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Dynamic and quasi-electromagnetostatic evolution of a thermoelectromagnetoelastic body

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ABSTRACT

A standard technique of evolution equations in Hilbert spaces of possible states with finite energy supplies results of existence and uniqueness for the dynamic evolution of a thermoelectromagnetoelastic body and for its "quasi-electromagnetostatic approximation" whose relevance is established through a convergence result as a parameter, accounting for the ratio of the speed of elastic wave propagation to the celerity of the light, goes to zero.

1. Introduction

Recently, thermoelectromagnetoelastic materials have been artificially engineered for the design of smart structures as thermoelectromagnetoelastic actuators or sensors. So it is of interest to propose an efficient mathematical model for the transient response of a body made of such materials to a given loading. First we will consider the fully dynamic situation that couples transient thermoelastic equations with the Maxwell equations. Due to the large discrepancy between the speed of elastic wave propagation and the celerity of light, a "quasi-electromagnetostatic approximation" has been proposed (see [1]) in order to practice computations. So, here, by using a technique of evolution equations in Hilbert spaces of possible states with finite energy, we intend to give results on the consistency of both models and on the relevance of the second one.

As we are mainly concerned with the status of the quasi-electromagnetostatic approximation, we directly consider the so-called non-dimensionalized equations (see [1,2] for their derivation), which involves a small parameter δ, accounting for the ratio of the maximum of speed of elastic wave propagation to the celerity of the light, and reads as:

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$$(\mathcal{P}_\delta)\begin{cases}\rho\dot{v}_\delta-\operatorname{div}\sigma_\delta=f,\quad \dot{\Delta}_\delta-(\operatorname{div}\kappa\nabla\theta_\delta,\frac{1}{\delta}\mathcal{M}z_\delta)=(r,-J,-K)\quad\text{in }\Omega\\ u_\delta=0\text{ on }\Gamma_{\mathrm{MD}},\quad \sigma_\delta n=g_{\mathrm{M}}\text{ on }\Gamma_{\mathrm{MN}}\\ \theta_\delta=0\text{ on }\Gamma_{\Theta\mathrm{D}},\quad \kappa\nabla\theta_\delta\cdot n=g_\Theta\text{ on }\Gamma_{\Theta\mathrm{N}}\\ E_\delta\wedge n=k\text{ on }\Gamma_{\mathrm{E}},\quad H_\delta\wedge n=j\text{ on }\Gamma_{\mathrm{H}}\\ (\sigma_\delta,\Delta_\delta)=M(e(u_\delta),\theta_\delta,z_\delta)\text{ in }\Omega\\ U_\delta(\cdot,0):=(u_\delta,v_\delta,\theta_\delta,z_\delta)(\cdot,0)=U^0_\delta\text{ in }\Omega\end{cases}$$

Here Ω is a bounded simply connected open subset of $\mathbb{R}^3$ with a Lipschitz-continuous boundary Γ whose unit outer normal is denoted by n and which admits three partitions $(\Gamma_{\mathrm{MD}},\Gamma_{\mathrm{MN}})$, $(\Gamma_{\Theta\mathrm{D}},\Gamma_{\Theta\mathrm{N}})$ and $(\Gamma_{\mathrm{E}},\Gamma_{\mathrm{H}})$ such that the two-dimensional Hausdorff measures $\mathcal{H}_2(\Gamma_{\mathrm{MD}})$ and $\mathcal{H}_2(\Gamma_{\Theta\mathrm{D}})$ are positive. The symbols $\sigma_\delta,\Delta_\delta,v_\delta,u_\delta,e(u_\delta),\theta_\delta,z_\delta=(E_\delta,H_\delta)$ *represent* the stress tensor, the 'thermoelectromagnetic induction', the velocity field, the displacement field, the temperature field, and the electromagnetic field (the couple composed of the electrical field and the magnetic field), respectively; while $f,g_{\mathrm{M}},r,g_\Theta,J,j,K,k$ stand for densities of body and surface forces, heat supply, electric currents, and magnetic currents, respectively. Eventually upper dot $\dot{}$ denotes the derivative with respect to the time parameter t, U^0_δ is the given state of the body at $t=0$, $U_\delta(t):=(u_\delta,v_\delta,\theta_\delta,z_\delta)(t)$ is its state at t, the positive element (ρ,κ) of $L^\infty(\Omega)\times L^\infty(\Omega;\mathbb{S}^3)$, $\mathbb{S}^3$ being the space of symmetric 3×3 matrices, *represents* the density and the thermal conductivity tensor, $\mathcal{M}$ is the Maxwell operator $\mathcal{M}(E_\delta,H_\delta)=(\operatorname{curl}H_\delta,-\operatorname{curl}E_\delta)$, and M is an element of $L^\infty(\Omega;\mathrm{Lin}(\mathbb{S}^3\times\Pi))$, $\Pi:=\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3$, such that:

$$M=\begin{bmatrix}a & -b\\ b^{\mathrm{T}} & c\end{bmatrix},\quad c=\begin{bmatrix}\beta & \alpha\\ \alpha^{\mathrm{T}} & \hat{c}\end{bmatrix},\quad \hat{c}=\begin{bmatrix}\gamma & \nu\\ \nu^{\mathrm{T}} & \mu\end{bmatrix}\tag{1}$$

with a in $\mathrm{Lin}(\mathbb{S}^3)$, b in $\mathrm{Lin}(\Pi,\mathbb{S}^3)$, b^{T} the transpose of b, c in $\mathrm{Lin}(\Pi)$, β in $\mathbb{R}$, α in $\mathrm{Lin}(\mathbb{R}^3\times\mathbb{R}^3,\mathbb{R})$, γ, μ, ν in $\mathrm{Lin}(\mathbb{R}^3)$, which satisfies:

$$\exists c_M>0;\quad M(x)m\cdot m\geq c_M|m|^2\quad\forall m\in\mathbb{S}^3\times\Pi,\ \text{a.e. }x\in\Omega\tag{2}$$

where $\mathrm{Lin}(\mathcal{V}_1,\mathcal{V}_2)$ denotes the space of linear mappings between any finite dimensional spaces $\mathcal{V}_1$ and $\mathcal{V}_2$ whose canonical euclidean norm and inner product are systematically denoted by $|\ |$ and $\cdot$ (as for $\mathbb{R}^3$), $\mathrm{Lin}(\mathcal{V}_1,\mathcal{V}_1)$ is shorten in $\mathrm{Lin}(\mathcal{V}_1)$.

In what follows, any element h of Π may be written as $h=(h_\theta,h_z)$, h_θ in $\mathbb{R}$, $h_z=(h_E,h_H)$ in $\mathbb{R}^3\times\mathbb{R}^3$, while C will denote various constant *independent of* δ that may vary from line to line.

