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D'Alembert's paradox, 1900–1914: Levi-Civita and his Italian and French followers

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ABSTRACT

Before the First World War, Tullio Levi-Civita (1873–1941) was already a well-known mathematician in Italy and abroad, in particular in France. Professor at the University of Padua since 1898, he had published important contributions to tensor calculus, theory of relativity, differential geometry, hydrodynamics, and the three-body problem. In 1918, when he moved to the University of Rome, he created an international school of mathematics. In this paper, we focus on d'Alembert's paradox to which Levi-Civita and some of his Italian and French followers contributed remarkable solutions. This case-study is used to illustrate Levi-Civita's approach to hydrodynamics and its influence in Italy and France, especially in the period 1910–1914.

1. Introduction

Since ancient times, it had been known that a certain resistance is opposed to a body moving in a fluid. However, in 1768 Jean Le Rond d'Alembert published a paper entitled *Paradoxe proposé aux géomètres sur la résistance des fluides*, in which for incompressible and inviscid potential flow he obtained the result of zero drag on head-tail symmetric two-dimensional bodies moving with constant velocity relative to the fluid.[1] The contradiction to experimental facts is striking. D'Alembert wrote that it was "a singular paradox" he left to "future geometers" for elucidation.

Still in 1918, Villat wrote in his survey on recent advances of hydrodynamics [3] p. 44: "*Le groupe des faits expérimentaux qui a le plus longuement déconcerté les mathématiciens est celui qui se rattache à la résistance qui oppose un fluide à l'avancement d'un solide entièrement baigné par celui-ci.*" Here, he explicitly mentioned Levi-Civita's contributions to the solution to the paradox. Today we commonly recognize that the neglect of friction is the actual cause of d'Alembert's paradox.

In this paper, we consider the concept of discontinuity surface introduced by Hermann von Helmholtz, through which many scholars believed they could solve d'Alembert's paradox; in particular, we focus on Levi-Civita's wake hypothesis that improved Helmholtz's theory by introducing some rigorous assumptions on the motion of a body in a perfect fluid. We use this case study to illustrate Levi-Civita's approach to hydrodynamics and its influence in Italy and France, especially on Villat's research. We show that although Levi-Civita's works on wave theory and on the theory of jets were received in the whole of Europe, the spread of his main contribution to the wake hypothesis was essentially limited to Italy and France.

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[1] See [1] Appendix A for a modern statement of d'Alembert's paradox. See also [2].

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2. The wake hypothesis

In the second half of the 19th century, Helmholtz (1868), Gustav Kirchhoff (1869) and Lord Rayleigh (1876) had already contributed to the solution to d'Alembert's paradox by elaborating new hydrodynamic theories.[2] Helmholtz's approach used the concepts of vorticity and vortex filaments in the flow and introduced the so-called "surface of discontinuity" (or, in his words, "vortex sheet") in an inviscid fluid to predict a non-zero drag on bodies and thus solve d'Alembert's paradox. By supposing a discontinuity surface, the components of flow velocity tangent to the sheet are discontinuous. There is a dead-water region (the "wake") behind the moving body that separates the surfaces of discontinuity from the rest of the liquid. The flow pressure exerted over the head of the body then has a high value when compared with the pressure exerted on the rear of the body. So, qualitatively, the "wake hypothesis" solves d'Alembert's paradox. In his 1869 paper, Kirchhoff considered the surfaces of discontinuity for expressing the force exerted on a lamina moving in a liquid – both perpendicular and inclined at an oblique angle to the flow. A few years later, Rayleigh was able to explicitly calculate the forces given by Kirchhoff for some actual cases. By "operating in the mode of an engineer", as Anderson wrote [5] p. 105, Rayleigh elaborated tables and graphs in "excellent agreement" with experimental data.[3]

In 1901, Levi-Civita, then professor of rational mechanics at the University of Padua, published his first paper on hydrodynamics written as a letter to Siacci and communicated by the latter to the *Rendiconti dell'Accademia dei Lincei* [6]. Francesco Siacci taught ballistics at the Scuola di Applicazione and then became professor of mechanics at the University of Turin; he was the author of celebrated treatises on ballistics translated into several languages. Levi-Civita addressed his letter to Siacci, since similar mathematical methods are sometimes used in ballistics and hydrodynamics – in some cases, the motion of a projectile can be dealt with as the motion of a body in a fluid.

Levi-Civita's 1901 paper drew attention to the paradox of d'Alembert by proposing a new and "rigorous" wake hypothesis [6]. Levi-Civita aimed to re-consider the previous results on the "wake" behind a moving body in a liquid due to Helmholtz, Kirchhoff and Rayleigh, and to put them on a solid mathematical basis. In particular, Levi-Civita considered a body S moving with constant velocity relatively to the fluid supposed to be infinite, frictionless and incompressible; he remarked that the velocity should be small enough, since for very high velocity it is difficult to mathematically describe "the drag behavior".

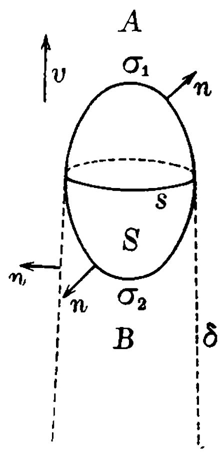

Fig. 1. [6] p. 131.

Levi-Civita assumed the following (see Fig. 1):

"1. The motion of the fluid produced by the body S [stationary with respect to S] has a surface of discontinuity δ behind the body; such a surface extends to an infinite distance behind the body and starts from a certain curve s on the boundary σ of S.

