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An inverse Hölder inequality and its application in lower bound estimates for blow-up time

Bin Guo

School of Mathematics, Jilin University, Changchun 130012, PR China

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ABSTRACT

This paper deals with the lower bound for blow-up solutions to a nonlinear viscoelastic hyperbolic equation. An inverse Hölder inequality with the correction constant is employed to overcome the difficulty caused by the failure of the embedding inequality $W_0^{1,r}(\Omega) \hookrightarrow L^{2\alpha-2}(\Omega)(\frac{r^*+2}{2} < \alpha < r^* = \frac{Nr}{N-r})$ and the lack of a version of the Gagliardo–Nirenberg inequality. Moreover, a lower bound for blow-up time is obtained by establishing a first-order differential inequality. This result is a continuation of an earlier work [1].

1. Introduction

In this paper, we study the following quasilinear hyperbolic equation with strong damping

$$\begin{cases} u_{tt} - \operatorname{div}(|\nabla u|^{p(x,t)-2}\nabla u) - \Delta u_t = |u|^{q(x,t)-2}u, & (x,t) \in \Omega \times (0,T) := Q_T \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T) := \Gamma_T \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega \end{cases} \tag{1}$$

where $\Omega \subset R^N$ $(N \geqslant 1)$ is a bounded domain, $\partial\Omega$ is Lipschitz continuous. It will be assumed throughout the paper that the exponents $p(x,t)$, $q(x,t)$ satisfy the following conditions:

$$1 < p^- \leqslant p(x,t) \leqslant p^+ < \infty, \; 1 < q^- \leqslant q(x,t) \leqslant q^+ < \infty \tag{2}$$

The problem with variable exponent occurs in many mathematical models of applied science, for example, viscoelastic fluids, electro-rheological fluids, processes of filtration through a porous media, fluids with temperature-dependent viscosity, etc. The interested readers may refer to [2–4] and the references therein. To the best of our knowledge, when p varies in space and in time, S.N. Antontsev [5] and B. Guo [1], etc., discussed the blowing-up properties of solutions to the initial and homogeneous boundary value problem of quasilinear wave equations involving a $p(x,t)$-Laplacian operator. However, it is natural to ask whether, if the solution blows up in finite time, we can give an estimate of a lower bound to the blow-up time? In fact, we all know that the upper bound ensures blowing-up of the solution, and the importance of the lower bound is that it may provide us with a safe time interval for operation if we use Problem (1) to model a physical process. Only in [6–9], the authors obtained a lower bound estimate about the blow-up time to a damped semilinear wave equation.

E-mail addresses: bguo@jlu.edu.cn, bguo1029@gmail.com.

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Such results are seldom seen for the problem with variable exponents. Indeed, in dealing with our problem, we must face some difficulties. The first difficulty is that the method to estimate the derivative of the control functional in parabolic cases [10,11] is no longer effective. The second one is that the embedding relationship $W_0^{1,r}(\Omega) \hookrightarrow L^{2\alpha-2}(\Omega)(\frac{r^*+2}{2} < \alpha < r^* = \frac{Nr}{N-r})$ does not hold. The third difficulty is that the monotonicity of the energy functional fails (Lemma 1.2). The last one is that we could not find a version of the Gagliardo–Nirenberg inequality for non-constant cases (Inequality (11)), which is the reason why we needed to establish an inverse Hölder inequality. Especially, we find that the latter three difficulties lead to the fact that our methods used in [6–9] are no longer applicable. In order to bypass these difficulties, we have to develop some new ideas or techniques. In this paper, we apply the modified interpolation inequality and a new energy estimate method to prove the inverse Hölder inequality with correction constant (Lemma 1.5) and then construct the suitable control functional to establish a differential inequality.

