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A high-order corrector estimate for a semi-linear elliptic system in perforated domains

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ABSTRACT

We derive in this Note a high-order corrector estimate for the homogenization of a microscopic semi-linear elliptic system posed in perforated domains. The major challenges are the presence of nonlinear volume and surface reaction rates. This type of correctors justifies mathematically the convergence rate of formal asymptotic expansions for the two-scale homogenization settings. As the main tool, we use energy-like estimates to investigate the error estimate between the micro and macro concentrations and between the corresponding micro- and macro-concentration gradients. This work aims at generalizing the results reported in [1,2].

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1. Introduction and problem setting

This Note is a follow-up of [2], in which the derivation of a high-order corrector for a microscopic semi-linear elliptic system posed in heterogeneous/perforated domains is concentrated. In the terminology of homogenization, a *corrector* or *corrector estimate* wants to quantify the error between the approximate solution (governed by a certain asymptotic procedure) and the exact solution. Typically, this kind of estimate is helpful also in controlling the approximation error of numerical methods for multiscale problems (e.g., [3,4]). The main result of this Note is Theorem 3.1, where we report the upper bound of the corrector up to an arbitrary high order.

We consider the semi-linear elliptic boundary value problem

$$\mathcal{A}^{\varepsilon} u_i^{\varepsilon} \equiv \nabla \cdot \left(-d_i^{\varepsilon} \nabla u_i^{\varepsilon}\right) = R_i\left(u_1^{\varepsilon}, \ldots, u_N^{\varepsilon}\right) \quad \text{in } \Omega^{\varepsilon}$$

associated with the boundary conditions

$$d_i^{\varepsilon} \nabla u_i^{\varepsilon} \cdot \mathrm{n} = \varepsilon\left(a_i^{\varepsilon} u_i^{\varepsilon} - b_i^{\varepsilon} F_i\left(u_i^{\varepsilon}\right)\right) \quad \text{across } \Gamma^{\varepsilon}$$
$$u_i^{\varepsilon} = 0 \quad \text{across } \Gamma^{ext}$$

for $i \in \{1, \ldots, N\}$, with $N \geq 2$ being the number of involved concentrations. For simplicity, we refer to this problem as $\left(P^{\varepsilon}\right)$.

This problem is connected to the Smoluchowski–Soret–Dufour modeling of the evolution of temperature and colloid concentrations [5,6]. Here, $u^{\varepsilon} := \left(u_1^{\varepsilon}, \ldots, u_N^{\varepsilon}\right)$ denotes the vector of the concentrations, d_i^{ε} represents the molecular diffusion with R_i being the volume reaction rate and $a_i^{\varepsilon}, b_i^{\varepsilon}$ are deposition coefficients, whilst F_i indicates a surface chemical reaction for the immobile species. Notice that the quantity ε is called the homogenization parameter or the scale factor. Denote by

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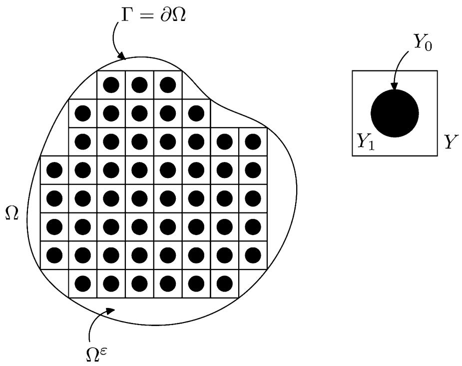

Fig. 1.1. Admissible 2-D perforated domain (left) and basic geometry of the microstructure (right). (By courtesy of Mai Thanh Nhat Truong, Hankyong National University, Republic of Korea.)

$x \in \Omega^\varepsilon$ the macroscopic variable and by $y = x/\varepsilon$ the microscopic variable representing high oscillations at the microscopic geometry. Henceforward, we understand throughout this paper the following convention:

$$d_i^\varepsilon(x) = d_i\left(\frac{x}{\varepsilon}\right) = d_i(y), \quad x \in \Omega^\varepsilon, y \in Y_1$$

with the same meaning for all the oscillating data such as a_i^ε, b_i^ε, etc.

The perforated domain $\Omega^\varepsilon \subset \mathbb{R}^d$ is thought to approximate a porous medium and its precise description can be found in [1,7,2]. As an example, we depict in Fig. 1.1 an admissible geometry of our medium and the corresponding microstructure.