2. The particles of the fluid of the region B behind the body behave as if they were rigidly connected to S.

3. The motion of the fluid in the region A [in front of the body] is irrotational, and satisfies the usual conditions at infinity." [6] p. 130–131

Levi-Civita translated his assumptions into mathematical terms; he remarked that viscosity is to be considered as a secondary dissipative phenomenon, and is then distinct from resistance. Finally, he deduced from his "wake hypothesis" a resistance proportional to square speed in accordance to Newton's law for incompressible fluids (drag is proportional to square speed) in a rigorous way "without leaving pure hydrodynamics".[4]

[2] See [4] for further details.

[3] For details on the contributions by Helmholtz, Kirchhoff and Rayleigh see Chapter 4 of [5].

[4] Levi-Civita quoted, for instance, the treatise on practical hydraulics by the civil engineer Ildebrando Nazzani, *Trattato di idraulica pratica* [7].

Although this work is not widely cited by historians, it had a strong influence in Italy and France. Inspired by Levi-Civita's results, Umberto Cisotti (1882–1946), one of Levi-Civita's students in Padua, published a series of papers on d'Alembert's paradox. In some of them, which appeared in the *Atti dell'Istituto Veneto*, Cisotti generalized the original statement of d'Alembert's paradox to bodies with various shapes [8]. Cisotti proved his results without discussing the reason for the gap between "mathematical representation and experimental facts" or the way to overcome this gap ([8] p. 423); he referred to Levi-Civita's memoir for a discussion of these subjects. Cisotti deduced from the general principles of hydrodynamics (i.e. Euler's equations and Bernoulli's law) that the drag force is "rigorously zero" for bodies with any shape [8] and also in any flow in all space [9] or in cylindrical canals [10]. Most of these results were already well known at that time; however, Levi-Civita and Cisotti aimed at showing that they descended naturally from Levi-Civita's wake hypothesis, and without exiting rational hydrodynamics.

In France, Jacques Hadamard (1865–1963) reacted to Levi-Civita's 1901 paper [6] very soon by writing a series of letters to his Italian colleague. Since 1901 Hadamard had been editing his own lectures on hydrodynamics given in the academic years 1898–1899 and 1899–1900, and then published in 1903 as *Leçons sur la propagation des ondes et les équations de l'hydrodynamique* [11]. In a letter to Levi-Civita dated 19 April 1902, Hadamard formulated a critical objection to the wake hypothesis by remarking that discontinuity surfaces were not legitimate for liquids, while "in the case of gases one must introduce the discontinuities." In particular, Hadamard wrote: "*Seulement, il y a un cas dans lequel je ne puis pas être d'accord avec vous: c'est celui des liquides. La véritable théorie du phénomène ne me paraît pas pouvoir être cherchée* (*pour les liquides*) *dans les discontinuités de l'espèce que vous introduisez*." We do not have Levi-Civita's answer, but in a following undated letter Hadamard recognized that his "objection" to the existence of a discontinuity in the case of the liquids was "a simple inadvertence".[5]

However, Levi-Civita's idea of wake did not take real fluids into account. Various scientists immediately pointed out that it was desirable to also consider actual examples in viscous fluids. In a letter to Levi-Civita dated 19 February 1903, Hadamard remarked the existence of phenomena concerning friction, turbulence or "swirls", which rational mechanics failed to explain – and nor did Levi-Civita's theory.[6]

3. Levi-Civita's analytical method

In 1907 Levi-Civita published a paper in the *Rendiconti del Circolo matematico di Palermo* [13], in which he described a new method that allowed him to find the general integral of a plane motion with wake if the shape of the body moving in the fluid is polygonal or curvilinear. He considered the two-dimensional case where the fluid is perfect. So, this paper completes the 1901 note [6] in which Levi-Civita had assumed his three hypotheses describing a moving body with wake, but had not been able to solve the differential equations leading to the shape of the surface of discontinuity for a given curve s (see Fig. 1). We focus on Levi-Civita's wake hypothesis that allowed him to solve d'Alembert's paradox.

As in his 1901 paper, Levi-Civita considered resistance (induced by pressure differences) and viscosity as two distinct and independent phenomena – viscosity being a marginal in his view (he called it "second-order effect"). In fact, experience shows that resistance is proportional to square speed; if the main part of resistance depended on the fluid viscosity, then resistance would be proportional to speed, as proved by Lamb in his *Hydrodynamics*.[7] Thus, Levi-Civita remarked ([13], p. 520), d'Alembert's paradox can be explained "without going out rational hydrodynamics" as a consequence of the "wake hypothesis", as Helmholtz had already shown by introducing the idea of discontinuity surface. In comparison to his 1901 paper [6], the shape of the body C moving in the fluid is now polygonal or curvilinear with a unique angular point O called the bow (see Fig. 2). He still considered the fluid to be divided into two parts: the region A, in front of the body, is irrotational, while the region B, behind the body, behaves as if it were rigidly connected to C. But now he pointed out that actually "the nature of the discontinuity is more complicated than the one which is supposed theoretically"; although experience showed that the region B did not exactly move with the body C and did not extend to infinity ([13] p. 522), his wake hypothesis was necessary for dealing mathematically with motions with wake. Apparently, the works published in those years induced Levi-Civita to be more cautious in explaining the wake hypothesis and its possible physical implications. We mention, for instance, that in 1904 Ludwig Prandtl (1875–1953) had introduced the so-called boundary-layer theory in his lecture at the Third International Congress of Mathematicians in Heidelberg, which very probably Levi-Civita attended.[8] Prandtl did not omit the viscous term even for slightly-viscous fluid as water or air, but he neglected this term everywhere except in a boundary layer of fluid near the solid walls to which the fluid adheres. According to Prandtl's theory, the flow can be divided into two parts: (i) the thin boundary layer ("Grenzschicht" or "Übergangsschicht") near the surface, where the effects of friction are dominant; (ii) an inviscid flow outside the boundary layer, where friction is negligible. The two regions interact with each other when the flow separates; in that case the separation region greatly affects the outer inviscid flow. So, the boundary-layer creates vorticity, which is then diffused and advected in the wake. Nowadays, Prandtl's theory is the commonly accepted resolution of d'Alembert's paradox.

