C. R. Mecanique 345 (2017) 903–907

Contents lists available at ScienceDirect

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A folded plate clamped along one side only ☆

Une plaque pliée encastrée d'un côté seulement

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ARTICLE INFO

Article history:
Received 23 June 2017
Accepted after revision 6 July 2017
Available online 19 October 2017

Keywords:
Folded thin elastic plate
Asymptotic model
Non-local transmission conditions

Mots-clés :
Plaque mince élastique pliée
Modèle asymptotique
Conditions de transmission non locales

ABSTRACT

An asymptotic model of a folded thin elastic plate is posed on two plane domains and contains transmission conditions at the common line segment of their boundaries. These conditions become non-local and inhomogeneous if only one side of the plate is fixed. Solvability and smoothness results and error estimates for the model are derived.

RÉSUMÉ

On considère un modèle asymptotique de plaque mince élastique pliée reposant sur deux domaines plans et mettant en jeu des conditions de transmission à l'interface entre les deux domaines. Ces conditions deviennent non locales et inhomogènes lorsqu'un seul bord de la plaque est encastré. On fournit des résultats concernant le caractère bien posé du problème, on établit des résultats de régularité des solutions, et on prouve des estimations d'erreur pour le problème modèle.

1. Formulation of the problem

Let

$$\Omega_j^h = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : y^j \in \Sigma_j, |z| < h/2\}, j = 2, 3 \tag{1}$$

be two elastic plates with rectangular mid-sections

$$\Sigma_j = \{y^j = (y_1^j, y_2^j) : y_1^j \in (0, \ell_1), y_2^j \in (0, \ell_j)\}, \ell_j > 0, j = 2, 3 \tag{2}$$

☆ The work is supported by the Russian Scientific Foundation, grant No. 17-11-01003.
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http://dx.doi.org/10.1016/j.crme.2017.07.003
1631-0721/

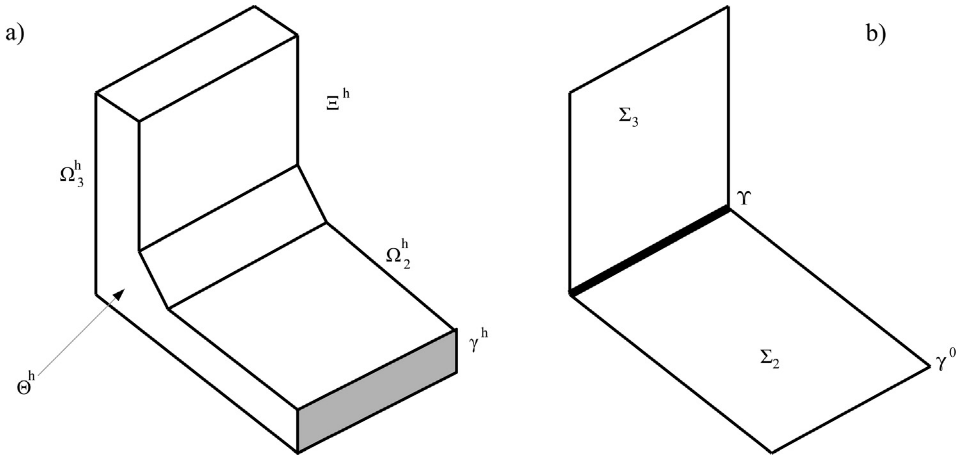

Fig. 1. The junction of plates (a) and its two-dimensional image (b).

where the local coordinates are given by $y^j = (x_1, x_j)$ and $z^2 = x_3, z^3 = x_2$. By rescaling, we set $\ell_1 = 1$ to make the coordinates and geometrical parameters dimensionless, in particular $h \ll 1$. The homogeneous isotropic elastic junction $\Xi^h = \Theta^h \cup \Omega_2^h \cup \Omega_3^h$ (Fig. 1a) consists of the plates (1) and the coupling bar

$$\Theta^h = \{x : x_1 \in (0, \ell_1), \xi = (\xi_2, \xi_3) = (h^{-1}x_2, h^{-1}x_3) \in \theta\} \tag{3}$$

where the domain $\theta \subset \mathbb{R}^2$ is bounded by a piecewise smooth contour $\partial\theta$ and includes the square $\{\xi : |\xi_j| < 1/2\}$ so that $\Omega_2^h \cap \Omega_3^h \subset \Theta^h$. The lateral side $\gamma^h = \{x : x_1 \in (0, \ell_1), x_2 = \ell_3, |x_3| < h/2\}$ (shaded in Fig. 1a) is clamped but the remaining part $\Gamma^h = \partial\Xi^h \setminus \gamma^h$ of the surface is traction-free. A volume force f^h is applied, e.g., due to gravity. The deformation of the solid Ξ^h is described by the boundary value problem