Our corrector estimate evaluation starts from the two-scale asymptotic expansion up to Mth-level ($M \geq 2$) given by

$$u_i^\varepsilon(x) = \sum_{m=0}^{M} \varepsilon^m u_{i,m}(x,y) + \mathcal{O}\left(\varepsilon^{M+1}\right), \quad x \in \Omega^\varepsilon \tag{1}$$

where $u_{i,m}(x,\cdot)$ is Y-periodic for $0 \leq m \leq M$ and $i \in \{1,...,N\}$.

It is worth noting that in [2], we have analyzed the solvability of (P^ε) using the energy minimization approach and derived the upscaled equations as well as the corresponding effective coefficients. Furthermore, we showed that using the separation of variables, the functions $u_{i,m}(x,y)$ for $0 \leq m \leq M$ can be structured as, e.g.,

$$u_{i,0}(x,y) = \tilde{u}_{i,0}(x)$$
$$u_{i,1}(x,y) = -\chi_{i,1}(y) \cdot \nabla_x \tilde{u}_{i,0}(x)$$
$$u_{i,2}(x,y) = \chi_{i,2}(y) \nabla_x^2 \tilde{u}_{i,0}(x)$$

with $\tilde{u}_{i,0}(x)$ being determined uniquely from the auxiliary problem and $\chi_{i,m}$ satisfying the corresponding cell problems. One can also rule out the $\tilde{u}_{i,0}$-based construction of $u_{i,m}$ that $u_{i,m}(x,y) = (-1)^m \chi_{i,m}(y) \nabla_x^m \tilde{u}_{i,0}(x)$ for $1 \leq m \leq M$.

In this scenario, we wish to obtain the error estimate up to a high-order expansion for the differences of concentrations and their gradients, albeit some types have been investigated so far. In particular, we prove in this Note a corrector in the form of

$$u^\varepsilon - \sum_{k=0}^{K} \varepsilon^k u_k - m^\varepsilon \sum_{m=K+1}^{M} \varepsilon^m u_m \tag{2}$$

in which we fix $K \in \mathbb{N}$ such that $0 \leq K \leq M-2$ and $m^\varepsilon \in C_c^\infty(\Omega)$ is a cut-off function such that $\varepsilon\left|\nabla m^\varepsilon\right| \leq C$ and

$$m^\varepsilon(x) := \begin{cases} 1, & \text{if } \operatorname{dist}(x,\Gamma) \leq \varepsilon \\ 0, & \text{if } \operatorname{dist}(x,\Gamma) \geq 2\varepsilon \end{cases}$$

(see [1] for more properties of m_ε).

With the above definition of m^ε, the second term in (2) vanishes everywhere except in a neighborhood of the boundary of Ω^ε. In other words, the appearance of m^ε provides that the speed of convergence in the interior of the material is better than the rate at the vicinity of the boundary, albeit the standard result expected that $\left\|u^\varepsilon - u_0\right\|_{H^1(\Omega^\varepsilon)} \leq C\varepsilon^{1/2}$. It is then easy to see that (2) includes the cases

$$u^\varepsilon - \sum_{m=0}^{M} \varepsilon^m u_m \quad \text{and} \quad u^\varepsilon - u_0 - m^\varepsilon \sum_{m=1}^{M} \varepsilon^m u_m, \quad M \geq 2$$

reported in [2] and further in [1].

The similarity between Theorem 3.1 and Theorem 11 in [2] is that under the energy-type method, we employ the cut-off function m^{ε} to distinguish the speeds of convergence in H^1-norm of the limit u_0 in the interior part of the perforated material and at the boundary of inclusions. The main difference consists in showing that if $K = M - 2$, the corrector (5) yields the order of $\mathcal{O}(\varepsilon^{M-\frac{3}{2}})$, whilst it only gives the order $\mathcal{O}(\varepsilon^{\frac{1}{2}})$ in Theorem 11 in [2].

Further comments can be found in Remark 3.1 and Remark 3.2, discussing the *a priori* assumptions on the smoothness of the limit u_0 and on the structure of the cell problems for arbitrarily high-order correctors.