2. Existence and uniqueness result for dynamic evolution problem $(\mathcal{P}_\delta)$

Classically, we seek U_δ in the form:

$$U_\delta=U^{\mathrm{e}}_{\mathrm{dyn}}+U^{\mathrm{r}}_\delta\tag{3}$$

where $U^{\mathrm{e}}_{\mathrm{dyn}}$ is the solution to a steady-state problem taking into account part of the external loading, while U^{r}_δ is the solution to a linear evolution equation governed by an m-dissipative operator A_δ in a Hilbert space $\mathbb{H}_{\mathrm{dyn}}$ of possible states with finite energy.

In the sequel, $H^1_{\Gamma'}(\Omega)$ and $H^1_{\Gamma'}(\Omega;\mathbb{R}^3)$ will denote the subspaces of the Sobolev spaces $H^1(\Omega)$ and $H^1(\Omega;\mathbb{R}^3)$ made of the elements with vanishing traces on Γ' included in Γ. We introduce the following spaces:

$$\begin{cases}V:=H^1_{\Gamma_{\mathrm{MD}}}(\Omega;\mathbb{R}^3)\times H^1_{\Gamma_{\Theta\mathrm{D}}}(\Omega)\\ L^2(\Omega,\operatorname{curl})=\left\{\xi\in L^2(\Omega;\mathbb{R}^3);\operatorname{curl}\xi\in L^2(\Omega;\mathbb{R}^3)\right\}\\ L^2_{\Gamma_{\mathrm{E}}}(\Omega,\operatorname{curl})=\left\{\xi\in L^2(\Omega;\mathbb{R}^3);\displaystyle\int_\Omega\xi\cdot\operatorname{curl}\psi'-\operatorname{curl}\xi\cdot\psi'\,\mathrm{d}x=0\quad\forall\psi'\in H^1_{\Gamma_{\mathrm{H}}}(\Omega;\mathbb{R}^3)\right\}\\ L^2_{\Gamma_{\mathrm{H}}}(\Omega,\operatorname{curl})=\left\{\eta\in L^2(\Omega;\mathbb{R}^3);\displaystyle\int_\Omega\eta\cdot\operatorname{curl}\varphi'-\operatorname{curl}\eta\cdot\varphi'\,\mathrm{d}x=0\quad\forall\varphi'\in H^1_{\Gamma_{\mathrm{E}}}(\Omega;\mathbb{R}^3)\right\}\\ L^2(\Omega,\mathcal{M})=L^2(\Omega,\operatorname{curl})\times L^2(\Omega,\operatorname{curl})\\ Z:=L^2_{\Gamma_{\mathrm{E}}}(\Omega,\operatorname{curl})\times L^2_{\Gamma_{\mathrm{H}}}(\Omega,\operatorname{curl})\end{cases}\tag{4}$$

which are equipped with their usual norm.

For p in $\{0,1\}$, we introduce the assumption ($H_{\text{dyn, p}}$):

$$\begin{cases} \text{(i)} \quad (f,r,J,K)\in C^{2p,1}([0,T];L^2(\Omega;\mathbb{R}^3\times\Pi)) \\ \qquad (g_M,g_\Theta)\in C^{2p+1,1}([0,T];L^2(\Gamma_{MN};\mathbb{R}^3)\times L^2(\Gamma_{\Theta N})) \\ \text{(ii)} \quad (j,k) \text{ belongs to } L^2(\Gamma_H;\mathbb{R}^3)\times L^2(\Gamma_E;\mathbb{R}^3) \text{ with } \int_{\Gamma_H} j\cdot n\,\mathrm{d}\mathcal{H}_2=\int_{\Gamma_E} k\cdot n\,\mathrm{d}\mathcal{H}_2=0 \text{ and} \\ \qquad \exists z^{k,j}=(E^k,H^j)\in C^{2p+1,1}([0,T];L^2(\Omega,\mathcal{M})) \text{ s.t.} \\ \qquad\quad \int_\Omega E^k\cdot\operatorname{curl}\psi'-\operatorname{curl}E^k\cdot\psi'\,\mathrm{d}x=\int_{\Gamma_E} k\cdot\psi'\,\mathrm{d}\mathcal{H}_2 \quad \forall\psi'\in H^1_{\Gamma_H}(\Omega;\mathbb{R}^3) \\ \qquad\quad \int_\Omega H^j\cdot\operatorname{curl}\varphi'-\operatorname{curl}H^j\cdot\varphi'\,\mathrm{d}x=\int_{\Gamma_H} j\cdot\varphi'\,\mathrm{d}\mathcal{H}_2 \quad \forall\varphi'\in H^1_{\Gamma_E}(\Omega;\mathbb{R}^3) \end{cases} \tag{$H_{\text{dyn, p}}$}$$

and, taking $p=0$ in ($H_{\text{dyn, p}}$), we will define $U^{\text{e}}_{\text{dyn}}$ in (11) through the following result.

Proposition 2.1. *There exists a unique solution* $((u^{\text{e}}_\delta,\theta^{\text{e}}_\delta),z^{\text{e}}_\delta)$ *in* $C^{2,1}([0,T];V\times L^2(\Omega,\mathcal{M}))$ *to*

$$\begin{cases} ((u^{\text{e}}_\delta,\theta^{\text{e}}_\delta),z^{\text{e}}_\delta)\in V\times(z^{k,j}+Z) \\ \displaystyle\int_\Omega M(e(u^{\text{e}}_\delta),\theta^{\text{e}}_\delta,z^{\text{e}}_\delta)\cdot(e(u'),\theta',z')+(\kappa\nabla\theta^{\text{e}}_\delta,-\frac{1}{\delta}\mathcal{M}z^{\text{e}}_\delta)\cdot(\nabla\theta',z')\,\mathrm{d}x=\mathcal{L}(u',\theta') \quad \forall((u',\theta'),z')\in V\times Z \\ \displaystyle\mathcal{L}(u',\theta')=\int_{\Gamma_{MN}} g_M\cdot u'\,\mathrm{d}\mathcal{H}_2+\int_{\Gamma_{\Theta N}} g_\Theta\theta'\,\mathrm{d}\mathcal{H}_2 \end{cases} \tag{5}$$

Proof. Let $z^{e0}_\delta:=z^{\text{e}}_\delta-z^{k,j}$, as $(u^{\text{e}}_\delta,\theta^{\text{e}}_\delta,z^{e0}_\delta)$ has to satisfy

$$z^{e0}_\delta\in Z;\ \int_\Omega (c(\theta^{\text{e}}_\delta,z^{e0}_\delta+z^{k,j})+b^{\mathrm{T}}e(u^{\text{e}}_\delta))\cdot(0,z')-\frac{1}{\delta}\mathcal{M}(z^{e0}_\delta+z^{k,j})\cdot z'\,\mathrm{d}x=0 \quad \forall z'\in Z \tag{6}$$

we introduce the key lemma:

Lemma 2.1. *For all* $h=(h_1,h_2)$ *in* $L^2(\Omega;\mathbb{R}^3\times\mathbb{R}^3)$ *there exists a unique* $\zeta=(\xi,\eta):=\Xi(h)$ *in* Z *such that*

$$\hat{c}\zeta-\frac{1}{\delta}\mathcal{M}\zeta=h \tag{7}$$

with

- $$|\Xi(h)|_{L^2(\Omega;\mathbb{R}^3\times\mathbb{R}^3)}\le c_M^{-1}|h|_{L^2(\Omega;\mathbb{R}^3\times\mathbb{R}^3)} \tag{8a}$$
- $$\int_\Omega \begin{bmatrix} \tilde{\mu} & \frac{1}{\delta}\nu^{\mathrm{T}}\gamma^{-1} \\ -\frac{1}{\delta}\gamma^{-1}\nu & \frac{1}{\delta^2}\gamma^{-1}\end{bmatrix}(\eta,\operatorname{curl}\eta)\cdot(H',\operatorname{curl}H')\,\mathrm{d}x = \int_\Omega \begin{bmatrix} -\nu^{\mathrm{T}}\gamma^{-1} & I \\ -\frac{1}{\delta}\gamma^{-1} & 0\end{bmatrix}h\cdot(H',\operatorname{curl}H')\,\mathrm{d}x \quad \forall H'\in L^2_{\Gamma_H}(\Omega,\operatorname{curl}) \tag{8b}$$
- $$\xi=\gamma^{-1}(-\nu\eta+\frac{1}{\delta}\operatorname{curl}\eta+h_1) \tag{8c}$$
- $$\frac{1}{\delta}\operatorname{curl}\xi=-(\tilde{\mu}\eta+\nu^{\mathrm{T}}\gamma^{-1}\left(\frac{\operatorname{curl}\eta}{\delta}+h_1\right)-h_2) \tag{8d}$$
- $$\tilde{\mu}:=\mu-\nu^{\mathrm{T}}\gamma^{-1}\nu \tag{8e}$$

Proof. An obvious variational elimination of ξ in (7) yields that η has to solve variational equation (8b), which, by Lax Milgram lemma, has a unique solution. If ξ is defined by (8c), then (8b) implies (8d), so that ζ belongs to Z and solves (7). □

For all (w,τ) in $H^1_{\Gamma_{MD}}(\Omega;\mathbb{R}^3)\times L^2(\Omega)$ we denote $-\Xi(\alpha^{\mathrm{T}}\tau+(b^{\mathrm{T}}e(w))_z)$ by $\mathcal{S}_\delta(w,\tau)$ with of course:

$$|\mathcal{S}_\delta(w,\tau)|_{L^2(\Omega;\mathbb{R}^3\times\mathbb{R}^3)}\le C|(w,\tau)|_{H^1_{\Gamma_{MD}}(\Omega;\mathbb{R}^3)\times L^2(\Omega)} \tag{9}$$

Hence the element $(u_\delta^{\rm e}, \theta_\delta^{\rm e})$ of V has to satisfy the variational equation:

$$\int_\Omega M(e(u_\delta^{\rm e}), \theta_\delta^{\rm e}, \mathcal{S}_\delta(u_\delta^{\rm e}, \theta_\delta^{\rm e})) \cdot (e(u'), \theta', \mathcal{S}_\delta(u', \theta')) + (\kappa \nabla \theta_\delta^{\rm e}, -\frac{1}{\delta}\mathcal{M}\mathcal{S}_\delta(u_\delta^{\rm e}, \theta_\delta^{\rm e})) \cdot (\nabla \theta', \mathcal{S}_\delta(u', \theta'))\,{\rm d}x$$
$$= -\int_\Omega M(0, 0, z^{k,j} + \Xi(-\hat{c}(z^{k,j}) + \frac{1}{\delta}\mathcal{M}z^{k,j})) \cdot (e(u'), \theta', 0)\,{\rm d}x + \int_{\Gamma_{\rm MD}} g_{\rm M} \cdot u' \,{\rm d}\mathcal{H}_2 + \int_{\Gamma_{\Theta{\rm D}}} g_\Theta \cdot \theta' \,{\rm d}\mathcal{H}_2 \quad \forall (u', \theta') \in V$$

which, by Lax Milgram lemma, has a unique solution, so that if

$$z_\delta^{\rm e} := \mathcal{S}_\delta(u_\delta^{\rm e}, \theta_\delta^{\rm e}) + z^{k,j} + \Xi(-\hat{c}(z^{k,j}) + \frac{1}{\delta}\mathcal{M}z^{k,j}) \tag{10}$$

$(u_\delta^{\rm e}, \theta_\delta^{\rm e}, z_\delta^{\rm e})$ is solution to (5). □

Finally we define $U_{\rm dyn}^{\rm e}$ by

$$U_{\rm dyn}^{\rm e} = (u_\delta^{\rm e}, \dot{u}_\delta^{\rm e}, \theta_\delta^{\rm e}, z_\delta^{\rm e}) \tag{11}$$

Clearly $U_{\rm dyn}^{\rm e}$ belongs to $C^{2,1}([0,T]; H^1_{\Gamma_{\rm MD}}(\Omega;\mathbb{R}^3)) \times C^{1,1}([0,T]; H^1_{\Gamma_{\rm MD}}(\Omega;\mathbb{R}^3)) \times C^{2,1}([0,T]; H^1_{\Gamma_{\Theta{\rm D}}}(\Omega) \times L^2(\Omega;\mathbb{R}^3 \times \mathbb{R}^3))$.

Next, the Hilbert space $\mathbb{H}_{\rm dyn}$ is:

$$\mathbb{H}_{\rm dyn} := H^1_{\Gamma_{\rm MD}}(\Omega;\mathbb{R}^3) \times L^2(\Omega;\mathbb{R}^3) \times L^2(\Omega;\Pi) \tag{12}$$

and is equipped with the inner product and norm:

$$\begin{aligned} (U^1, U^2)_{\rm dyn} &:= \int_\Omega ae(u^1)\cdot e(u^2)\,{\rm d}x + \int_\Omega \rho v^1 \cdot v^2\,{\rm d}x + \int_\Omega c(\theta^1, z^1)\cdot(\theta^2, z^2)\,{\rm d}x \\ |U^i|^2_{\rm dyn} &= (U^i, U^i)_{\rm dyn}, \quad \forall U^i \in \mathbb{H}_{\rm dyn}, \forall i \in \{1, 2\} \end{aligned} \tag{13}$$

while operator A_δ is defined by:

$$\begin{cases} D(A_\delta) := \Big\{ U = (u, v, \theta, z) \in \mathbb{H}_{\rm dyn}; \\ \text{(i)}\ (v, \theta, z) \in V \times Z \\ \text{(ii)}\ \exists! (w, \tau, \zeta) \in L^2(\Omega; \mathbb{R}^3 \times \Pi) \text{ s.t.} \\ \quad \int_\Omega \rho w \cdot v'\,{\rm d}x + \int_\Omega (ae(u) - b(\theta, z)) \cdot e(v')\,{\rm d}x = 0 \quad \forall v' \in H^1_{\Gamma_{\rm MD}}(\Omega;\mathbb{R}^3) \\ \quad \int_\Omega (c(\tau, \zeta) + b^{\rm T} e(v)) \cdot (\theta', z') + \kappa \nabla\theta \cdot \nabla\theta' - \frac{1}{\delta}\mathcal{M}z \cdot z'\,{\rm d}x = 0 \quad \forall (\theta', z') \in H^1_{\Gamma_{\Theta{\rm D}}}(\Omega) \times Z \\ \quad A_\delta U = (v, w, \tau, \zeta) \Big\} \end{cases} \tag{14}$$

and satisfies:

Proposition 2.2. *Operator A_δ is m-dissipative.*

Proof. First the very definition of A_δ implies that for all U in $D(A_\delta)$ we have

$$\begin{aligned} (A_\delta U, U)_{\rm dyn} &= \int_\Omega ae(v)\cdot e(u)\,{\rm d}x - \int_\Omega (ae(u) - b(\theta, z))\cdot e(v)\,{\rm d}x + \int_\Omega -(b^{\rm T} e(v)\cdot(\theta, z) + \kappa\nabla\theta\cdot\nabla\theta - \frac{1}{\delta}\mathcal{M}z\cdot z)\,{\rm d}x \\ &= -\int_\Omega \kappa \nabla\theta\cdot\nabla\theta\,{\rm d}x \le 0 \end{aligned}$$

Second, as for all $\phi = (\phi_u, \phi_v, \phi_\theta, \phi_z)$ in $\mathbb{H}_{\rm dyn}$, the possible, but necessary unique, solution $\overline{U} = (\bar{u}, \bar{v}, \bar{\theta}, (\overline{E}, \overline{H}))$ to

$$\overline{U} - A_\delta \overline{U} = \phi \tag{15}$$

does satisfy:

$$\int_\Omega \rho\bar{v}\cdot v' + (ae(\bar{v}) - b(\bar{\theta},\bar{z}))\cdot e(v')\,\mathrm{d}x = \int_\Omega \rho\phi_v\cdot v' - ae(\phi_u)\cdot e(v')\,\mathrm{d}x \quad \forall v'\in H^1_{\Gamma_{\mathrm{MD}}}(\Omega;\mathbb{R}^3)$$
$$\int_\Omega (c(\bar{\theta},\bar{z}) + b^{\mathrm{T}}e(\bar{v}))\cdot(\theta',z') + \kappa\nabla\bar{\theta}\cdot\nabla\theta' - \frac{1}{\delta}\mathcal{M}\bar{z}\cdot z'\,\mathrm{d}x = \int_\Omega c(\phi_\theta,\phi_z)\cdot(\theta',z')\,\mathrm{d}x \quad \forall(\theta',z')\in H^1_{\Gamma_{\Theta\mathrm{D}}}(\Omega)\times Z \tag{16}$$

Lemma 2.1 implies that $(\bar{v},\bar{\theta})$ is determined as the solution to

$$\begin{cases} (\bar{v},\bar{\theta})\in V \\ \displaystyle\int_\Omega \rho\bar{v}\cdot v' + M(e(\bar{v}),\bar{\theta},\mathcal{S}_\delta(\bar{v},\bar{\theta}))\cdot(e(v'),\theta',\mathcal{S}_\delta(v',\theta')) + (\kappa\nabla\bar{\theta}, -\frac{1}{\delta}\mathcal{M}(\mathcal{S}_\delta(\bar{v},\bar{\theta}))\cdot(\nabla\theta',\mathcal{S}_\delta(v',\theta'))\,\mathrm{d}x \\ \qquad = \displaystyle\int_\Omega \rho\phi_v\cdot v' + (b(0,\Xi(c(\phi_\theta,\phi_z)_z)) - ae(\phi_u))\cdot e(v')\,\mathrm{d}x \quad \forall(v',\theta')\in V \end{cases} \tag{17}$$

which, by Lax Milgram lemma, exists and is unique. So $\overline{U} := (\bar{u}+\phi_u, \bar{v}, \bar{\theta}, \mathcal{S}_\delta(\bar{v},\bar{\theta}) + \Xi(c(\phi_\theta,\phi_z)_z))$ belongs to $D(A_\delta)$ and is solution to (15). □

Thus, as $(\mathcal{P}_\delta)$ is formally equivalent to

$$\begin{cases} \dfrac{\mathrm{d}U^{\mathrm{r}}_\delta}{\mathrm{d}t} = A_\delta U^{\mathrm{r}}_\delta + F_\delta \\ U^{\mathrm{r}}_\delta(0) = U^{\mathrm{r}0}_\delta := U^0_\delta - U^{\mathrm{e}}_{\mathrm{dyn}}(0) \\ F_\delta := (0, f/\rho - \ddot{u}^{\mathrm{e}}_\delta, c^{-1}((r,-J,-K) + (\operatorname{div}\kappa\nabla(\theta^{\mathrm{e}}_\delta, -\dot{\theta}^{\mathrm{e}}_\delta), \frac{1}{\delta}\mathcal{M}(z^{\mathrm{e}}_\delta - \dot{z}^{\mathrm{e}}_\delta)))) \in C^{0,1}([0,T];\mathbb{H}_{\mathrm{dyn}}) \end{cases} \tag{18}$$

one has Theorem 2.1.

Theorem 2.1. *Under the assumption* $(\mathrm{H}_{\mathrm{dyn},0})$*, and if* $U^{\mathrm{r}0}_\delta \in D(A_\delta)$*, then* $(\mathcal{P}_\delta)$ *has a unique solution in* $C^1([0,T];\mathbb{H}_{\mathrm{dyn}})$ *with* $(v_\delta,\theta_\delta,z_\delta)$ *in* $C^0([0,T]; V\times L^2(\Omega,\mathcal{M}))$.

3. The quasi-electromagnetostatic approximation

3.1. Existence and uniqueness result for quasi-electromagnetostatic evolution problem $(\mathcal{P})$

Computing a numerical approximation of the solution to $(\mathcal{P}_\delta)$ may be difficult because the speed of propagation of elastic waves is rather lower than the light celerity, the parameter δ being of order 2×10^{-5} for a $BaTiO_3$–$CoFe_2O_4$ composite with 0.6 volume fraction of barium titanate (see [1]). Thus in [1] is introduced the so-called quasi-static evolution problem which consists in assuming that there exists an electromagnetic potential (φ,ψ) in $H^1(\Omega)\times H^1(\Omega)$ such that the electromagnetic field $z=(E,H)$ reduces to $(E,H)=(\nabla\varphi,\nabla\psi)$. We also add an assumption on $z^{k,j}$, more precisely let us introduce $(\mathrm{H}_{\mathrm{qst}})$:

$$\begin{cases} \text{(i)} \quad (f,r)\in C^{0,1}([0,T];L^2(\Omega;\mathbb{R}^3\times\mathbb{R})), \\ \qquad ((J,K),g_{\mathrm{M}},g_\Theta)\in C^{1,1}([0,T];L^2(\Omega;\mathbb{R}^3\times\mathbb{R}^3)\times L^2(\Gamma_{\mathrm{MN}};\mathbb{R}^3)\times L^2(\Gamma_{\Theta\mathrm{N}})), \\ \text{(ii)} \quad \exists(\varphi^k,\psi^j)\in C^{1,1}([0,T],H^1_{\Gamma_E}(\Omega)\times H^1_{\Gamma_H}(\Omega)); z^{k,j} = (\nabla\varphi^k,\nabla\psi^j) \end{cases} \tag{$\mathrm{H}_{\mathrm{qst}}$}$$