Define the energy functional as

$$E(t) = \frac{1}{2}\int_\Omega |u_t|^2 \mathrm{d}x + \int_\Omega \frac{1}{p(x,t)}|\nabla u|^{p(x,t)}\mathrm{d}x - \int_\Omega \frac{1}{q(x,t)}|u|^{q(x,t)}\mathrm{d}x$$

Set

$$L(t) = \frac{1}{2}\int_\Omega |u|^2 \mathrm{d}x + \frac{1}{2}\int_0^t\int_\Omega |\nabla u|^2 \mathrm{d}x\mathrm{d}\tau - \frac{t}{2}\|\nabla u_0\|_2^2 + \beta(t+t_0)^2, \beta = \frac{\alpha_1}{2(q^-+1)p^-}$$

$$t_0 = 2\max\left\{\frac{\|\nabla u_0\|_2^2}{\beta}, \frac{\|u_1 u_0\|_1}{2\beta}\right\} > 0, E_1 = \frac{(q^+-p^+)}{q^+p^+}\alpha_1, \alpha_1 = \left(\frac{1}{B_1^{q^+}}\right)^{\frac{p^+}{q^+-p^+}}, B_1 = \max\{1, B\}$$

$$B = (1+|\Omega|)|\Omega|^{1-\frac{(N-p^-)r}{Np^-}}\pi^{-\frac{1}{2}}N^{-\frac{1}{p^-}}\left[\frac{p^--1}{N-p^-}\right]^{1-\frac{1}{p^-}}\left\{\frac{\Gamma(1+\frac{N}{2})\Gamma(N)}{\Gamma(\frac{N}{p^-})\Gamma(1+N-\frac{N}{p^-})}\right\}^{\frac{1}{N}}$$

The blowing-up property of the solution and energy estimates are given in the following theorem and lemma.

Theorem 1.1 *([1]). Assume that the exponents $p(x,t)$, $q(x,t)$ satisfy (2) and the following conditions hold*

$$(H_1)\ u_0 \in W_0^{1,p(x)}(\Omega),\ u_1 \in L^2(\Omega),\ E(0) + \frac{|\Omega|}{p^-} + \frac{|\Omega|}{q^-} < E_1,\ \min\{\|\nabla u_0\|_{p(x)}^{p^-}, \|\nabla u_0\|_{p(x)}^{p^+}\} > \alpha_1$$

$$(H_2)\ \max\{2, p^+\} < q^- \leqslant q(x,t) \leqslant q^+ < \frac{Np^-}{N-p^-},\ \forall x \in \Omega,\ t \geqslant 0$$

$$(H_3)\ p_t \leqslant 0,\ q_t \geqslant 0,\ \left|\frac{p_t}{p^2}\right| + \left|\frac{q_t}{q^2}\right| \in L^1_{loc}((0,\infty); L^1(\Omega))$$

then there exists a $T^* \leqslant \frac{-L^{1-\frac{q^++2}{4}}(0)}{(L^{1-\frac{q^++2}{4}})'(0)}$ *such that* $\lim_{t\to T^*} L(t) = +\infty$.

Lemma 1.2 *([1]). Suppose that $u \in L^{q(x,t)}(Q_T) \cap L^\infty(0,T; W_0^{1,p(x,t)}(\Omega))$, $u_t \in L^2(0,T; H^1(\Omega))$ is a solution to Problem (1), then $E(t)$ satisfies the following identity*

$$E(t) + \int_0^t\int_\Omega |\nabla u_s|^2 \mathrm{d}x\,\mathrm{d}s = E(0) + \int_0^t\int_\Omega \frac{p_s(.)}{p^2(.)}|\nabla u|^{p(.)}\Big(\ln|\nabla u|^{p(.)} - 1\Big)\mathrm{d}x\,\mathrm{d}s$$
$$- \int_0^t\int_\Omega \frac{q_s(.)}{q^2(.)}|u|^q\Big(\ln|u|^q - 1\Big)\mathrm{d}x\,\mathrm{d}s$$

Lemma 1.3 *([1]). If u is the solution to Problem (1) and (H_3) is satisfied, then the energy functional $E(t)$ satisfies*

$$E(t) + \int_0^t\int_\Omega |\nabla u_s|^2 \mathrm{d}x\,\mathrm{d}s \leqslant E(0) + \Big(\frac{1}{p^-} + \frac{1}{q^-}\Big)|\Omega| := E_2,\ \ t \geqslant 0$$

The following corollary may follow from Theorem 1.1.