2. Assumptions. Function spaces

We introduce the function space

$$V^{\varepsilon} := \left\{ v \in H^1\left(\Omega^{\varepsilon}\right) | v = 0 \text{ on } \Gamma^{ext} \right\}$$

which is a closed subspace of the Hilbert space $H^1\left(\Omega^{\varepsilon}\right)$ with the semi-norm

$$\|v\|_{V^{\varepsilon}} = \left(\sum_{i=1}^{d} \int_{\Omega^{\varepsilon}} \left| \frac{\partial v}{\partial x_i} \right|^2 \mathrm{d}x \right)^{1/2} \quad \text{for all } v \in V^{\varepsilon}$$

We define $\mathcal{V}^{\varepsilon} := V^{\varepsilon} \times \ldots \times V^{\varepsilon}$ as well as some other function spaces such as $\mathcal{W}^{p,q}\left(\Omega^{\varepsilon}\right) := W^{p,q}(\Omega^{\varepsilon}) \times \ldots \times W^{p,q}(\Omega^{\varepsilon})$ the Sobolev space of functions with index of differentiability $p \in \mathbb{N}$ and integrability q and $\mathcal{W}^{q}\left(\Omega^{\varepsilon}\right) := L^q\left(\Omega^{\varepsilon}\right) \times \ldots \times L^q\left(\Omega^{\varepsilon}\right)$ for $q \in (2, \infty]$.

To handle the corrector estimates, we need the following assumptions.

(A_1) The diffusion coefficient $d_i^{\varepsilon} \in L^{\infty}\left(\mathbb{R}^d\right)$ is Lipschitz and Y-periodic, and there exists a positive constant α_i such that

$$d_i\left(y\right) \xi_i \xi_j \geq \alpha_i \left|\xi\right|^2 \quad \text{for any } \xi \in \mathbb{R}^d$$

(A_2) The deposition coefficients $a_i^{\varepsilon}, b_i^{\varepsilon} \in L^{\infty}\left(\Gamma^{\varepsilon}\right)$ are positive and Y-periodic.

(A_3) The reaction rates $R_i : \Omega^{\varepsilon} \times [0,\infty)^N \to \mathbb{R}$ and $F_i : \Gamma^{\varepsilon} \times [0,\infty) \to \mathbb{R}$ are Carathéodory functions. Moreover, they satisfy the structural assumptions:

$$R_i\left(\sum_{m=0}^{M} \varepsilon^m u_{1,m}, \ldots, \sum_{m=0}^{M} \varepsilon^m u_{N,m} \right) = \sum_{m=0}^{M} \varepsilon^m \bar{R}_i\left(u_{1,m}, \ldots, u_{N,m}\right) + \mathcal{O}\left(\varepsilon^{M+1}\right) \tag{3}$$

$$F_i\left(\sum_{m=0}^{M} \varepsilon^m u_{i,m} \right) = \sum_{m=0}^{M} \varepsilon^m \bar{F}_i\left(u_{i,m}\right) + \mathcal{O}\left(\varepsilon^{M+1}\right) \tag{4}$$

where $\bar{R}_i$ and $\bar{F}_i$ are global Lipschitz functions with the Lipschitz constants L_i and K_i for $i \in \{1, \ldots, N\}$, in the sense that

$$\left|\bar{R}_i\left(u_{1,m}, \ldots, u_{N,m}\right) - \bar{R}_i\left(v_{1,m}, \ldots, v_{N,m}\right)\right| \leq L_i \sum_{i=1}^{N} \left|u_{i,m} - v_{i,m}\right|$$

$$\left|\bar{F}_i\left(u_{1,m}, \ldots, u_{N,m}\right) - \bar{F}_i\left(v_{1,m}, \ldots, v_{N,m}\right)\right| \leq K_i \sum_{i=1}^{N} \left|u_{i,m} - v_{i,m}\right|$$

for every $0 \leq m \leq M$.

Unless otherwise specified, all the constants C appearing in this Note are independent of the homogenization parameter ε, but the respective values may differ from line to line and may change even within a single chain of estimates.