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[5] The detailed discussion between Hadamard and Levi-Civita is reported in [12] pp. 87–90.

[6] For the complete text of the letters see [12] pp. 88–90.

[7] Levi-Civita quoted Lamb's *Hydrodynamics*, Chapter XIX, § 318, of the edition of 1906.

[8] Levi-Civita, indeed, participated in the 1904 International Congress, where Prandtl gave his celebrated lecture [14]. For a description of Prandtl's theory, see, for instance, [1] pp. 283–286.

To sum up Prandtl's 1905 paper, Anderson writes ([5] p. 255): "Prandtl gave the boundary-layer equations for steady two-dimensional flow, suggested some solution approaches for those equations, made a rough calculation of friction drag on a flat plate, and discussed aspects of boundary-layer separation under the influence of an adverse pressure gradient. Those were all pioneering contributions." In the twenties and thirties, Prandtl's boundary-layer theory was largely applied by his followers for calculations in actual hydrodynamics and in aeronautics.

In a footnote aiming at highlighting what is a discontinuity surface and how it is created, Levi-Civita ([13] p. 522) pointed out that the discontinuity surface is a "vortex surface", where new vortex rings must continually come out from the first part of the surface of discontinuity, and descend along the surface, in substitution for those that separate at the end. This remark sounds much like Prandtl's concept of boundary-layer. However, Levi-Civita's mathematical approach to fluid dynamics is very different from Prandtl's heuristic method.[9] Levi-Civita ([13] p. 520) claimed that his mathematical solution to d'Alembert's paradox was very satisfactory because "until now the theory of perfect fluids has corresponded the factual circumstances", and therefore "it would be very strange that, in the same order of questions, some other phenomena – escaping the ordinary scheme of mechanism – intervene unexpectedly and dominate." Apparently, in Levi-Civita's opinion, previous solutions to d'Alembert's paradox were not rigorous enough.

Furthermore, Levi-Civita quoted the experiences of Friedrich Ahlborn and Étienne Marey, who published their interesting photographs of wakes taken some years before.[10] These photos show that there are turbulent flows in the region *B* close to the body, while the flow becomes irrotational at a distance from the body. After mentioning all these relevant experimental data, Levi-Civita ([13] p. 523) argued that, both from an experimental and a theoretical point of view, "[his] schematic wake hypothesis very probably leads to an accurate assessment of direct resistance", as the resistance only depends on "the state of motion of [the discontinuity surface[π[made by the two discontinuity surfaces π_1 and π_2, see Fig. 2]".

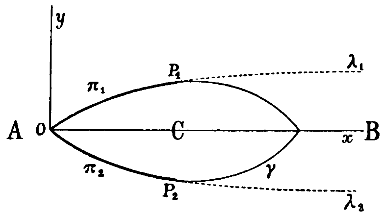

Fig. 2. [13] p. 525.

Though we do not illustrate his method here,[11] we just mention that Levi-Civita employed sophisticated results of complex analysis, in particular Cauchy's residue theory, in order to extend "the Helmholtz–Kirchhoff method" by introducing a suitable conformal representation.[12] In an article published in the *Encyclopédie des sciences mathématiques* on the development of hydrodynamic theories, the authors Love, Appell, Beghin, Villat ([18] p. 118) wrote: "*En 1907 T. Levi-Civita [13] réussit d'une façon très élégante à limiter le champ des formes analytiques pouvant correspondre au mouvement permanent d'un fluide indéfini autour d'un obstacle courbe ; il suppose les parois de l'obstacle continues à l'exception d'un point anguleux ; c'est en ce point (hypothèse qui semblait naturelle) qu'il place le point mort, où le courant se divise pour entourer l'obstacle. Sous forme très simple, T. Levi-Civita calcule la résistance éprouvée par l'obstacle. Mais aucun exemple n'est traité explicitement, et il n'existe encore aucun moyen de rattacher la fonction arbitraire dont dépend le problème à la forme de la paroi de l'obstacle supposé connue.*" Levi-Civita seems to be aware that actual examples would be relevant for corroborating his theory. He hoped to be able to deal "soon" with "a technical problem of practical application to ships." ([13] p. 522) Though he claimed that "we do not miss the practical interest [of his method]" ([13] p. 521), he was concerned with rational hydrodynamics rather than its applications to actual cases.

4. The reception of Levi-Civita's wake hypothesis

Not everybody shared Levi-Civita's mathematical approach to hydrodynamics – in particular his wake hypothesis. Some scholars – for instance Lord Kelvin in England or Joseph Boussinesq in France – strongly criticized the concept of discontinuity surfaces, which seemed to oppose the principle of minimum energy.[13] As Ghosh ([19] p. 75) wrote, the essential objection against Helmholtz's theory lies in the fact that "the introduction of the body should modify the flow at infinity, i.e.