Taking into account these assumptions in the equations associated with $(\mathcal{P}_\delta)$ *implies* that the thermoelectromagnetoelastic state $U := (u,v,\theta,(\nabla\varphi,\nabla\psi))$, with $(u,v,\theta,(\varphi,\psi))$ in $H^1_{\Gamma_{\mathrm{MD}}}(\Omega;\mathbb{R}^3)\times L^2(\Omega;\mathbb{R}^3)\times H^1_{\Gamma_{\Theta\mathrm{D}}}(\Omega)\times((\varphi^k,\psi^j)+W)$, $W := H^1_{\Gamma_{\mathrm{E}}}(\Omega)\times H^1_{\Gamma_{\mathrm{H}}}(\Omega)$, has to solve the following problem $(\mathcal{P})$:

$$(\mathcal{P})\begin{cases} (\sigma,\Delta) = M(e(u),\theta,(\nabla\varphi,\nabla\psi)) \\ \displaystyle\int_\Omega \Delta\cdot(0,(\nabla\varphi',\nabla\psi'))\,\mathrm{d}x = \int_\Omega (\Delta^0 - (0,(\mathbf{J},\mathbf{K})))\cdot(0,(\nabla\varphi',\nabla\psi'))\,\mathrm{d}x \quad \forall(\varphi',\psi')\in W \\ \displaystyle\int_\Omega \dot{\Delta}\cdot(\theta',0) + \kappa\nabla\theta\cdot\nabla\theta'\,\mathrm{d}x = \int_\Omega r\cdot\theta'\,\mathrm{d}x + \int_{\Gamma_{\Theta\mathrm{N}}} g_\Theta\cdot\theta'\,\mathrm{d}\mathcal{H}_2 \quad \forall\theta'\in H^1_{\Gamma_{\Theta\mathrm{D}}}(\Omega) \\ \displaystyle\int_\Omega \rho\dot{v}\cdot v' + \sigma\cdot e(v')\,\mathrm{d}x = \int_\Omega f\cdot v'\,\mathrm{d}x + \int_{\Gamma_{\mathrm{MN}}} g_{\mathrm{M}}\cdot v'\,\mathrm{d}\mathcal{H}_2 \quad \forall v'\in H^1_{\Gamma_{\mathrm{MD}}}(\Omega;\mathbb{R}^3) \\ U(0) = U^0 := (u^0,v^0,(\nabla\varphi^0,\nabla\psi^0)) \text{ with } (u^0,v^0,\theta^0,(\varphi^0,\psi^0)) \\ \quad \text{given in } H^1_{\Gamma_{\mathrm{MD}}}(\Omega;\mathbb{R}^3)\times L^2(\Omega;\mathbb{R}^3)\times L^2(\Omega)\times((\varphi^k,\psi^j)(0)+W) \end{cases}$$

where

$$\Delta^0 := b^{\mathrm{T}} e(u^0) + c(\theta^0, (\nabla\varphi^0, \nabla\psi^0)), \quad (\mathbf{J}, \mathbf{K})(t) := \int_0^t (J, K)(s)\, \mathrm{d}s \tag{19}$$

As in [3,4], we seek U in the form:

$$U = U^{\mathrm{e}}_{\mathrm{qst}} + U^{\mathrm{r}} \tag{20}$$

with $U^{\mathrm{e}}_{\mathrm{qst}}$ defined by:

$$U^{\mathrm{e}}_{\mathrm{qst}} = (u^{\mathrm{e}}, \dot{u}^{\mathrm{e}}, \theta^{\mathrm{e}}, (\nabla\varphi^{\mathrm{e}}, \nabla\psi^{\mathrm{e}})) \tag{21}$$

where $((u^{\mathrm{e}}, \theta^{\mathrm{e}}), (\varphi^{\mathrm{e}}, \psi^{\mathrm{e}}))$ in $V \times ((\varphi^k, \psi^j) + W)$ is uniquely determined by:

$$\begin{aligned}&\int_\Omega M(e(u^{\mathrm{e}}), \theta^{\mathrm{e}}, (\nabla\varphi^{\mathrm{e}}, \nabla\psi^{\mathrm{e}})) \cdot (e(u'), \theta', (\nabla\varphi', \nabla\psi')) + \kappa\nabla\theta^{\mathrm{e}} \cdot \nabla\theta' \,\mathrm{d}x = \\ &= \mathcal{L}(u', \theta') + \int_\Omega ((\Delta^0 - (0, (J, K))) \cdot (0, (\nabla\varphi', \nabla\psi')))\,\mathrm{d}x \quad \forall((u', \theta'), (\varphi', \psi')) \in V \times W\end{aligned} \tag{22}$$

Next we note that if $U^{\mathrm{r}} = (u^{\mathrm{r}}, v^{\mathrm{r}}, \theta^{\mathrm{r}}, (\nabla\varphi^{\mathrm{r}}, \nabla\psi^{\mathrm{r}}))$ then $(\varphi^{\mathrm{r}}, \psi^{\mathrm{r}})$ satisfies:

$$\begin{cases} (\varphi^{\mathrm{r}}, \psi^{\mathrm{r}}) \in W \text{ s.t.} \\ \displaystyle\int_\Omega M(e(u^{\mathrm{r}}), \theta^{\mathrm{r}}, (\nabla\varphi^{\mathrm{r}}, \nabla\psi^{\mathrm{r}})) \cdot (0, 0, (\nabla\varphi', \nabla\psi'))\,\mathrm{d}x = 0 \quad \forall(\varphi', \psi') \in W \end{cases} \tag{23}$$

so that there exists a linear continuous mapping $\mathcal{S}$ from $H^1_{\Gamma_{\mathrm{MD}}}(\Omega; \mathbb{R}^3) \times L^2(\Omega)$ into G,

$$G := \left\{ (\xi, \eta) \in L^2(\Omega; \mathbb{R}^3 \times \mathbb{R}^3); \exists(\varphi', \psi') \in W \text{ s.t. } (\xi, \eta) = (\nabla\varphi', \nabla\psi') \right\} \tag{24}$$