3. Corrector estimate

Theorem 3.1. *Assume that (A_1)–(A_3) hold. Let u^{ε} be the vector of solutions of the elliptic system $\left(P^{\varepsilon}\right)$. Consider the high-order asymptotic expansion (1) up to M-level ($M \geq 2$) and take $u_0 \in \mathcal{W}^{M+2,\infty}(\Omega^{\varepsilon}) \cap \mathcal{W}^{M+1,\infty}(\Gamma^{\varepsilon})$ and $u_m \in \mathcal{W}^{\infty}\left(\Omega^{\varepsilon}; H^1_{\#}\left(Y_1\right)/\mathbb{R}\right)$ for all $0 \leq m \leq M$. For a fixed $K \in \mathbb{N}$ such that $0 \leq K \leq M-2$, the following corrector estimate holds:*

$$\left\| u^{\varepsilon} - \sum_{k=0}^{K} \varepsilon^k u_k - m^{\varepsilon} \sum_{m=K+1}^{M} \varepsilon^m u_m \right\|_{\mathcal{V}^{\varepsilon}} \leq C \left(\varepsilon^{M-1} + \varepsilon^M + \sum_{m=K+1}^{M} \left(\varepsilon^{m-\frac{1}{2}} + \varepsilon^{m+\frac{1}{2}} \right) \right) \tag{5}$$

where $C > 0$ is a generic ε-independent constant.

Proof. Before giving the proof, let us recall the structural inequalities of the cut-off function m^{ε}. The following useful estimates (cf. [8]) hold true:

$$\left\|1-m^{\varepsilon}\right\|_{L^2(\Omega^{\varepsilon})} \le C\varepsilon^{1/2}, \quad \varepsilon\left\|\nabla m^{\varepsilon}\right\|_{L^2(\Omega^{\varepsilon})} \le C\varepsilon^{1/2} \tag{6}$$

To bound from above in terms of ε the quantity (2), we can reduce the discussion to the corrector at ith concentration, which is defined as

$$\Psi_i^{\varepsilon} := u_i^{\varepsilon} - \sum_{k=0}^{K} \varepsilon^k u_{i,k} - m^{\varepsilon} \sum_{m=K+1}^{M} \varepsilon^m u_{i,m} \quad \text{for } i \in \{1, \dots, N\}$$

We observe that Ψ_i^{ε} can be decomposed further as

$$\Psi_i^{\varepsilon} = \underbrace{u_i^{\varepsilon} - \sum_{m=0}^{M} \varepsilon^m u_{i,m}}_{\varphi_i^{\varepsilon}} + \underbrace{\left(1-m^{\varepsilon}\right) \sum_{m=K+1}^{M} \varepsilon^m u_{i,m}}_{\gamma_i^{\varepsilon}} \tag{7}$$

As in [2,1], we use the auxiliary problems

$$\begin{cases} \mathcal{A}_0 u_{i,0} = 0, & \text{in } Y_1 \\ -d_i(y)\nabla_y u_{i,0}\cdot \mathrm{n} = 0, & \text{on } \partial Y_0 \\ u_{i,0} \text{ is } Y - \text{periodic in } y \end{cases} \tag{8}$$

$$\begin{cases} \mathcal{A}_0 u_{i,1} = -\mathcal{A}_1 u_{i,0}, & \text{in } Y_1 \\ -d_i(y)\left(\nabla_x u_{i,0} + \nabla_y u_{i,1}\right)\cdot \mathrm{n} = 0, & \text{on } \partial Y_0 \\ u_{i,1} \text{ is } Y - \text{periodic in } y \end{cases} \tag{9}$$

$$\begin{cases} \mathcal{A}_0 u_{i,m+2} = \bar{R}_i(u_m) - \mathcal{A}_1 u_{i,m+1} - \mathcal{A}_2 u_{i,m}, & \text{in } Y_1 \\ -d_i(y)\left(\nabla_x u_{i,m+1} + \nabla_y u_{i,m+2}\right)\cdot \mathrm{n} = b_i(y)\bar{F}_i\left(u_{i,m}\right) - a_i(y)u_{i,m}, & \text{on } \partial Y_0 \\ u_{i,m+2} \text{ is } Y - \text{periodic in } y \end{cases} \tag{10}$$

for $0 \le m \le M-2$.