$$-\mu\Delta u^h - (\lambda + \mu)\nabla\nabla \cdot u^h = f^h \quad \text{in } \Xi^h \tag{4}$$

$$\sigma^{(n)}(u^h) = 0 \text{ on } \Gamma^h \setminus \Upsilon^h \tag{5}$$

$$u^h = 0 \text{ on } \gamma^h \tag{6}$$

where ∇ =grad, $\nabla\cdot$ =div and Δ is the Laplace operator. Moreover, $\lambda \geq 0$ and $\mu > 0$ are the Lamé constants, $u = (u_1, u_2, u_3)$ is the displacement vector, and stresses are determined as follows:

$$\begin{aligned} &\sigma_{pq} = \mu(\partial_p u_q + \partial_q u_p) + \delta_{p,q}\lambda(\partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3), \quad \partial_p = \partial/\partial x_p \\ &\sigma^{(n)} = (\sigma_1^{(n)}, \sigma_2^{(n)}, \sigma_3^{(n)}), \quad \sigma_p^{(n)} = n_1\sigma_{p1} + n_2\sigma_{p2} + n_3\sigma_{p3} \end{aligned} \tag{7}$$

Here, $\delta_{p,q}$ is the Kronecker symbol and $n = (n_1, n_2, n_3)$ is the unit vector of the outward normal defined everywhere on $\partial\Xi^h$, with the exception of the union Υ^h of closed edges on the surface.

The main goal of this paper is to present the asymptotics as $h \to +0$ of elastic fields in the folded plate Ξ^h clamped along the only lateral face γ^h.

2. Motivation

In the pioneering paper [1], the asymptotics of elastic fields in a cantilever, a junction of two perpendicular thin rods, is examined and the transmission conditions are derived at the joint point of two line segments substituting for the spacial rods in the model. The most intriguing observation in [1] is that the obtained one-dimensional models of the cantilevers with one or two clamped exterior ends differ from each other, namely they involve distinct groups of transmission conditions. Further effects of the interaction between elements of anisotropic thin elastic rod construction are found out in the paper [2], where the classification "fixed/movable" for rods and nodes in the junction is introduced and a procedure to detect its movable fragments is developed. The latter require a certain modification of asymptotic Ansätze for elastic fields and introduces new algebraic unknowns and orthogonality conditions for unknown functions on edges of the "skeleton" of the elastic junction. An adaptation of this procedure and asymptotic analysis of the spectral elasticity problem for various junctions of rods is given in [3,4].

Similar interaction effects can be readily predicted in thin-wall structures. This direction of mathematical studies is started with the paper [5], where a folded plate is considered, that is, a junction of two perpendicular thin straight plates Ω_2^h and Ω_3^h. The resultant transmission conditions at the common side Υ of the rectangles Σ_2 and Σ_3, i.e. the plate mid-sections, are derived under the assumption that both plates are fixed along the lateral sides, which are parallel to the axis Υ of the parallelepiped $\Theta^{\rm h} = \Omega_2^{\rm h} \cap \Omega_3^{\rm h}$. In the present paper, we also deal with such type of the plate junction in Fig. 1a, and describe its two-dimensional model with a certain group of transmission conditions at the segment Υ in Fig. 1b, which differs from the one in [5] because only one plate gets a clamped side in the problem (4)–(6). It is remarkable that the principle force F and torque M, (20), applied to the attached plate $\Omega_3^{\rm h}$, enter the transmission conditions and in some sense reflect a far interaction of the junction elements.

We selected in Section 1 the simplest geometry and physical properties for $\Xi^{\rm h}$ in order to reduce the technicality to the necessary minimum. Available generalizations are discussed in Section 7.

The mathematical field of elastic junctions is much bigger than the two specific creatures mentioned above, but in the short note we are not able to provide an exhaustive description of solved and open problems as well as of the existing literature.

3. The Korn inequality

For the plate $\Omega_3^{\rm h}$ with the clamped side $\gamma^{\rm h}$ and the attached rod $\Theta^{\rm h}$, the following inequality is valid:

$$|||u^{\rm h}; \Omega_2^{\rm h} \cup \Theta^{\rm h}|||_h \le c D(u^{\rm h}; \Omega_2^{\rm h} \cup \Theta^{\rm h}) \tag{8}$$

see the original paper [6] and, e.g., the monographs [7,8]. Here, $D(u^{\rm h}; \Xi^{\rm h})$ is the ersatz of elastic energy

$$D(u^{\rm h}; \Xi^{\rm h}) = \sum_{p,q}^{3} \int_{\Xi^{\rm h}} |\varepsilon_{pq}(u)|^2 \, {\rm d}x, \quad \varepsilon_{pq}(u) = \frac{1}{2}\left(\frac{\partial u_p}{\partial x_q} + \frac{\partial u_p}{\partial x_q}\right) \tag{9}$$