Corollary 1.4. *Suppose that all the conditions of Theorem 1.1 hold and $p^- \geqslant 2$, then*

$$\lim_{t\to T^*} \int_\Omega |u|^{q(x,t)} \mathrm{d}x = +\infty \tag{3}$$

Proof. Step 1. We follow the lines of the proof of Theorem 1.1 of [1] to obtain

$$L(t)L''(t) - \frac{q^+ + 2}{4}(L'(t))^2 \geqslant 0, t > 0 \tag{4}$$

On the one hand, by (4) and the positivity of the function $L(t)$, we get $L''(t) \geqslant 0$. On the other hand, using the condition $L'(0) > 0$, we have $L'(t)$ is nonnegative and nondecreasing with respect to time variable t. Then we may claim that

$$\lim_{t\to T^*} L'(t) = +\infty \tag{5}$$

As a matter of fact, the above conclusion follows from the following inequality.

$$L(t) - L(0) = \int_0^t L'(s)\mathrm{d}s \leqslant tL'(t)$$

Step 2. We utilize Hölder inequality and Lemmas 2.1–2.2 of [12] or Lemmas 3.2.4 and 3.2.20 of [13]

$$\begin{cases} \left|\int_\Omega u(.,t)u_t(.,t)\mathrm{d}x\right| \leqslant \|u\|_2\|u_t\|_2, \quad \int_\Omega |\nabla u|^2\mathrm{d}x \leqslant |\Omega|^{1-\frac{2}{p^-}}\left(\int_\Omega |\nabla u|^{p^-}\mathrm{d}x\right)^{\frac{2}{p^-}} \\ \|u\|_2 \leqslant (1+|\Omega|)\|u\|_{q(.)} \leqslant (1+|\Omega|)\max\left\{\left(\int_\Omega |u|^{q(.)}\mathrm{d}x\right)^{\frac{1}{q^+}}, \left(\int_\Omega |u|^{q(.)}\mathrm{d}x\right)^{\frac{1}{q^-}}\right\} \\ \|\nabla u\|_{p^-} \leqslant (1+|\Omega|)\|\nabla u\|_{p(.)} \leqslant (1+|\Omega|)\max\left\{\left(\int_\Omega |\nabla u|^{p(.)}\mathrm{d}x\right)^{\frac{1}{p^+}}, \left(\int_\Omega |\nabla u|^{p(.)}\mathrm{d}x\right)^{\frac{1}{p^-}}\right\} \end{cases} \tag{6}$$

By Lemma 1.3 and the definition of $E(t)$, we have

$$\int_\Omega \frac{1}{p(.)}|\nabla u|^{p(.)}\mathrm{d}x + \frac{1}{2}\int_\Omega |u_t|^2\mathrm{d}x \leqslant E_2 + \int_\Omega \frac{1}{q(.)}|u|^{q(.)}\mathrm{d}x \tag{7}$$

Inequalities (6) and (7) indicate that

$$\begin{aligned} &\left|\int_\Omega u(.,t)u_t(.,t)\mathrm{d}x\right| \leqslant (1+|\Omega|)\max\left\{\left(E_2 + \int_\Omega |u|^{q(.)}\mathrm{d}x\right)^{\frac{1}{2}+\frac{1}{q^+}}, \left(E_2 + \int_\Omega |u|^{q(.)}\mathrm{d}x\right)^{\frac{1}{2}+\frac{1}{q^-}}\right\} \\ &\int_\Omega |\nabla u|^2\mathrm{d}x \leqslant |\Omega|^{1-\frac{2}{p^-}}(1+|\Omega|)\max\left\{\left(\int_\Omega |\nabla u|^{p(.)}\mathrm{d}x\right)^{\frac{2}{p^+}}, \left(\int_\Omega |\nabla u|^{p(.)}\mathrm{d}x\right)^{\frac{2}{p^-}}\right\} \\ &\qquad \leqslant |\Omega|^{1-\frac{2}{p^-}}(1+|\Omega|)\max\left\{\left(E_2 + \int_\Omega |u|^{q(.)}\mathrm{d}x\right)^{\frac{2}{p^+}}, \left(E_2 + \int_\Omega |u|^{q(.)}\mathrm{d}x\right)^{\frac{2}{p^-}}\right\} \end{aligned} \tag{8}$$

Step 3. A simple computation shows that

$$L'(t) = \int_\Omega u(.,t)u_t(.,t)\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla u(.,t)|^2\mathrm{d}x - \frac{1}{2}\int_\Omega |\nabla u_0|^2\mathrm{d}x + 2\beta(t+t_0) \tag{9}$$

So, it is easy to obtain from (5), (8) and (9)

$$\lim_{t\to T^*} \int_\Omega |u(.,t)|^{q(.,t)}\mathrm{d}x = +\infty \qquad \square$$

The following inverse Hölder inequality with the correction constant plays an important role in proving our main results.