[9] See, for instance, [1] pp. 286–289.

[10] Levi-Civita quotes [15] and [16].

[11] For technical details of Levi-Civita's method, see [12]. On the late nineteenth-century hydrodynamic theories of Helmholtz, Kirchhoff and Rayleigh, see [4].

[12] Levi-Civita himself referred to "la méthode de Helmholtz–Kirchhoff" in the letter dated 19 June 1911 to Villat: "*Je me réjouis infiniment que le perfectionnement de la méthode de Helmholtz-Kirchhoff signalé par moi ait donné lieu à bien des recherches savantes.*" Levi-Civita's letters to Villat are in the dossier Villat, in the "Archives de l'Académie des sciences" in Paris and published in [17] pp. 371–410.

[13] See, for instance, [1] for a general survey on different contributions to hydrodynamics.

the wake should extend to infinity – a contradiction to observed facts." The same objection remains valid for Levi-Civita's wake hypothesis.

Nevertheless, several students and colleagues followed his research, especially in Italy and France. Cisotti, Tommaso Boggio (1877–1963), Emilio Almansi (1869–1948), Gustavo Colonnetti (1886–1968), Bruto Caldonazzo (1886–1960), Attilio Palatini (1889–1949), and Giuseppe Picciati (1868–1908) generalized Levi-Civita's results on plane motions with wake in several papers. In his lectures held at the University of Pavia in the academic year 1918–1919 and collected in the book *Idromeccanica piana*, Cisotti wrote ([20] p. V): *"The development of plane hydrodynamics in recent years – due for the most part to Italian contributions, concerning both the development of methods and the large number of results – made me feel that time is appropriate not only to talk about this school, but also to extend the knowledge to a wider circle of scientists who are outside the field of mathematics, more particularly among the specialists of hydraulic disciplines."*

We focus on France, where Levi-Civita's ideas inspired particularly the works of Marcel Brillouin (1854–1948) and of his student Henri Villat (1879–1972). In his lectures on theoretical physics at the "Collège de France", Brillouin dealt with "jets" (1908–1909) and "viscous fluids" (1909–1910). Villat attended the lectures of his *patron* (supervisor) of 1909, as he explicitly remarked in his thesis [21]. As appears from reading the notes of his course, Brillouin analysed Levi-Civita's 1907 paper [13] in March 1909, and some papers by Boggio and Picciati in the following months.[14] It is interesting to follow the genesis and progress of Villat's ideas, then developed in his thesis, through the exchanges between Villat, Brillouin, and Boussinesq.[15]

The concept of discontinuity surface soon attracted Brillouin's interest. In a memoir published in *Annales de Chimie et Physique* [25] (Brillouin, 1911), Brillouin mentioned the definition of the vortex surface given by Helmholtz, the "new" contributions by Prandtl, and discussed Levi-Civita's wake hypothesis and his analytical method [13] (Levi-Civita 1907). In his paper, Brillouin showed the necessity of discontinuity surfaces in a fluid, but defined the limits of validity of the wake hypothesis. He proved that, in a two-dimensional flow, the discontinuity surfaces must be infinite, otherwise some points of fluid would have a negative pressure and then d'Alembert's paradox would be recovered. Then Duhem was to denote that statement as "the Brillouin paradox" and to extend it from two to three dimensions [26].

"*C'est ce Mémoire de M. Levi-Civita* [13] – Brillouin wrote ([25] p. 153) – *qui m'a ramené à l'étude de ces questions*." However, Brillouin employed physical considerations for calculating the resistance in spite of Levi-Civita's analytical method. "*En outre, la mode d'exposition diffère sensiblement de celui de M. Levi-Civita : en particulier pour la recherche de l'expression de la résistance, rattachée non aux problèmes analytiques de la solution, mais aux propriétés physiques du courant*", he remarked ([25] pp. 153–154). Brillouin was proud to point out that the theoretical results of the resistance are in accordance with the experimental data that appeared in some tables due to Alfred Kannapell and reported in his article.

In a letter from 20 June 1910, Brillouin persuaded his student Villat to consider Levi-Civita's 1907 paper, which "deserves" most interest, though in a previous letter from 10 June he had suggested his pupil avoid "the Italian fashion of publication" by referring to the lack of examples in Levi-Civita's works.[16] In his dissertation, essentially published in *Annales de l'ENS* in 1911 [21], Villat followed Brillouin's advice and reconsidered, developed and generalized such crucial points of Levi-Civita's 1907 paper [13] connected to the wake hypothesis. We do not illustrate Villat's work here,[17] which completes Levi-Civita's paper by solving the following question: "To find the motion and all its elements, if the shape of the obstacle is given." ([21] p. 206) We point out that Villat's mathematical methods are directly inspired by those of Levi-Civita. Some of Villat's students analysed topics connected to Levi-Civita's hydrodynamic research. For instance, René Thiry (1886–1968) was directly inspired by "an important memoir of Levi-Civita [13]" ([28], p. 1). He started from Villat's result [29] that hydrodynamic equations can lead to more than one solution – in particular, if the obstacle is made by two concurrent segments with concavity towards the current, then two solutions are possible and neither is better a priori than the other. Thiry proved that, under certain conditions, the solutions constitute a continuous succession between the two solutions deduced by Villat.