verifying $(\nabla\varphi^{\mathrm{r}}, \nabla\psi^{\mathrm{r}}) = \mathcal{S}(u^{\mathrm{r}}, \theta^{\mathrm{r}})$. Hence U^{r} reduces to $\mathcal{U}^{\mathrm{r}} = (u^{\mathrm{r}}, v^{\mathrm{r}}, \theta^{\mathrm{r}})$, and this multi-physical constraint therefore suggests the introduction of the following Hilbert space $\mathbb{H}_{\mathrm{qst}}$ of possible states with finite energy, isomorph to a closed subspace of $\mathbb{H}_{\mathrm{dyn}}$:

$$\mathbb{H}_{\mathrm{qst}} := H^1_{\Gamma_{\mathrm{MD}}}(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^3) \times L^2(\Omega) \tag{25}$$

equipped with the inner product and norm:

$$\begin{aligned}&(\mathcal{U}^1, \mathcal{U}^2)_{\mathrm{qst}} := \int_\Omega a e(u^1) \cdot e(u^2)\,\mathrm{d}x + \int_\Omega \rho v^1 \cdot v^2\,\mathrm{d}x + \int_\Omega c(\theta^1, \mathcal{S}(u^1, \theta^1)) \cdot (\theta^2, \mathcal{S}(u^2, \theta^2))\,\mathrm{d}x \\ &|\mathcal{U}^i|^2_{\mathrm{qst}} = (\mathcal{U}^i, \mathcal{U}^i)_{\mathrm{qst}} \quad \forall\mathcal{U}^i \in \mathbb{H}_{\mathrm{qst}}\ i = 1, 2\end{aligned} \tag{26}$$

The unbounded operator A which will govern the evolution of $\mathcal{U}^{\mathrm{r}}$ is then defined by:

$$\begin{cases} D(A) = \Big\{\mathcal{U} = (u, v, \theta) \in \mathbb{H}_{\mathrm{qst}}; \\ \quad \text{(i) } (v, \theta) \in V \\ \quad \text{(ii) } \exists!(w, \tau) \in L^2(\Omega; \mathbb{R}^3 \times \mathbb{R}^3) \text{ s.t.} \\ \quad \displaystyle\int_\Omega \rho w \cdot v' + (a e(u) - b(\theta, \mathcal{S}(u, \theta))) \cdot e(v')\,\mathrm{d}x = 0 \qquad \forall v' \in H^1_{\Gamma_{\mathrm{MD}}}(\Omega; \mathbb{R}^3) \\ \quad \displaystyle\int_\Omega (c(\tau, \mathcal{S}(v, \tau)) + b^{\mathrm{T}} e(v)) \cdot (\theta', 0) + \kappa\nabla\theta \cdot \nabla\theta'\,\mathrm{d}x = 0 \quad \forall\theta' \in H^1_{\Gamma_{\Theta\mathrm{D}}}(\Omega)\Big\} \\ A\mathcal{U} = (v, w, \tau) \end{cases} \tag{27}$$

Of course one has the following.

Proposition 3.1. *Operator A is m-dissipative.*

Proof. First, the very definitions of A *and* $\mathcal{S}$ imply that for all $\mathcal{U}$ in $D(A)$:

$$(A\mathcal{U},\mathcal{U})_{\text{qst}} = \int_\Omega ae(v)\cdot e(u)\,\mathrm{d}x - \int_\Omega (ae(u) - b(\theta,\mathcal{S}(u,\theta)))\cdot e(v)\,\mathrm{d}x + \int_\Omega c(\tau,\mathcal{S}(v,\tau))\cdot(\theta,\mathcal{S}(u,\theta))\,\mathrm{d}x$$
$$= \int_\Omega (\theta,\mathcal{S}(u,\theta))\cdot b^{\mathrm{T}}e(v) - b^{\mathrm{T}}e(v)\cdot(\theta,\mathcal{S}(u,\theta)) - \kappa\nabla\theta\cdot\nabla\theta\,\mathrm{d}x \le 0$$

Second, for all $\Psi = (\Psi_u, \Psi_v, \Psi_\theta)$ in $\mathbb{H}_{\text{qst}}$, the possible and unique $\overline{\mathcal{U}} = (\bar{u}, \bar{v}, \bar{\theta})$ such that $\overline{\mathcal{U}} - A\overline{\mathcal{U}} = \Psi$ has to satisfy:

$$\begin{cases} (\bar{v},\bar{\theta}) \in V \\ \displaystyle\int_\Omega \rho\bar{v}\cdot v' + M(e(\bar{v}),\bar{\theta},\mathcal{S}(\bar{v},\bar{\theta}))\cdot(e(v'),\theta',\mathcal{S}(v',\theta')) + \kappa\nabla\bar{\theta}\cdot\nabla\theta'\,\mathrm{d}x \\ \displaystyle= \int_\Omega \rho\Psi_v\cdot v' - ae(\Psi_u)\cdot e(v') + c(\Psi_\theta,\mathcal{S}(0,\Psi_\theta))\cdot\theta' + b(0,\mathcal{S}(\Psi_u,0))\cdot e(v')\,\mathrm{d}x \quad \forall (v',\theta')\in V \end{cases} \tag{28}$$

Hence $(\bar{v},\bar{\theta})$ is determined in a unique way and $\overline{\mathcal{U}} := (\bar{v}+\Psi_u, \bar{v}, \bar{\theta})$ belongs to $D(A)$ and $\overline{\mathcal{U}} - A\overline{\mathcal{U}} = \Psi$! □

Eventually, as $(\mathcal{P})$ is formally equivalent to

$$\begin{cases} \dfrac{\mathrm{d}\mathcal{U}^{\mathrm{r}}}{\mathrm{d}t} = A\mathcal{U}^{\mathrm{r}} + \mathcal{F}, \quad \mathcal{F} = (0, f/\rho - \ddot{u}^{\mathrm{e}}, \beta^{-1}(\operatorname{div}\kappa\nabla(\theta^{\mathrm{e}} - \dot{\theta}^{\mathrm{e}}) + r)) \in C^{0,1}([0,T];\mathbb{H}_{\text{qst}}) \\ \mathcal{U}^{\mathrm{r}}(0) = \mathcal{U}^{\mathrm{r}0} := (u^0 - u^{\mathrm{e}}(0), v^0 - v^{\mathrm{e}}(0), \theta^0 - \theta^{\mathrm{e}}(0)) \end{cases} \tag{29}$$

one has:

Theorem 3.1. *Under assumptions* $(\mathrm{H}_{\text{dyn},0})$ *and* $(\mathrm{H}_{\text{qst}})$*, and if* $\mathcal{U}^{\mathrm{r}0}$ *belongs to* $D(A)$ *then* $(\mathcal{P})$ *has a unique solution* $C^1([0,T]; H^1_{\Gamma_{\mathrm{MD}}} \times L^2(\Omega;\mathbb{R}^3)\times L^2(\Omega)\times((\nabla\varphi^k,\nabla\psi^j)+G))$ *with* $(v,\theta)\in C^0([0,T];V)$.