Lemma 1.5. *Assume that u is the solution to Problem (1). Then there exists a positive constant C depending on* $|\Omega|$, p^-, N *such that for any* $\frac{N(q^+-p^-)}{p^-} < k < q^+$

$$\begin{aligned}\int_\Omega \frac{1}{q^+}|u|^{q^+}\mathrm{d}x \leqslant \frac{1}{q^- - p^+}\max\left\{C^{\mu(k)}, C^{\nu(k)}\right\}\max\left\{\Big(\int_\Omega |u|^k \mathrm{d}x\Big)^{\alpha(k)}, \Big(\int_\Omega |u|^k\mathrm{d}x\Big)^{\beta(k)}\right\} \\ + \frac{p^+}{q^- - p^+}\Big(E_2 + \frac{|\Omega|}{q^-}\Big)\end{aligned} \tag{10}$$

where μ, ν, α, β *are defined as the following*

$$\mu(k) = \frac{N(q^+ - k)}{kp^- - N(q^+ - p^-)}, \nu(k) = \frac{Np^-(q^+ - k)}{k(Np^- - Np^+ + p^+p^-) - Np^-(q^+ - p^+)}$$
$$\alpha(k) = \frac{Np^- - q^+(N - p^-)}{kp^- - N(q^+ - p^-)}, \beta(k) = \frac{[Np^- - q^+(N - p^-)]p^+}{k(Np^- - Np^+ + p^+p^-) - Np^-(q^+ - p^+)}$$

Proof. By the interpolation inequality and the embedding inequality $L^{p(.)} \hookrightarrow L^{p^-}$, we know that there exists a positive constant C_1 depending on $|\Omega|$, p^-, q^+, N such that

$$\begin{aligned}\int_\Omega |u|^{q^+}\mathrm{d}x &\leqslant C_1\Big(\int_\Omega |u|^k\mathrm{d}x\Big)^{\frac{(1-\theta)q^+}{k}}\Big(\int_\Omega |\nabla u|^{p^-}\mathrm{d}x\Big)^{\frac{\theta q^+}{p^-}} \\ &\leqslant C\Big(\int_\Omega |u|^k\mathrm{d}x\Big)^{\frac{(1-\theta)q^+}{k}}\max\left\{\Big(\int_\Omega |\nabla u|^{p(x,t)}\mathrm{d}x\Big)^{\frac{\theta q^+}{p^+}}, \Big(\int_\Omega |\nabla u|^{p(x,t)}\mathrm{d}x\Big)^{\frac{\theta q^+}{p^-}}\right\}\end{aligned} \tag{11}$$

where

$$\frac{1}{q^+} = \frac{1-\theta}{k} + \frac{\theta}{p^{-*}}, \quad p^{-*} = \frac{Np^-}{N - p^-} \; C = C_1(1 + |\Omega|)$$

Noticing that $0 < \frac{\theta q^+}{p^-} < 1$ and combining Young inequality with (11), we have

$$\begin{aligned}\int_\Omega |u|^{q^+}\mathrm{d}x \leqslant C\max\left\{\delta^{\frac{p^-}{\theta q^+ - p^-}}, \delta^{\frac{p^+}{\theta q^+ - p^+}}\right\}\max\left\{\Big(\int_\Omega |u|^k\mathrm{d}x\Big)^{\alpha(k)}, \Big(\int_\Omega |u|^k\mathrm{d}x\Big)^{\beta(k)}\right\} \\ + C\max\left\{\delta^{\frac{p^-}{\theta q^+}}, \delta^{\frac{p^+}{\theta q^+}}\right\}\int_\Omega |\nabla u|^{p(.)}\mathrm{d}x\end{aligned} \tag{12}$$

where

$$\alpha(k) = \frac{Np^- - q^+(N - p^-)}{kp^- - N(q^+ - p^-)} > 1, \beta(k) = \frac{[Np^- - q^+(N - p^-)]p^+}{k(Np^- - Np^+ + p^+p^-) - Np^-(q^+ - p^+)} > 1$$

and then choosing $C\max\left\{\delta^{\frac{p^-}{\theta q^+}}, \delta^{\frac{p^+}{\theta q^+}}\right\} = 1$ and using Inequality (7), we obtain the following inequality

$$\begin{aligned}\int_\Omega |u|^{q^+}\mathrm{d}x \leqslant \max\left\{C^{\frac{p^-}{p^- - \theta q^+}}, C^{\frac{p^+}{p^+ - \theta q^+}}\right\}\max\left\{\Big(\int_\Omega |u|^k\mathrm{d}x\Big)^{\alpha}, \Big(\int_\Omega |u|^k\mathrm{d}x\Big)^{\beta}\right\} \\ + \int_\Omega \frac{p^+}{q(.)}|u|^{q(.)}\mathrm{d}x + p^+E_2\end{aligned}$$