Pure analysis had a prominent role in Levi-Civita's and Villat's approach to hydrodynamics. We have already noticed that Levi-Civita used methods of complex analysis in his paper on the wake hypothesis [13]. Furthermore, in Villat's thesis [21] and in [13], a special functional relation due to Ulisse Dini appears. This formula connects the values of a function f on the circumference of a circle with the values of its normal derivative on the same circumference, if f is assumed to be harmonic inside the circle. In their letters, Levi-Civita and Villat largely discussed the hydrodynamic consequences of Dini's formula.[18] In connection with this research, Villat published several notes on the Dirichlet principle in a circle, an annulus and other special figures, and exchanged some letters with Émile Picard (1856–1941) on that subject. In the period 1900–1920, especially in Italy and France, mathematicians successfully applied the so-called Fredholm method in order to solve some special classes of integral equations that describe physical phenomena, including heat theory, theory of elasticity, and hydrodynamics. Some of the works published by Levi-Civita and Villat are part of this new trend.[19]

[14] In particular, Brillouin considered [22] and [23] concerning the motion of a sphere on a viscous fluid. See Brillouin's Papers at the "Bibliothèque de l'Institut de France" (MS5601 and MS5602).

[15] On these exchanges and generally on the influence of Levi-Civita's works on hydrodynamics in France, see [24].

[16] The letters by Brillouin to Villat are in Dossier Villat, "Archives de l'Académie des sciences" in Paris.

[17] See [24] and [27] for a discussion of Villat's work on hydrodynamics.

[18] For a detailed analysis of the correspondence between Villat and Levi-Civita, and its connection to Dini's formula, see [12] pp. 97–108.

[19] See [30] for a detailed analysis.

Villat played a very important role in applied hydrodynamics and aeronautics in France after World War I – in 1920 he became professor at the University of Strasbourg, in 1927 he obtained the chair of mechanics at the Sorbonne, and in 1929 was appointed director of the Institute of fluid mechanics in Paris.[20] In the 1930s, he was more and more involved in several institutional tasks, including the teaching at the "École supérieure d'aéronautique" in Paris. Nevertheless, in his publications as well as in his lectures, Villat preferred a mathematical approach to hydrodynamics without mentioning any physical experiments explicitly, even when he dealt with aerodynamics. His taste for abstract mathematical methods, especially those linked to the theory of functions and complex analysis, explains his admiration for the work of Levi-Civita and his school.

As Darrigol ([1] p. 270) writes: *"In summary, at the turn of the century, discontinuity surfaces remained the main analytical approach to the resistance problem for a slightly viscous fluid. Yet they had well-identified shortcomings, namely: they led to utterly instable and physically impossible motions, they gave smaller resistances than in reality, and they were essentially indeterminate in the case of smoothly-shaped bodies. For a Kelvin, these defects were fatal. For a Rayleigh, a Föppl, or a Levi-Civita, discontinuity surfaces marked a significant step toward a successful theory of resistance."* Together with Rayleigh, Föppl, and Levi-Civita, we could also add the name of Henri Villat, who closely followed Levi-Civita's research, especially on the wake hypothesis. In his *Aperçu théoriques sur la résistance des fluides* published in 1920 [32], Villat's aim "*est de donner, en un abrégé aussi succinct que possible, les résultats théoriques que l'on peut obtenir en appliquant les équations de l'hydrodynamique des fluides parfaits à l'étude du mouvement d'un solide dans un fluide*" ([32] p. 1). After describing the mathematical tools employed in the theory – such as Dini's formula which we mentioned above and some results of the theory of integral equations – Villat illustrated Levi-Civita's wake hypothesis and his method, the contributions of Cisotti, Brillouin, and other followers of Levi-Civita, including his own works. Villat's dealing is purely theoretical. In the book, there is no reference to any theory different from that based on the wake hypothesis; Prandtl's theory is not even mentioned.

The exchanges between Levi-Civita and Villat allow us to highlight their common search for a mathematical approach to hydrodynamics. Their close friendship was mainly founded on their shared scientific ideas. Levi-Civita supported his French well-reputed colleague by accepting to give lectures in his Institute of Fluid Mechanics, and to publish in the journal of pure and applied mathematics (*Journal de mathématiques pures et appliquées*) and in *Mémorial des sciences mathématiques*, both directed by Villat.[21]

5. A French–Italian affair

Marcel Brillouin and Villat belonged to the small group of Italian and French mathematicians who accepted Levi-Civita's ideas on the wake hypothesis and tried to show the legitimacy and relevance of discontinuity surfaces in hydrodynamics. The following words of Villat sound like a piece of propaganda to promote discontinuity surfaces by highlighting their accordance with actual facts ([3] p. 45): "*Pour échapper à des conséquences évidemment contradictoires avec l'expérience la plus courante, il est nécessaire de supposer dans le fluide la présence de surfaces de discontinuité: à l'arrière du corps il se forme une région du fluide mort, immobile par rapport à ce corps. Au reste l'observation, même superficielle, montre que dans la réalité de telles surfaces ont une existence effective (qui n'en a fait l'expérience, en wagon ou derrière le coupe-vent d'une automobile ?). L'hypothèse introduite répond donc à la matérialité des faits.*" Cisotti, the main follower of Levi-Civita's ideas on hydrodynamics in Italy, promoted the wake theory due to Levi-Civita and then developed by the various members of the group in his aforementioned lectures on two-dimensional hydrodynamics held at the University of Pavia, as well as in various scientific publications.

