilon}(\mathbf{u}_\eta^1(t)), \theta_\lambda^1(t), \zeta_\mu^1(t)\big), \Theta^2\big(\boldsymbol{\sigma}_{\eta\lambda\mu}^2(t), \boldsymbol{\varepsilon}(\mathbf{u}_\eta^2(t)), \theta_\lambda^2(t), \zeta_\mu^2(t)\big)\Big)$$

$$\Lambda^3(\eta,\theta,\mu)(t) = \Big(\phi^1\big(\boldsymbol{\sigma}_{\eta\lambda\mu}^1(t), \boldsymbol{\varepsilon}(\mathbf{u}_\eta^1(t)), \theta_\lambda^1(t), \zeta_\mu^1(t)\big),\ \phi^2\big(\boldsymbol{\sigma}_{\eta\lambda\mu}^2(t), \boldsymbol{\varepsilon}(\mathbf{u}_\eta^2(t)), \theta_\lambda^2(t), \zeta_\mu^2(t)\big)\Big)$$

By using arguments similar to those in the proof of lemma 3.8 in [1], we have the following result.

Lemma 3.7. *The operator* Λ *has a unique fixed point* $(\eta^*, \lambda^*, \mu^*) \in L^2(0,T; \mathbf{V}' \times E_0 \times E_0)$.

We can now prove the existence of a solution to problem PV. To this aim, denote

$$\boldsymbol{\sigma}_*^\ell = \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\mathbf{u}}_*^\ell) + (\mathcal{E}^\ell)^* \nabla \varphi_*^\ell + \boldsymbol{\sigma}_{\eta^*\lambda^*\mu^*}^\ell \text{ and } \mathbf{D}_*^\ell = \mathcal{E}^\ell \varepsilon(\mathbf{u}_*^\ell) - \mathcal{B}^\ell \nabla \varphi_*^\ell$$

We use equalities $\Lambda^1(\eta^*,\lambda^*,\mu^*) = \eta^*$, $\Lambda^2(\eta^*,\lambda^*,\mu^*) = \lambda^*$ and $\Lambda^3(\eta^*,\lambda^*,\mu^*) = \mu^*$, it follows that

$$(\eta^*(t), v)_{\mathbf{V}'\times\mathbf{V}} = \sum_{\ell=1}^{2} \big(\mathcal{G}^\ell \varepsilon(\mathbf{u}_*^\ell(t)),\ \varepsilon(\mathbf{v}^\ell)\big)_{\mathcal{H}^\ell} + \sum_{\ell=1}^{2} \big((\mathcal{E}^\ell)^* \nabla \varphi_*^\ell,\ \varepsilon(\mathbf{v}^\ell)\big)_{\mathcal{H}^\ell} + j_{\mathrm{ad}}(\beta_*(t), \mathbf{u}_*(t), \mathbf{v}) +$$

$$j_{\nu c}(\mathbf{u}_*(t), \mathbf{v}) + \sum_{\ell=1}^{2} \Big(\int_0^t \mathcal{F}^\ell\Big(\boldsymbol{\sigma}_*^\ell(s) - \mathcal{A}^\ell \varepsilon(\dot{\mathbf{u}}_*^\ell(s)) - (\mathcal{E}^\ell)^* \nabla \varphi_*^\ell(s), \varepsilon(\mathbf{u}_*^\ell(s)), \zeta_*^\ell(s)\Big)\, \mathrm{d}s,\ \varepsilon(\mathbf{v}^\ell)\Big)_{\mathcal{H}^\ell}, \ \forall \mathbf{v} \in \mathbf{V}$$

$$\lambda_*^\ell(t) = \Theta^\ell\big(\boldsymbol{\sigma}_*^\ell(t) - \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\mathbf{u}}_*^\ell(t)) - (\mathcal{E}^\ell)^* \nabla \varphi_*^\ell(t), \boldsymbol{\varepsilon}(\mathbf{u}_*^\ell(t)), \theta_*^\ell(t), \zeta_*^\ell(t)\big),\ \ell = 1,2$$

$$\mu_*^\ell(t) = \phi^\ell\big(\boldsymbol{\sigma}_*^\ell(t) - \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\mathbf{u}}_*^\ell(t)) - (\mathcal{E}^\ell)^* \nabla \varphi_*^\ell(t), \boldsymbol{\varepsilon}(\mathbf{u}_*^\ell(t)), \theta_*^\ell(t), \zeta_*^\ell(t)\big),\ \ell = 1,2$$

The uniqueness part of Theorem 3.1 is a consequence of the uniqueness of the fixed point of the operators Λ and the unique solvability of the problems $\mathbf{PV}_\eta$, $\mathbf{PV}_\lambda$, $\mathbf{PV}_\mu$ and $\mathbf{PV}_{\eta\lambda\mu}$, which completes the proof of Theorem 3.1.

References

[1] A. Azeb Ahmed, S. Boutechebak, Analysis of a dynamic thermo-elastic-viscoplastic contact problem, Electron. J. Qual. Theory Differ. Equ. 71 (2013) 1.
[2] M. Frémond, B. Nedjar, Damage in concrete: the unilateral phenomenon, Nucl. Eng. Des. 156 (1995) 323–335.
[3] M. Sofonea, W. Han, M. Shillor, Analysis and Approximation of Contact Problems with Adhesion or Damage, Pure Appl. Math., vol. 276, Chapman-Hall/CRC Press, New York, 2006.
[4] T. Hadj ammar, B. Benyattou, S. Drabla, A dynamic contact problem between elasto-viscoplastic piezoelectric bodies, Electron. J. Qual. Theory Differ. Equ. 49 (2014) 1.

Further reading

[1] M. Frémond, Équilibre des structures qui adhèrent à leur support, C. R. Acad. Sci. Paris, Ser. II 295 (1982) 913–916.