This completes the proof of Lemma 1.5. □

Due to the lack of a version of the Gagliardo–Nirenberg inequality for non-constant cases, we have to apply the Gagliardo–Nirenberg inequality for constant cases to establish an inverse Hölder inequality.

Corollary 1.6. *Assume that u is the solution to Problem (1). Then there exists a positive constant C depending on $|\Omega|$, p^-, N such that for any $\frac{N(q^+-p^-)}{p^-} < k < q^+$*

$$\begin{aligned}\int_\Omega \frac{1}{q(.)}|u|^{q(.)}\mathrm{d}x \leqslant \frac{1}{q^- - p^+}\max\left\{C^{\mu(k)}, C^{\nu(k)}\right\}\max\left\{\Big(\int_\Omega |u|^k \mathrm{d}x\Big)^{\alpha(k)}, \Big(\int_\Omega |u|^k \mathrm{d}x\Big)^{\beta(k)}\right\} \\ + \frac{p^+}{q^- - p^+}\Big(E_2 + \frac{|\Omega|}{q^-}\Big)\end{aligned} \tag{13}$$

In fact, we apply Corollary 1.4 and Lemma 1.5 to obtain the following conclusion:

Theorem 1.7. *Suppose that all the conditions of Theorem 1.1 hold and $p^- \geqslant 2$, then*

$$\lim_{t\to T^*}\int_\Omega |u(.,t)|^r \mathrm{d}x = +\infty, \ \ \textit{for any } r > \frac{N(q^+-p^-)}{p^-}$$

Proof. Case 1. When $\frac{N(q^+-p^-)}{p^-} < r < p^+$, we apply Corollary 1.6 to get

$$\begin{aligned}\int_\Omega \frac{|u(.,t)|^{q(.,t)}}{q(.,t)}\mathrm{d}x \leqslant \frac{1}{q^- - p^+}\max\left\{C^{\mu(r)}, C^{\nu(r)}\right\}\max\left\{\Big(\int_\Omega |u|^r \mathrm{d}x\Big)^{\alpha(r)}, \Big(\int_\Omega |u|^r \mathrm{d}x\Big)^{\beta(r)}\right\} \\ + \frac{p^+}{q^- - p^+}\Big(E_2 + \frac{|\Omega|}{q^-}\Big)\end{aligned} \tag{14}$$

Combining Inequality (14) with Corollary 1.4, we know that the conclusion remains true.

Case 2. When $r \geqslant p^+$, it follows from Hölder inequality

$$\int_\Omega |u(.,t)|^{q^+}\mathrm{d}x \leqslant |\Omega|^{1-\frac{q^+}{r}}\Big(\int_\Omega |u(.,t)|^r \mathrm{d}x\Big)^{\frac{q^+}{r}} \tag{15}$$

It is obvious that the conclusion follows from Corollary 1.4 and Inequality (15). □

Next, we first consider the case when $\frac{p^{-*}}{2} + 1 < q^+ < p^{-*} = \frac{Np^-}{N-p^-}(1 < p^- < N)$. Our main result follows.

Theorem 1.8. *If all the conditions of Theorem 1.1 hold and $2 < p^- < q^+ < p^-\Big(1 + \frac{(2+p^{-*})}{2N}\Big)$, then for any $\frac{N(q^+-p^-)}{p^-} < k \leqslant \frac{p^{-*}}{2} + 1$, the blow-up time T^* satisfies the following estimate*

$$T^* \geq \int_{H(0)}^{+\infty} \frac{1}{C_1 \max\{y^{\eta(k)}, y^{\theta(k)}\} + C_2 \max\left\{y^{\alpha(k)}, y^{\beta(k)}\right\} + C_3}\mathrm{d}y$$

where the constants C_1, C_2, C_3 and the exponents $\alpha(k)$, $\beta(k)$, $\eta(k)$, $\theta(k)$ are defined in (21) and $H(0) = \int_\Omega |u_0|^k \mathrm{d}x$.