In (8)–(10), the operators $\mathcal{A}_0$, $\mathcal{A}_1$ and $\mathcal{A}_2$ are defined, respectively, as follows:

$$\mathcal{A}_0 := \nabla_y \cdot \left(-d_i(y)\nabla_y\right)$$
$$\mathcal{A}_1 := \nabla_x \cdot \left(-d_i(y)\nabla_y\right) + \nabla_y \cdot \left(-d_i(y)\nabla_x\right)$$
$$\mathcal{A}_2 := \nabla_x \cdot \left(-d_i(y)\nabla_x\right)$$

By induction, one can easily obtain that the first part of decomposition (7), the function φ_i^{ε}, satisfies the following equation:

$$\mathcal{A}^{\varepsilon}\varphi_i^{\varepsilon} = R_i\left(u^{\varepsilon}\right) - \sum_{m=0}^{M-2} \varepsilon^m \bar{R}_i(u_m) - \varepsilon^{M-1}\left(\mathcal{A}_1 u_{i,M} + \mathcal{A}_2 u_{i,M-1}\right) - \varepsilon^M \mathcal{A}_2 u_{i,M} \text{ in } \Omega^{\varepsilon} \tag{11}$$

associated with the following boundary condition at Γ^{ε}

$$-d_i^{\varepsilon}\nabla_x \varphi_i^{\varepsilon}\cdot \mathrm{n} = \varepsilon^M d_i^{\varepsilon}\nabla_x u_{i,M}\cdot \mathrm{n} + \varepsilon\left[a_i^{\varepsilon}\left(\sum_{m=0}^{M-2}\varepsilon^m u_{i,m} - u_i^{\varepsilon}\right) + b_i^{\varepsilon}\left(F_i\left(u_i^{\varepsilon}\right) - \sum_{m=0}^{M-2}\varepsilon^m \bar{F}_i\left(u_{i,m}\right)\right)\right] \tag{12}$$

Multiplying (11) by $\varphi_i \in V^{\varepsilon}$, integrating the result by parts, and finally using (12), we arrive at

$$\begin{aligned} \int_{\Omega^{\varepsilon}} d_i^{\varepsilon}\nabla\varphi_i^{\varepsilon}\nabla\varphi_i \,\mathrm{d}x = & \left\langle R_i\left(u^{\varepsilon}\right) - \sum_{m=0}^{M-2}\varepsilon^m \bar{R}_i(u_m), \varphi_i \right\rangle_{L^2(\Omega^{\varepsilon})} \\ & - \varepsilon^{M-1}\left\langle \mathcal{A}_1 u_{i,M} + \mathcal{A}_2 u_{i,M-1} + \varepsilon\mathcal{A}_2 u_{i,M}, \varphi_i\right\rangle_{L^2(\Omega^{\varepsilon})} \\ & - \varepsilon\left\langle a_i^{\varepsilon}\left(\sum_{m=0}^{M-2}\varepsilon^m u_{i,m} - u_i^{\varepsilon}\right) + b_i^{\varepsilon}\left(F_i\left(u_i^{\varepsilon}\right) - \sum_{m=0}^{M-2}\varepsilon^m \bar{F}_i\left(u_{i,m}\right)\right), \varphi_i\right\rangle_{L^2(\Gamma^{\varepsilon})} \\ & - \varepsilon^M \int_{\Gamma^{\varepsilon}} d_i^{\varepsilon}\nabla_x u_{i,M}\cdot \mathrm{n}\varphi_i \,\mathrm{d}S_{\varepsilon} \end{aligned} \tag{13}$$

We can now gain the first part of the corrector (5), i.e. we shall estimate each integral on the right-hand side of (13), which we denote by $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$ and $\mathcal{I}_4$, respectively.

Let $\bar{L} := \max\left\{\bar{L}_1, ..., \bar{L}_N\right\}$. Using (3) in combination with the structural inequality $\left\|\bar{R}_i\left(u_m\right)\right\|_{L^2(\Omega^\varepsilon)} \le \bar{L}\left\|u_m\right\|_{\mathcal{W}^2(\Omega^\varepsilon)} + \left\|\bar{R}_i\left(0\right)\right\|_{L^2(\Omega^\varepsilon)}$ for all $0 \le m \le M$, we see that