In this section we consider a controversy, which took place in volume 159 of the *Comptes rendus de l'Académie des sciences*, in order to show that this group composed by Levi-Civita and his Italian and French followers was compact and very reactive against those who put their ideas in discussions. The controversy involved the mathematician and physicist Pierre Duhem (1861–1916) on one side, and Levi-Civita's group on the other – Picard, Villat, Levi-Civita himself and, indirectly Cisotti, Brillouin, and Boggio. Duhem was a reputed physicist, who had been interested in hydrodynamics since the late eighties when he had published his lectures held at the University of Lille. Significantly, he seems not to be aware of Levi-Civita's theory of wake, nor of the papers and books published by Brillouin and Villat on d'Alembert's paradox from the beginning of the century. Apparently, their works were not very popular, even in France.

Duhem published a note in the *Comptes rendus* [33], in which he started from the following statement by Cisotti [34] on the validity of d'Alembert's paradox: "*Un corps solide est plongé dans un fluide indéfini de toutes parts, dénué de viscosité, compressible suivant une loi quelconque et soustrait à l'action de toute force. Le mouvement du solide est une translation uniforme, de vitesse* V*, parallèle à* Ox *; le mouvement du fluide est un régime permanent; à l'infini le fluide est en repos. La somme des projections sur* Ox *des forces qu'il faut appliquer au solide pour entretenir un tel mouvement est égale à zéro.*" ([33] p. 592)

Duhem pointed out that Cisotti had proved the theorem by assuming that there is no surface of discontinuity in the fluid. He went on ([33] p. 592): "*Nous nous proposons de démontrer ici, en suivant la même voie que M. Cisotti, que le théorème énoncé demeure vrai, même si le fluide est partagé par des surfaces de discontinuité.*" Picard's reaction was immediate; his brief note is published in the same issue of the journal [35]. Duhem, he wrote, thinks that discontinuity surfaces do not allow us

[20] A wide image of Villat's role in French interwar period is in [31]. An analysis of Villat's role in France is in [27].

[21] For Levi-Civita's activity in France, see section 5.2 in [24].

to avoid d'Alembert's paradox. But discontinuity surfaces go to infinity – and he quoted the works by Helmholtz, Levi-Civita, and Villat – while Duhem implicitly assumes that these surfaces are finite.

Duhem answered with two notes, still published in volume 159 of the *Comptes rendus*, where he tried to reinterpret the meaning of his conclusion by bringing it into agreement with Levi-Civita's ideas. He admitted a mistake, even if implicitly. In [36] Duhem wrote that his last note [33] led to a merely apparent contradiction as, "*d'une manière presque simultanée*", Picard, Villat and Levi-Civita pointed out.[22] Duhem claimed that his conclusion was "*sous une forme d'une précision insuffisante*". The right statement, Duhem wrote, had to be the following: "*Ce qu'on doit regarder comme inadmissible, c'est, au sein d'un fluide indéfini qui contient un corps solide, l'hypothèse d'un régime permanent* avec repos du fluide à l'infini." ([36] p. 639) Indeed, this statement is not in contradiction with the research developed by Levi-Civita and his followers ("*ceux qui ont suivi sa manière de voir*"), according to whom the fluid is not "*partout en repos à l'infini*". In fact, discontinuity surfaces persist at infinity. "*Notre démonstration paraît donc prouver que la voie suivie par M. Levi-Civita et par ses continuateurs est bien la seule qui permette d'échapper au paradoxe de D'Alembert*", wrote Duhem ([36] p. 639). In his third and last note published in the same volume of the journal [26], Duhem considered a "remarkable" theorem by Brillouin [25]: d'Alembert's paradox is valid if discontinuity surfaces are not infinite. In [26] Duhem proved that "the Brillouin paradox", as he called this result, is also valid in three dimensions.

The last note published in volume 159 of the *Comptes rendus* concerning d'Alembert's paradox is due to Villat [37] and is communicated by Duhem – as Duhem himself had promised in a letter to Villat (see the Appendix). In order to show the inconsistency of Duhem's first note [33], Villat referred to Brillouin's fundamental paper of 1911 that Duhem had extended to the case of a three-dimensional fluid in his third note of the *Comptes rendus* [26]. Villat was a young *maître de conférences* in Montpellier, while Duhem was already a professor of physics at the University of Bordeaux.

To summarize, after the publication of the notes by Picard and Villat and an exchange of letters between Duhem and Levi-Civita, Picard and Villat, the controversy ends with the decisive defeat of Duhem, who admitted that his result was valid only under an assumption ("*l'absence de toute surface de discontinuité*") that opposed to Levi-Civita's wake hypothesis. Levi-Civita's followers seem then to compose a close-knit group – they shared the same theory on the wake hypothesis and on the way of solving d'Alembert's paradox, referred to the works published by their small circle of scholars, and were ready to support each other. The letter by Duhem to Villat dated 10 November 1914 and reported in the Appendix shows well how Duhem was under attack. He wrote: "*Peu après votre première lettre, j'en ai reçu une de M. Émile Picard et une autre de M. Levi-Civita; toutes deux attiraient mon attention sur l'apparente contradiction entre ma conclusion et vos travaux. Cela m'a décidé à envoyer une note à l'Académie pour dissiper le malentendu. Je regrette maintenant de n'avoir pas attendu davantage, car votre note eût rendu la mienne inutile.*"

6. Epilogue

1. From 1918 onwards, Levi-Civita was professor at the University of Rome, where he contributed to creating a mathematical school at an international level. In hydrodynamics, his research on the theory of jets and water-wave theory attracted several students (PhD-students and Rockefeller students) – such as the French Robert Mazet and Marie-Louise Dubreil-Jacotin, but also the Dutch Dirk Struik and the German Alexander Weinstein – while his contributions to the wake hypothesis remained confined to his small group of Italian–French collaborators.[23]