Proof. Define $H(t) = \int_\Omega |u(t)|^k \mathrm{d}x$ with $\frac{N(q^+-p^-)}{p^-} < k \leqslant \frac{p^{-*}}{2} + 1$, then

$$H'(t) = k\int_\Omega |u|^{k-2}uu_t\,\mathrm{d}x \leq \frac{k}{2}\int_\Omega |u|^{2k-2}\mathrm{d}x + \frac{k}{2}\int_\Omega |u_t|^2 \mathrm{d}x := I_1 + I_2 \tag{16}$$

Due to $2(k-1) \leqslant p^{-*}$, the embedding theorem $W_0^{1,p^-}(\Omega) \hookrightarrow L^{p^{-*}}(\Omega)$ shows that

$$I_1 = \frac{k}{2}\int_\Omega |u|^{2k-2}\mathrm{d}x \leqslant \frac{kC}{2}\Big(\int_\Omega |\nabla u|^{p^-}\mathrm{d}x\Big)^{\frac{2k-2}{p^-}} \leqslant \frac{kC}{2}\Big(\int_\Omega |\nabla u|^{p(.)}\mathrm{d}x + |\Omega|\Big)^{\frac{2k-2}{p^-}} \tag{17}$$

By (13), (17) and Lemma 1.3, one has

$$
\begin{aligned}
I_1 &\leqslant \frac{kC}{2}\left(p^+E_2 + \int_\Omega \frac{p^+}{q(.)}|u|^{q(.)}\mathrm{d}x + |\Omega|\right)^{\frac{2k-2}{p^-}} \\
&\leqslant \frac{kC}{2}\left[p^+E_2 + \frac{p^+}{q^- - p^+}\max\left\{C^{\mu(k)}, C^{\nu(k)}\right\}\max\left\{H^{\alpha(k)}, H^{\beta(k)}\right\}\right. \\
&\quad \left. + \frac{p^+p^+}{q^- - p^+}\left(E_2 + \frac{|\Omega|}{q^-}\right) + |\Omega|\right]^{\frac{2k-2}{p^-}}
\end{aligned}
\tag{18}
$$

For I_2, we have the following estimate

$$
I_2 \leqslant \frac{kE_2}{2} + \frac{k}{2q^- - 2p^+}\max\left\{C^{\mu(k)}, C^{\nu(k)}\right\}\max\left\{H^{\alpha(k)}, H^{\beta(k)}\right\} + \frac{kp^+}{2q^- - 2p^+}\left(E_2 + \frac{|\Omega|}{q^-}\right)
\tag{19}
$$

Furthermore, by (17)–(19), we have

$$
H'(t) \leqslant C_1 \max\{H^{\eta(k)}, H^{\theta(k)}\} + C_2 \max\left\{H^{\alpha(k)}, H^{\beta(k)}\right\} + C_3
\tag{20}
$$

where the exponents $\mu(k)$, $\nu(k)$, $\alpha(k)$, $\beta(k)$, $\eta(k)$, $\theta(k)$ and the coefficients C_i $(i = 1, 2, 3)$ are defined as the following

$$
\begin{aligned}
&\mu(k) = \frac{N(q^+ - k)}{kp^- - N(q^+ - p^-)},\ \nu(k) = \frac{Np^-(q^+ - k)}{k(Np^- - Np^+ + p^+p^-) - Np^-(q^+ - p^+)} \\
&\alpha(k) = \frac{Np^- - q^+(N - p^-)}{kp^- - N(q^+ - p^-)},\ \beta(k) = \frac{[Np^- - q^+(N - p^-)]p^+}{k(Np^- - Np^+ + p^+p^-) - Np^-(q^+ - p^+)} \\
&\eta(k) = \frac{2(k-1)\alpha(k)}{p^-},\ \ \theta(k) = \frac{2(k-1)\beta(k)}{p^-} \\
&C_1 = \frac{kC}{4}\left(\frac{p^+}{2q^- - 2p^+}\max\left\{C^{\mu(k)}, C^{\nu(k)}\right\}\right)^{\frac{2k-2}{p^-}},\ C_2 = \frac{k}{2q^- - 2p^+}\max\left\{C^{\mu(k)}, C^{\nu(k)}\right\} \\
&C_3 = \frac{kC}{4}\left[2p^+E_2 + \frac{2p^+p^+}{q^- - p^+}\left(E_2 + \frac{2|\Omega|}{q^-}\right) + 2|\Omega|\right]^{\frac{2k-2}{p^-}} + \frac{kE_2}{2} + \frac{kp^+}{2q^- - 2p^+}\left(E_2 + \frac{|\Omega|}{q^-}\right)
\end{aligned}
\tag{21}
$$