Remark 3.1. Under the assumption

$$\begin{cases} \exists (Q_{\mathrm{E}}, Q_{\mathrm{H}}) \text{ in } C^1([0,T]; L^2(\Omega)\times L^2(\Omega)) \text{ s.t.} \\ (\dot{Q}_{\mathrm{E}}, \dot{Q}_{\mathrm{H}}) + (\operatorname{div} J, \operatorname{div} K) = 0 \end{cases} \tag{H1}$$

electric induction Δ_{E} and magnetic induction Δ_{H} satisfy

$$\begin{cases} \operatorname{div}\dot{\Delta}_{\mathrm{E}} + \operatorname{div} J = 0 \quad \operatorname{div}\dot{\Delta}_{\mathrm{H}} + \operatorname{div} K = 0 \\ \dot{\Delta}_{\mathrm{E}}\cdot n = -J\cdot n \quad \dot{\Delta}_{\mathrm{H}}\cdot n = -K\cdot n \end{cases} \tag{30}$$

thus, if the data of the problem is $(Q_{\mathrm{E}}, Q_{\mathrm{M}})$, the density of electric and magnetic body charges, it is not necessary to introduce initial conditions (φ^0, ψ^0), which then satisfy

$$\begin{cases} (\varphi^0,\psi^0) \in (\varphi^k,\psi^j)(0) + W \\ \displaystyle (c(\theta^0,\nabla\varphi^0,\nabla\psi^0) + b^{\mathrm{T}}e(v^0))\cdot(0,\nabla\varphi',\nabla\psi') = \int_\Omega (Q_{\mathrm{E}}(0), Q_{\mathrm{H}}(0))\cdot(\varphi',\psi)\,\mathrm{d}x \quad \forall(\varphi',\psi')\in W \end{cases} \tag{31}$$

so that

$$(\operatorname{div}\Delta_{\mathrm{E}}(t), \operatorname{div}\Delta_{\mathrm{H}}(t)) = (Q_{\mathrm{E}}(t), Q_{\mathrm{H}}(t)) \quad \forall t\in[0,T] \tag{32}$$

Moreover, classically, (j,k), (J,K) and $(q_{\mathrm{E}}, q_{\mathrm{H}})$, the so-called surface electric and magnetic charges assumed to belong to $C^1([0,T]; L^2(\Gamma_{\mathrm{E}})\times L^2(\Gamma_{\mathrm{H}}))$ are linked by:

$$\dot{q}_{\mathrm{H}} + \operatorname{div}_\Gamma k - K\cdot n = 0 \quad \text{on } \Gamma_{\mathrm{E}}, \quad \dot{q}_{\mathrm{E}} + \operatorname{div}_\Gamma j - J\cdot n = 0 \text{ on } \Gamma_{\mathrm{H}} \tag{33}$$

where $\operatorname{div}_\Gamma$ is the surface divergence operator. As $(\mathrm{H}_{\text{qst}})$(ii) implies $\operatorname{div}_\Gamma j = 0$ on Γ_{H}, $\operatorname{div}_\Gamma k = 0$ on Γ_{E}, one has:

$$-\Delta_{\mathrm{E}}\cdot n = q_{\mathrm{E}} \quad \text{on } \Gamma_{\mathrm{H}}, \quad -\Delta_{\mathrm{H}}\cdot n = q_{\mathrm{H}} \quad \text{on } \Gamma_{\mathrm{E}} \tag{34}$$

3.2. $(\mathcal{P})$ as the limit of $(\mathcal{P}_\delta)$ when δ goes to zero

Assumption (H_{qst}) allows us to determine a new U^{e}_{dyn} denoted by $\hat{U}^{e}_{dyn}$ and a new F_δ denoted by F, *which are independent of* δ by:

$$\begin{aligned}
&\hat{U}^{e}_{dyn} := (\hat{u}^{e}, \dot{\hat{u}}^{e}, \hat{\theta}^{e}, (\nabla\hat{\varphi}^{e}, \nabla\hat{\psi}^{e}))\\
&F := (0, f/\rho - \ddot{u}^{e}, c^{-1}((r, -J, -K) + (\operatorname{div}\kappa\nabla(\theta^{e}_{\delta}, \dot{\theta}^{e}_{\delta}), 0, 0)))\\
&((\hat{u}^{e}, \hat{\theta}^{e}), (\hat{\varphi}^{e}, \hat{\psi}^{e})) \in V \times ((\varphi^{j}, \psi^{k}) + W) \text{ s.t.}\\
&\int_\Omega M(e(\hat{u}^{e}), \hat{\theta}^{e}, (\nabla\hat{\varphi}^{e}, \nabla\hat{\psi}^{e})) \cdot (e(u'), \theta', (\nabla\varphi', \nabla\psi')) + \kappa\nabla\hat{\theta}^{e}\cdot\nabla\theta' \,dx = \mathcal{L}(u', \theta') \quad \forall((u', \theta'), (\varphi', \psi')) \in V \times W
\end{aligned} \tag{35}$$

We introduce the additional assumption:

$$\begin{cases}
\exists U^{rq} := (u^{rq}, v^{rq}, \theta^{rq}, z^{rq}) \quad 0 \le q \le 2 \text{ s.t.}\\
\text{(i)}\quad U^{r0} := U^{0} - \hat{U}^{e}_{dyn}(0) \in D(A_\delta)\\
\qquad U^{r1} := A_\delta U^{r0} + F(0) \in D(A_\delta)\\
\qquad U^{r2} := A_\delta U^{r1} + \dot{F}(0) \in D(A_\delta)\\
\text{(ii)}\quad \mathcal{M}(\Xi(\alpha\theta^{r1} + (b^{T}e(v^{r0}))_z) + (J(0), K(0))) = 0
\end{cases} \tag{H_{conv}}$$

and adapt the strategy of [2] to show that the solution to $(\mathcal{P}_\delta)$ converges toward the one to $(\mathcal{P})$ in the following sense.

Theorem 3.2. *Under assumptions* $(H_{dyn,\,1})$, (H_{qst}), (H_{conv}) *and if* $\left(U^{r0}_{\delta}, U^{r0} := U^{0} - \hat{U}_{dyn}(0), \mathcal{U}^{r0}\right)$ *belongs to* $D(A_\delta) \times D(A_\delta) \times D(A)$*, then*

$$\begin{aligned}
\sup_{t\in[0,T]} |U_\delta(t) - U(t)|_{dyn} \le |U^{0}_{\delta} - U^{0}|_{dyn} + C\delta \Bigg(&|U^{r1}|_{dyn} + |(J, K)(t)|_{L^2(\Omega;\mathbb{R}^3\times\mathbb{R}^3)} + \int_0^t (|\dot{F}(s)|_{dyn}\,ds\\
&+ \int_0^t \left[|U^{r2}|_{dyn} + |(\dot{J}, \dot{K})(s)|_{L^2(\Omega;\mathbb{R}^3\times\mathbb{R}^3)} + \int_0^s |\ddot{F}(y)|_{dyn}\,dy \right] ds \Bigg)
\end{aligned} \tag{36}$$