$$\left|\left\langle R_i\left(u^\varepsilon\right) - \sum_{m=0}^{M-2}\varepsilon^m \bar{R}_i\left(u_m\right), \varphi_i\right\rangle_{L^2(\Omega^\varepsilon)}\right| \le \varepsilon^{M-1}\left(\bar{L}\left\|u_{M-1}\right\|_{\mathcal{W}^2(\Omega^\varepsilon)} + \left\|\bar{R}_i\left(0\right)\right\|_{L^2(\Omega^\varepsilon)}\right)\left\|\varphi_i\right\|_{L^2(\Omega^\varepsilon)}$$
$$+ \varepsilon^{M}\left(\bar{L}\left\|u_{M}\right\|_{\mathcal{W}^2(\Omega^\varepsilon)} + \left\|\bar{R}_i\left(0\right)\right\|_{L^2(\Omega^\varepsilon)}\right)\left\|\varphi_i\right\|_{L^2(\Omega^\varepsilon)}$$
$$\le C\left(\varepsilon^{M-1} + \varepsilon^M\right)\left\|\varphi_i\right\|_{L^2(\Omega^\varepsilon)} \tag{14}$$

Direct computations give

$$\mathcal{A}_1 u_{i,M} = (-1)^{M+1}\left[d_i\left(\frac{x}{\varepsilon}\right)\chi_{i,M}\left(\frac{x}{\varepsilon}\right) + \nabla_y\left(d_i\left(\frac{x}{\varepsilon}\right)\chi_{i,M}\left(\frac{x}{\varepsilon}\right)\right)\right]\nabla_x^{M+1}\tilde{u}_{i,0}$$
$$\mathcal{A}_2 u_{i,M-1} = (-1)^{M} d_i\left(\frac{x}{\varepsilon}\right)\chi_{i,M-1}\left(\frac{x}{\varepsilon}\right)\nabla_x^{M+1}\tilde{u}_{i,0}$$
$$\mathcal{A}_2 u_{i,M} = (-1)^{M+1} d_i\left(\frac{x}{\varepsilon}\right)\chi_{i,M}\left(\frac{x}{\varepsilon}\right)\nabla_x^{M+2}\tilde{u}_{i,0}$$

Due to $u_{i,0} \in W^{M+2,\infty}(\Omega^\varepsilon)$ and $u_{i,m} \in L^\infty\left(\Omega^\varepsilon; H^1_\#\left(Y_1\right)/\mathbb{R}\right)$ for all $0 \le m \le M$ in combination with (A_1), the second integral $\mathcal{I}_2$ can be bounded above by

$$\varepsilon^{M-1}\left|\left\langle \mathcal{A}_1 u_{i,M} + \mathcal{A}_2 u_{i,M-1} + \varepsilon\mathcal{A}_2 u_{i,M}, \varphi_i\right\rangle_{L^2(\Omega^\varepsilon)}\right| \le C\varepsilon^{M-1}\left\|\varphi_i\right\|_{L^2(\Omega^\varepsilon)} \tag{15}$$

Let $\bar{K} := 1 + \max\left\{\bar{K}_1, ..., \bar{K}_N\right\}$. For the integral $\mathcal{I}_3$, we proceed as in the proof of (14). We thus claim that

$$\varepsilon\left|\left\langle a_i^\varepsilon\left(\sum_{m=0}^{M-2}\varepsilon^m u_{i,m} - u_i^\varepsilon\right) + b_i^\varepsilon\left(F_i\left(u_i^\varepsilon\right) - \sum_{m=0}^{M-2}\varepsilon^m \bar{F}_i\left(u_{i,m}\right)\right), \varphi_i\right\rangle_{L^2(\Gamma^\varepsilon)}\right| \le C\left(\varepsilon^{M-1} + \varepsilon^M\right)\left\|\varphi_i\right\|_{L^2(\Omega^\varepsilon)} \tag{16}$$

in which we use (1) and (4) together with (A_2) and the Hölder inequality, as well as the trace inequality (cf. [1, Lemma 2.31]). On top of that, it yields for the last integral $\mathcal{I}_4$ that

$$\varepsilon^M\left|\int_{\Gamma^\varepsilon} d_i^\varepsilon \nabla_x u_{i,M}\cdot \mathrm{n}\varphi_i \,\mathrm{d}S_\varepsilon\right| \le \varepsilon^M\left\|d_i^\varepsilon \nabla_x u_{i,M}\cdot\mathrm{n}\right\|_{L^2(\Gamma^\varepsilon)}\left\|\varphi_i\right\|_{L^2(\Gamma^\varepsilon)}$$
$$\le C\varepsilon^{M-1}\left\|\varphi_i\right\|_{L^2(\Omega^\varepsilon)} \tag{17}$$

where we follow the computations that $\left\|d_i^\varepsilon \nabla_x u_{i,M}\cdot\mathrm{n}\right\|_{L^2(\Gamma^\varepsilon)} \le C\varepsilon^{-1/2}$ and apply again the trace inequality.