2. The search for rigour is typical of Levi-Civita's work in all fields of mathematical physics to which he contributed – hydrodynamics, three-body problem, analytical mechanics, tensor calculus, and theory of relativity. In accordance with his ideas on mathematical-physical disciplines, Levi-Civita aimed to solve d'Alembert's paradox without leaving the field of "rational hydrodynamics". Villat shared the same approach to hydrodynamics both in his publications and in his lectures. In this regard, we report the following section of a letter from Theodore von Kármán to Jerome Clarke Hunsaker[24]: "I agree that Villat and Pérès are excellent men; however, Villat is far beyond the line of what we would call the frontier of useful or applied mechanics. I really believe that the man we could use best for a general lecture is Kampé de Fériet, director of the Institute for fluid mechanics, in Lille. In the last years, he published two reviews on recent progress concerning waves and turbulence. Both reports were excellent and just on the limit between the practical and theoretical viewpoint as we like it. Besides that, he follows the experimental research, whereas Villat, in spite of the fact that he is director of an experimental institution, has no idea of experimental questions".

3. The solution to d'Alembert's paradox has been and still is the object of controversies. For instance, in his book on hydrodynamics published in 1950 [39], Garrett Birkhoff harshly criticized the solution to d'Alembert's paradox that followed from Prandtl's boundary-layer theory. Other recent works show a new approach to the solution to d'Alembert's paradox different from Prandtl's theory.[25] The inherent complexity of fluid mechanics and the diversity of backgrounds of the actors who try to solve its problems still lead to predictable disagreements.

[22] We report in the Appendix the letters concerning this polemics sent by Picard, Villat, Levi-Civita to Duhem, and by Duhem to Villat.

[23] We refer to [24] for a more detailed analysis of Levi-Civita's school of hydrodynamics at the University of Rome. See also [38].

[24] The letter dated 2 March 1937 is contained in the von Kármán Papers, box 47, folder 3, California Institute of Technology. I thank Giovanni Battimelli for giving me the content of this letter.

[25] See, for instance, [40,41].

7. Appendix[26]

Henri Villat to Pierre Duhem
Montpellier, le 30 Octobre 1914

Monsieur

Voulez-vous me permettre de vous soumettre quelques réflexions que me suggère la lecture de votre très intéressante communication contenue dans les *Comptes-Rendus* du 19 octobre, et dont je prends connaissance à l'instant. Il ne saurait être question de mettre en doute le théorème que vous démontrez, mais il me semble que la conclusion finale en est un peu trop compréhensive. Même si l'on admet l'existence de discontinuités dans le fluide, le paradoxe de D'Alembert subsiste, mais à la condition essentielle que les surfaces de discontinuité ne s'étendent pas jusqu'à l'infini. Or j'ai justement démontré, incidemment, dans un théorème imprimé depuis plusieurs semaines aux *Annales* de Toulouse, et qui paraîtra sans doute bientôt, que l'hypothèse de l'existence de surfaces de discontinuité entraîne comme conséquence inéluctable que ces surfaces s'étendent à l'infini, et cela même si le fluide n'est pas illimité dans tous les sens. De cette circonstance il résulte que les intégrales de surface, relatives à une surface tracée toute entière à grande distance du solide immergé, ne s'annulent plus lorsque celle-ci disparaît à l'infini, et qu'on peut obtenir pour l'intégrale

$$\iint p \cos(n, x)\, \mathrm{d}S$$

étendue à la surface du solide, une expression quelconque non nulle. Le théorème des forces vives, en pareil cas, ne permet d'ailleurs de plus rien conclure, la force vive totale du fluide extérieur au solide, étant infinie (ce qui semble s'expliquer par le fait que le régime permanent demande pour s'établir un temps infini).

Dans les exemples qui ont pu être formés au moyen de la théorie des mouvements discontinus, par MM. Levi-Civita, Cisotti, et aussi par moi-même, ces circonstances se sont toujours rencontrées, et la pression totale, dans le sens de la vitesse du solide, sur la surface de celui-ci, n'y est pas nulle.

Par suite, ce qui doit résulter des considérations en question, ce serait, non pas l'impossibilité de l'établissement d'un régime permanent dans un fluide indéfini renfermant un solide, mais l'impossibilité d'admettre, à l'état permanent, la présence de surfaces de discontinuité dans un tel fluide, sans que ces surfaces s'étendent indéfiniment – fait qui peut se démontrer d'une façon tout à fait différente, comme je vous le disais plus haut. Il n'y avait donc pas lieu de condamner d'une façon absolue la théorie des mouvements discontinus, qui, au moins en première approximation, donne parfois des résultats satisfaisants.

Je suis extrêmement heureux si vous vouliez bien me communiquer votre avis sur les considérations qui précèdent, et qui sont relatives à un ordre de sujet qui, comme vous vous en doutez, est pour moi d'un intérêt tout particulier. Je profite de la circonstance pour vous exprimer les sentiments d'admiration, et au même temps de reconnaissance que la lecture de vos travaux m'a bien souvent inspirée, et je vous prie de croire, Monsieur, à l'assurance de mes sentiments les plus respectueux et les plus dévoués
H. Villat
104, avenue de Toulouse
Montpellier

Tullio Levi-Civita to Pierre Duhem
Modern Hôtel
Rome
le 31 Octobre 1914

Monsieur et illustre Collègue,

Au moment de rentrer à Padoue, je lis dans les *Comptes Rendus* du 19 de ce mois votre note sur le paradoxe de Dalembert.