Proof. First we choose $U^{0}_{\delta} = U^{0}$ and by using the unique decomposition:

$$\begin{cases}
z_\delta = (\nabla\varphi_\delta, \nabla\psi_\delta) + \tilde{z}_\delta, \quad (\varphi_\delta, \psi_\delta) \in (\varphi^{e}, \psi^{e}) + W\\
\tilde{z}_\delta \in Z_{div,flux,0} = \left\{ (\xi, \eta) \in Z; \int_\Omega (\gamma\xi, \mu\eta) \cdot (\nabla\varphi', \nabla\psi')\,dx = 0 \quad \forall(\varphi', \psi') \in W \right\}
\end{cases} \tag{37}$$

we observe that $U^{*}_{\delta} := (u_\delta, v_\delta, \theta_\delta, (\nabla\varphi_\delta, \nabla\psi_\delta)) - U$ is solution to a problem similar to $(\mathcal{P})$, but with vanishing initial data, 0 in place of (φ^{k}, ψ^{j}), and a loading reduced to

$$-\int_\Omega b(0, \tilde{z}_\delta) \cdot e(v') - \begin{bmatrix} 0 & \nu \\ \nu^{T} & 0 \end{bmatrix} \tilde{z}_\delta \cdot (\nabla\varphi', \nabla\psi')\,dx - \int_\Omega c(0, \dot{\tilde{z}}_\delta) \cdot (\theta', 0)\,dx$$

so that

$$|U^{*}_{\delta}(t)|_{dyn} \le c\left(|\tilde{z}_\delta(t)|_{L^2(\Omega;\mathbb{R}^3\times\mathbb{R}^3)} + \int_0^t |\dot{\tilde{z}}_\delta(s)|_{L^2(\Omega;\mathbb{R}^3\times\mathbb{R}^3)}\,ds \right) \quad \forall t \in [0, T] \tag{38}$$

Next, as (H_{conv})(ii) implies that U^{rq} does not depend on δ, $(H_{dyn,\,1})$ and (H_{conv})(i) yield the uniform bound:

$$\left| \frac{d^q}{dt^q} U^{r}_{\delta}(t) \right|_{dyn} \le |U^{rq}|_{dyn} + \int_0^t \left| \frac{d^q}{dt^q} F(s) \right|_{dyn} ds \quad \forall(q, t) \in \{1, 2\} \times [0, T] \tag{39}$$

Finally, as

$$\left|\mathcal{M}\frac{\mathrm{d}^{q-1}}{\mathrm{d}t^{q-1}}\tilde{z}_\delta(t)\right|_{L^2(\Omega;\mathbb{R}^3\times\mathbb{R}^3)} \le C\delta\left(\left|\frac{\mathrm{d}^q}{\mathrm{d}t^q}U^{\mathrm{r}}_\delta(t)\right|_{\mathrm{dyn}} + \left|\frac{\mathrm{d}^{q-1}}{\mathrm{d}t^{q-1}}(J,K)(t)\right|_{L^2(\Omega;\mathbb{R}^3\times\mathbb{R}^3)}\right) \quad \forall (q,t)\in\{1,2\}\times[0,T] \tag{40}$$

the proof is achieved by using the crucial inequality in the mathematical analysis of electromagnetism (see [5]):

$$\exists C>0;\ |\xi|_{L^2(\Omega;\mathbb{R}^3)} \le C|\operatorname{curl}\xi|_{L^2(\Omega;\mathbb{R}^3)},\quad |\eta|_{L^2(\Omega;\mathbb{R}^3)} \le C|\operatorname{curl}\eta|_{L^2(\Omega;\mathbb{R}^3)} \quad \forall(\xi,\eta)\in Z_{\mathrm{div,flux},0} \tag{41}$$

as, of course, if the initial data of $(\mathcal{P}_\delta)$ differs from U^0, the additional term is bounded by $|U^0_\delta - U^0|_{\mathrm{dyn}}$, the semi-group generated by A_δ being of contraction. □

4. Concluding remarks

This mere exercise on the use of the theory of semi-groups of linear operators in Hilbert spaces shows that the quasi-electromagnetostatic evolution is a rather good approximation of the dynamic evolution. When (J,j), (K,k) are smooth enough (see Remark 3.1), problem $(\mathcal{P})$ involves the electromagnetic boundary conditions:

$$\begin{cases}\nabla\varphi\wedge n = k \quad \text{on } \Gamma_{\mathrm{E}}, & -\Delta_{\mathrm{E}}\cdot n = q_{\mathrm{E}} \quad \text{on } \Gamma_{\mathrm{H}}\\ \nabla\psi\wedge n = j \quad \text{on } \Gamma_{\mathrm{H}}, & -\Delta_{\mathrm{H}}\cdot n = q_{\mathrm{H}} \quad \text{on } \Gamma_{\mathrm{E}}\end{cases} \tag{42}$$

In term of smart devices, an electric actuator condition *and* a magnetic sensor condition are involved together on Γ_{E}, whereas a magnetic actuator condition *and* an electric sensor condition are involved in Γ_{H}. Hence, on each part Γ_{E} or Γ_{H}, we have mixed conditions. By arguing as in Section 3.1, it is easy to show the well-posedness of a quasi-electromagnetostatic evolution problem, with two *different* partitions $(\Gamma^{\mathrm{el}}_a, \Gamma^{\mathrm{el}}_s)$, $(\Gamma^{\mathrm{ma}}_a, \Gamma^{\mathrm{ma}}_s)$ where actuator/sensor electric and actuator/sensor magnetic conditions are imposed, respectively. Such a problem should be the approximation of a dynamic evolution problem with boundary electromagnetic data on the sole physically realistic partition $(\Gamma_{\mathrm{E}}, \Gamma_{\mathrm{H}})$ satisfying rather complex compatibility conditions, which makes the practical character of such a situation rather questionable!

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References

[1] F. Bonaldi, G. Geymonat, F. Krasucki, Modeling of smart materials with thermal effects: dynamic and quasi-static evolution, Math. Models Methods Appl. Sci. 25 (14) (2015) 2633–2667.
[2] S. Imperiale, P. Joly, Mathematical and numerical modeling of piezoelectric sensors, ESAIM: Mathematical Modelling and Numerical Analysis 46 (4) (2012) 875–909.
[3] T. Weller, C. Licht, Asymptotic modeling of thin piezoelectric plates, Ann. Solid Struct. Mech. 1 (3–4) (2010) 173–183.
[4] C. Licht, A. Ould Khaoua, T. Weller, Transient response of thermoelastic bodies linked by a thin layer of low stiffness and high thermal resistivity, C. R. Mecanique 343 (1) (2015) 18–26.
[5] P. Fernandes, G. Gilardi, Magnetostatic and electrostatic problems in inhomogeneous anisotropic media with irregular boundary and mixed boundary conditions, Math. Models Methods Appl. Sci. 7 (7) (1997) 957–991.