Combining (13)–(17), we observe that

$$\left|\left\langle \varphi_i^\varepsilon, \varphi_i\right\rangle_{V^\varepsilon}\right| \le C\left(\varepsilon^{M-1} + \varepsilon^M\right)\left\|\varphi_i\right\|_{L^2(\Omega^\varepsilon)} \quad \text{for } \varphi_i \in V^\varepsilon \text{ and } i \in \{1, ..., N\} \tag{18}$$

which then leads to $\left\|\varphi_i^\varepsilon\right\|_{V^\varepsilon} \le C\varepsilon^{M-1}$ by choosing $\varphi_i = \varphi_i^\varepsilon$ for $i \in \{1, ..., N\}$.

It remains to estimate the second part of decomposition (7). We consider the following quantity:

$$\left\langle \gamma_i^\varepsilon, \varphi_i\right\rangle_{V^\varepsilon} \quad \text{for } \varphi_i \in V^\varepsilon \text{ and } i \in \{1, ..., N\}$$

At this stage, the following estimate is straightforward due to (6):

$$\left|\left\langle \left(1 - m^\varepsilon\right)\sum_{m=K+1}^{M}\varepsilon^m u_{i,m}, \varphi_i\right\rangle_{V^\varepsilon}\right| \le C\left\|\nabla\left(1 - m^\varepsilon\right)\left(\sum_{m=K+1}^{M}\varepsilon^m u_{i,m}\right)\right\|_{L^2(\Omega^\varepsilon)}\left\|\varphi_i\right\|_{V^\varepsilon}$$
$$+ C\left\|\left(1 - m^\varepsilon\right)\nabla\left(\sum_{m=K+1}^{M}\varepsilon^m u_{i,m}\right)\right\|_{L^2(\Omega^\varepsilon)}\left\|\varphi_i\right\|_{V^\varepsilon}$$
$$\le C\sum_{m=K+1}^{M}\varepsilon^m\left\|\nabla\left(1 - m^\varepsilon\right)\right\|_{L^2(\Omega^\varepsilon)}\left\|\varphi_i\right\|_{V^\varepsilon}$$

$$+ C \sum_{m=K+1}^{M} \varepsilon^m \left\| 1 - m^\varepsilon \right\|_{L^2(\Omega^\varepsilon)} \|\varphi_i\|_{V^\varepsilon}$$
$$\leq C \sum_{m=K+1}^{M} \left(\varepsilon^{m-\frac{1}{2}} + \varepsilon^{m+\frac{1}{2}} \right) \|\varphi_i\|_{V^\varepsilon} \text{ for all } \varphi_i \in V^\varepsilon \tag{19}$$

Thanks to the triangle inequality, we combine (18) and (19) to get

$$\left| \left\langle \Psi_i^\varepsilon, \varphi_i \right\rangle_{V^\varepsilon} \right| \leq C \left(\varepsilon^{M-1} + \varepsilon^M + \sum_{m=K+1}^{M} \left(\varepsilon^{m-\frac{1}{2}} + \varepsilon^{m+\frac{1}{2}} \right) \right) \|\varphi_i\|_{V^\varepsilon} \quad \text{for } \varphi_i \in V^\varepsilon$$

By choosing $\varphi_i = \Psi_i^\varepsilon$ and then by simplifying both sides of the resulting estimate by $\left\| \Psi_i^\varepsilon \right\|_{V^\varepsilon}$, we obtain that

$$\left\| \Psi_i^\varepsilon \right\|_{V^\varepsilon} \leq C \left(\varepsilon^{M-1} + \varepsilon^M + \sum_{m=K+1}^{M} \left(\varepsilon^{m-\frac{1}{2}} + \varepsilon^{m+\frac{1}{2}} \right) \right)$$