Je m'empresse à vous faire part d'une remarque, assez petite si vous voulez, mais non sans importance pour [sic] qu'il s'agit de mettre ma recherche « Scie e leggi di resistenza »[27] à l'abri de votre dernière conclusion : « l'échappatoire des surfaces de discontinuité est désormais fermée ».

La remarque est tout simplement la suivante :

Si l'on admet que le sillage (*dead water* des angles [sic] lorsqu'il s'agit d'un solide immobile plongé dans un courant) ait une étendue finie, ou est bien conduit à votre résultat. Mais il n'en est plus de même si l'on admet avec moi (et avec tous les auteurs – par ex. Cisotti, Boggio, Brillouin, Villat – qui ont adopté mon point de vue) un sillage indéfini.

Vous le constatez à l'instant en réfléchissant que, d'après cette dernière hypothèse, une des conditions de l'énoncé au début de votre note « à l'infini le fluide est en repos » n'est plus satisfaite sic et simpliciter. Elle reste vérifiée, cela va sans

[26] The letters of Villat, Levi-Civita and Picard to Duhem are contained in the Fond Duhem, "Archives de l'Académie des sciences", Paris. The letters of Duhem to Villat are in Fond Villat, "Archives de l'Académie des sciences". We report the letters by following a chronological order. We thank the "Archives de l'Académie des sciences de Paris" and especially Isabelle Maurin-Joffre for permission to reproduce these letters.

[27] Rendiconti del Circolo Matematico di Palermo, T. XXIII, 1906. (Footnote by Levi-Civita.)

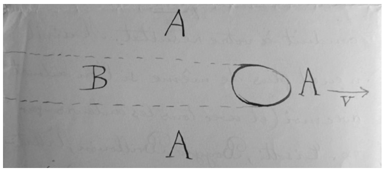

Fig. 3.

dire, pour la région A, mais elle ne l'est pas lorsqu'on s'éloigne indéfiniment du solide à l'intérieur du sillage B.[28] (Voir Fig. 3.)

Qu'il me soit permis de profiter de l'occasion pour exprimer chaleureusement mes vœux pour le triomphe de votre noble pays dans la lutte gigantesque où il se trouve à présent engagé en défense de l'humanité et de la justice.
Agréez mes hommages respectueux et mon souvenir toujours reconnaissant et cordial.
T. Levi-Civita

Pierre Duhem to Henri Villat
Bordeaux le 2 Nov. 1914

Mon cher Collègue,

Il va sans dire que ma conclusion est subordonnée à l'hypothèse que le repos du fluide à l'infini est tel que toutes les intégrales considérées s'annulent pour les surfaces ou volumes infiniment éloignés. S'il n'en est pas ainsi, mes conclusions tombent. On peut donc les regarder, si l'on veut, comme établissant qu'on ne saurait admettre un régime permanent où le repos à l'infini est de cette nature.

Je vous verrai avec un bien grand intérêt préciser dans une note aux C.R. cette conclusion que votre prochain mémoire permet de préciser. Ma conclusion, en effet, me jetais, pour les fluides incompressibles en particulier, dans un embarras dont je ne parvenais pas à me tirer et que vous allez, je crois, dissiper.
Croyez-moi, je vous prie, votre très dévoué,
P. Duhem

Émile Picard to Pierre Duhem
Paris, 3 Novembre 1914

Cher ami,

Je viens de lire dans les *Comptes Rendus* du 19 Oct. votre note sur le paradoxe de D'Alembert. À première vue, votre conclusion m'étonne un peu. Dans les exemples donnés (en particulier dans ces derniers temps sous l'influence d'un travail de M. Levi-Civita) les surfaces de discontinuités s'étendent à l'infini. Dans ces conditions votre raisonnement, où il pourrait y avoir des intégrales n'ayant pas de sens, est-il bien probant ?

Prenez ces remarques pour ce qu'elles valent. Je suis si inquiet de tant de choses en ce moment (deux fils et deux gendres au feu) que je n'ai pas la tête assez libre pour approfondir sérieusement quoi que ce soit.
Bien cordialement à vous,
Émile Picard

Pierre Duhem to Henri Villat
Bordeaux, le 10 Nov. 1914

Mon cher Collègue,

Peu après votre première lettre, j'en ai reçu une de M. Émile Picard et une autre de M. Levi-Civita ; toutes deux attiraient mon attention sur l'apparente contradiction entre ma conclusion et vos travaux. Cela m'a décidé à envoyer une note à l'Académie pour dissiper le malentendu. Je regrette maintenant de n'avoir pas attendu davantage, car votre note eût rendu la mienne inutile.

Quoiqu'il en soit, puisque vous remettez entre mes mains le sort de votre note, j'use de cette liberté pour l'envoyer à l'Académie. C'est la première fois que je fasse usage de mon droit de présenter des notes.

Je suis tout heureux que ces soient vos travaux, auxquels je porte un intérêt aussi vif que peu compétent, qui m'en fournissent l'occasion.
Croyez-moi, cher Collègue, à mes sentiments très dévoués,
P. Duhem

[28] Comparez, je vous en prie, la préface de mon mémoire cité. J'y montre que mon hypothèse n'est pas physiquement choquante comme représentation limite du mécanisme de la résistance. (Footnote by Levi-Civita.)

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