Applying Theorem 1.7, we get

$$
\lim_{t\to T^*}\int_\Omega |u|^k \mathrm{d}x = +\infty
\tag{22}
$$

According to (20) and (22), we obtain

$$
\int_{H(0)}^{+\infty} \frac{1}{C_1\max\{y^{\eta(k)}, y^{\theta(k)}\} + C_2\max\left\{y^{\alpha(k)}, y^{\beta(k)}\right\} + C_3}\,\mathrm{d}y \le T^*
$$

This completes the proof of this theorem. □

Remark 1.9. Since $p \in \left[p^-\left(1 + \frac{(2+p^{-*})}{2N}\right), p^{-*}\right]$, it seems that we can not obtain results similar to Lemma 1.5 unless we may obtain more information about $\|u_t\|_2$. So, we need to develop a new method or technique to discuss this problem.

Remark 1.10. As a matter of fact, from the process of the proof above, we found that an inverse Hölder inequality can ensure the possibility of blow-up of solutions and the establishment of a differential inequality. So, if we want to solve such problems, we have to establish an inverse Hölder inequality for $1 < k \leqslant \frac{N(q^+ - p^-)}{p^-}$. However, due to technical reasons, we now can not solve this problem.

Finally, we consider another case when $2 < p^- < q^+ \leqslant \frac{p^{-*}}{2} + 1$. For such cases, we may borrow some ideas of [8,6] to get the following theorem.

Theorem 1.11. *If all the conditions of Theorem 1.1 hold and* $2 < p^- < q^+ \leqslant \frac{2+p^{-*}}{2}$, $|q_t| \leqslant C_0$ *then the blow-up time* T^* *satisfies the following estimate*

$$T^* \geq \int\limits_{H(0)}^{+\infty} \frac{1}{C_9 y^\lambda + q^+ y + C_{10}} \mathrm{d}y$$

where the constants C_9, C_{10} *and the exponent* λ *are defined in (27) and* $H(0) = \int_\Omega |u_0|^{q(.,0)}\mathrm{d}x$.

Proof. Define $G(t) = \int_\Omega |u(.,t)|^{q(.,t)}\mathrm{d}x$. Then, a simple computation shows that

$$\begin{aligned} G'(t) &= \int\limits_\Omega q(.,t)|u(.,t)|^{q(.,t)-2}u(.,t)u_t(.,t)\mathrm{d}x + \int\limits_\Omega q_t(.,t)|u(.,t)|^{q(.,t)} \ln |u(.,t)|\mathrm{d}x \\ &\leqslant \frac{q^+}{2}\int\limits_\Omega |u(.,t)|^{2q(.,t)-2}\mathrm{d}x + \frac{q^+}{2}\int\limits_\Omega |u_t(.,t)|^2\mathrm{d}x + C_0 \int\limits_{\{x\in\Omega,|u|\geqslant 1\}} |u(.,t)|^{q(.,t)}\ln|u(.,t)|\mathrm{d}x \\ &\leqslant \frac{q^+}{2}\int\limits_\Omega |u(.,t)|^{2q(.,t)-2}\mathrm{d}x + \frac{q^+}{2}\int\limits_\Omega |u_t(.,t)|^2\mathrm{d}x + \frac{2C_0}{(p^{-*}-2)e}\int\limits_\Omega |u(.,t)|^{q(.,t)+\frac{p^{-*}-2}{2}}\mathrm{d}x \\ &\triangleq J_1 + J_2 + J_3 \end{aligned} \tag{23}$$

Here, we have used the fact that $q_t \geqslant 0$, $0 \leqslant \frac{\ln s}{s} \leqslant \frac{1}{e}$, $\forall\ s \geqslant 1$.