This completes the proof of Theorem 3.1. □

Remark 3.1. To obtain high-order corrector estimates, the limit u_0 has to be very smooth as stated e.g., in Theorem 3.1. The reason is that at the Mth level of expansion, we need ε-independent L^∞-bounds of the terms $\nabla_x^{M+1}\tilde{u}_{i,0}$, $\nabla_x^{M+2}\tilde{u}_{i,0}$ in Ω^ε and of $\nabla_x^M \tilde{u}_{i,M}$ on Γ^ε. To support this approach, recall that u_0 is solution of a homogenized system $\nabla_x \cdot (-q_i \nabla_x u_{i,0}) = \bar{R}_i(u_0), i \in \{1, ..., N\}$ in which q_i are (positive constant) homogenized coefficients given by

$$q_i = \frac{1}{|Y_1|} \int_{Y_1} d_i(y)(-\nabla_y \chi_{i,1} + \mathbb{I})\, \mathrm{d}y$$

while $\mathbb{I}$ stands for the identity matrix. This homogenized system is associated with the zero Dirichlet boundary condition at Γ^{ext} and still satisfies the ellipticity condition.

Note that if we suppose, for simplicity, that $\bar{R}_i$ are linear functions with respect to u_0; then the homogenized system becomes the nonhomogeneous elliptic equation in vectorial form. Therefore, we can apply the classical results in [9, Theorem 12.4] to guarantee that the derivatives of u_0 up to the desired order are in $L^\infty(\Omega)$. Thus, the needed smoothness of u_0 when dealing with the high-order correctors ($M \geq 2$) is obtainable. This result can be used similarly when we consider the correctors for $u^\varepsilon - u_0$ and $u^\varepsilon - u_0 - \varepsilon u_1$ derived from (2) with $K = 0$ and $K = 1$, respectively.

Remark 3.2. From the high-order auxiliary problems (8)–(10) and the fact that $u_{i,m}(x, y) = (-1)^m \chi_{i,m}(y) \nabla_x^m \tilde{u}_{i,0}(x)$ for $1 \leq m \leq M$, one can derive the corresponding cell problems for the high-order corrector:

$$\begin{cases} \mathcal{A}_0 \chi_{i,1} = \nabla_y d_i(y), & \text{in } Y_1 \\ -d_i(y) \nabla_y \chi_{i,1} \cdot \mathrm{n} = d_i(y) \cdot \mathrm{n}, & \text{on } \partial Y_0 \\ \chi_{i,1} \text{ is } Y-\text{periodic in } y \end{cases}$$

and

$$\begin{cases} \nabla_y \cdot \left(-d_i(y) \left(\nabla_y \chi_{i,m+2} - \chi_{i,m+1} \right) \right) \nabla_x^{m+2} \tilde{u}_{i,0} & \\ = (-1)^m \bar{R}_i \left((-1)^m \chi_{1,m} \nabla_x^m \tilde{u}_{1,0}, ..., (-1)^m \chi_{N,m} \nabla_x^m \tilde{u}_{N,0} \right) - (d_i(y) - \mathbb{I}) \nabla_y \chi_{i,m+1}(y) \nabla_x^{m+2} \tilde{u}_{i,0}, & \text{in } Y_1 \\ -d_i(y) \left(\nabla_y \chi_{i,m+2} - \chi_{i,m+1} \right) \nabla_x^{m+2} \tilde{u}_{i,0} \cdot \mathrm{n} = (-1)^m b_i(y) \bar{F}_i \left((-1)^m \chi_{i,m} \nabla_x^m \tilde{u}_{i,0} \right) - a_i(y) \chi_{i,m} \nabla_x^m \tilde{u}_{i,0}, & \text{on } \partial Y_0 \\ \chi_{i,m+2} \text{ is } Y-\text{periodic in } y \end{cases}$$

where $x \in \Omega$ is viewed here as the involved parameter, while $0 \leq m \leq M - 2$ with $i \in \{1, ..., N\}$. Obviously, these problems are linear, which ensures that their solvability is standard.

We also remark that from elliptic regularity theory [10,11], since Y_1 is a non-convex polygon, the above cell system for $\chi_{i,m}$ only admits a unique solution whose regularity is $H^{1+s}(Y_1)$ for $s \in (-1/2, 1/2)$ (cf. [10]), and we cannot go further from this regularity no matter how smooth the involved terms are. In addition, the non-existence result for this type of problems can be found in, e.g., [11, Theorem 14.11].

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