Next, we estimate the value of J_i $(i = 1,2,3)$, respectively. First of all, by Lemmas 2.1 of [14] and Theorem 8.3.1 of [13], there exists a positive constant $C_4 = C_4(|\Omega|, p^\pm, N)$ such that

$$\begin{aligned} J_1 &\leqslant \frac{q^+}{2}\max\left\{\|u\|^{2q^+-2}_{2(q(.)-2)}, \|u\|^{2q^--2}_{2(q(.)-2)}\right\} \leqslant \frac{q^+C_4}{2}\max\left\{\|\nabla u\|^{2q^+-2}_{p(.)}, \|\nabla u\|^{2q^--2}_{p(.))}\right\} \\ &\leqslant \frac{q^+C_4}{2}\max\left\{\Big(\int\limits_\Omega |\nabla u|^{p(.)}\mathrm{d}x\Big)^{\frac{2q^+-2}{p^-}}, \Big(\int\limits_\Omega |\nabla u|^{p(.)}\mathrm{d}x\Big)^{\frac{2q^--2}{p^+}}\right\} \\ &\leqslant \frac{q^+C_4}{2}\max\left\{\Big(p^+E_2 + \int\limits_\Omega \frac{p^+}{q(.)}|u|^{q(.)}\mathrm{d}x\Big)^{\frac{2q^+-2}{p^-}}, \Big(p^+E_2 + \int\limits_\Omega \frac{p^+}{q(.)}|u|^{q(.)}\mathrm{d}x\Big)^{\frac{2q^--2}{p^+}}\right\} \\ &\leqslant \frac{q^+C_4}{2}\max\left\{1, M_1^{\frac{2q^--2}{p^+}-\frac{2q^+-2}{p^-}}\right\}\Big(M_1 + \int\limits_\Omega \frac{p^+}{q(.)}|u|^{q(.)}\mathrm{d}x\Big)^{\frac{2q^+-2}{p^-}} \\ &\leqslant C_5 G^{\frac{2q^++p^*-2}{2p^-}}(t) + C_6 \end{aligned} \tag{24}$$

where C_i $(i = 5,6)$ is defined as follows

$$C_5 = 2^{\frac{2q^++p^*-8p^--2}{2p^-}}\max\left\{M_1^{\frac{2q^+-2}{p^-}-\frac{2q^++p^*-2}{2p^-}}, M_1^{\frac{2q^--2}{p^+}-\frac{2q^++p^*-2}{2p^-}}\right\}C_4q^+$$

$$C_6 = C_5M_1^{2q^++p^*-2}2p^-, M_1 = p^+|E_1| + 1 > 0$$

Noticing that $q^+ + \frac{p^{-*}-2}{2} \leqslant p^{-*}$, we follow the line of the proof of J_1 to get

$$J_3 \leqslant C_7\, G^{\frac{2q^++p^*-2}{2p^-}}(t) + C_8 \tag{25}$$

where C_i $(i = 7,8)$ is defined as follows

$$C_7 = \frac{2^{\frac{2q^++p^*-2}{2p^-}}C_0C_4}{(p^{-*}-2)e}\max\left\{1, M_1^{\frac{2q^++p^*-2}{2p^-}-\frac{2q^-+p^*-2}{2p^+}}\right\}, C_8 = C_7\, M_1^{\frac{2q^++p^*-2}{2p^-}}$$

Subsequently, Lemma 1.3 and the definition of $E(t)$ indicate that

$$J_2 \leqslant q^+\Big(E_2 + G(t)\Big) = q^+E_2 + q^+G(t) \tag{26}$$

By (23)–(26), we have

$$G'(t) \leqslant C_5 G^{\frac{2q^+ + p^* - 2}{2p^-}}(t) + C_6 + q^+ E_2 + q^+ G(t) + C_7 G^{\frac{2q^+ + p^* - 2}{2p^-}}(t) + C_8$$
$$= C_9 G^{\lambda}(t) + q^+ G(t) + C_{10}$$

where C_9, C_{10} and λ are defined as follows

$$C_9 = C_5 + C_7,\ C_{10} = q^+ E_2 + C_6 + C_8,\ \lambda = \frac{2q^+ + p^* - 2}{2p^-} > 1 \tag{27}$$

This completes the proof of Theorem 1.11. □

Remark 1.12. In fact, we apply the methods used in this paper to obtain similar results of the problem of [1] under certain assumptions.

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