and the anisotropic Sobolev norm $|||u^{\rm h}; \Omega_2|||_h$ looks as follows:

$$\begin{aligned}\Big(\int_{\Omega_2} \Big(\sum_{p=1}^{2} \Big(\Big|\frac{\partial u_p^{\rm h}}{\partial x_p}\Big|^2 + h^2\Big(\Big|\frac{\partial u_p^{\rm h}}{\partial x_3}\Big|^2 + \Big|\frac{\partial u_3^{\rm h}}{\partial x_p}\Big|^2 + |u_p^{\rm h}|^2\Big)\Big) \\ + \Big|\frac{\partial u_1^{\rm h}}{\partial x_2}\Big|^2 + \Big|\frac{\partial u_2^{\rm h}}{\partial x_1}\Big|^2 + \Big|\frac{\partial u_3^{\rm h}}{\partial x_3}\Big|^2 + h^2|u_3^{\rm h}|^2\Big) \, {\rm d}x\Big)^{1/2}\end{aligned} \tag{10}$$

To formulate the Korn inequality in the plate $\Omega_3^{\rm h}$, which was proved in [3,4], we set

$$u^{\rm h}(x) = \mathbf{u}^{\rm h}(x) + a_3^{\rm h}\varphi^3 + a_0^{\rm h}\varphi^0(x), \ (\mathbf{u}^{\rm h}, \varphi^m)_{\Omega_3^{\rm h}} = 0, m = 0, 3, \varphi^3 = e_{(3)}, \varphi^0(x) = e_{(3)}x_1 - e_{(1)}x_3 \tag{11}$$

where $e_{(p)}$ is the unit vector of the x_p-axis and $(\ ,\ )_\Omega$ is the scalar product in the Lebesgue space $L^2(\Omega)$. Let the norm $|||u^{\rm h}; \Omega_3^{\rm h}|||$ be defined by (10) with the replacement $2 \leftrightarrow 3$. Then

$$|||\mathbf{u}^{\rm h}; \Omega_3^{\rm h}|||_h^2 + h^{-1}|a_0^{\rm h}|^2 + h^{-1}|a_3^{\rm h}|^2 \le C D(u^{\rm h}; \Xi^{\rm h}) \tag{12}$$

The factors c and C in (8) and (12) are independent of $u^{\rm h}$ and $h \in (0, 1]$. These relations furnish the anisotropic weighted Korn inequality for the whole junction $\Xi^{\rm h}$, the asymptotic accuracy of which is verified in [3,4].

4. The asymptotic Ansätze

The well-known procedure of dimension reduction proposes the following asymptotic forms for the restrictions $u_{(j)}^{\rm h}$ on $\Omega_j^{\rm h}$ of the displacement field $u^{\rm h}$:

$$u_{(j)}^{\rm h}(x) = h^{-2}U_{(j)}^{-2}(y^j) + h^{-1}U_{(j)}^{-1}(y^j, \zeta_j) + h^0 U_{(j)}^0(y^j, \zeta_j) + \dots \tag{13}$$

where $\zeta_j = h^{-1}z^j$ is the stretched transversal coordinates, cf. (1), and $U_{(j)}^m$ is a vector function on Ω_j^1. The main asymptotic terms are different in the fixed ($j = 2$) and attached ($j = 3$) plates

$$U_{(2)}^{-2}(y^2) = e_{(3)}^2 w_3^2(y^2), \quad U_{(3)}^{-2}(y^3) = e_{(3)}^3 w_3^3(y^3) + e_{(2)}^3(t + y_1^3 r) - e_{(1)}^3 y_2^3 r \tag{14}$$

but the second and the third terms look quite similar:

$$\begin{aligned} U_{(j)}^{-1}(y^j, \zeta^j) &= e_{(1)}^j\Big(w_j^1(y^j) - \zeta_j \frac{\partial w_3^j}{\partial y_1^j}(y^j)\Big) + e_{(2)}^j\Big(w_j^2(y^j) - \zeta_j \frac{\partial w_3^j}{\partial j_1^j}(y^j)\Big) \\ U_{(j)}^0(y^j, \zeta_j) &= \frac{\mu}{2}\lambda_\bullet e_{(3)}^j\Big(\Big(\frac{\zeta_j^2}{2} - \frac{1}{24}\Big)\Delta_y w_3^j(y^j) - \zeta_j \nabla_{y^j} \cdot w_\bullet^j(y^j)\Big), \quad \lambda_\bullet = \frac{2\lambda\mu}{\lambda + 2\mu} \end{aligned} \tag{15}$$

Here, $w^j = (w^j_\bullet, w^j_3) = (w^j_1, w^j_2, w^j_3)$ and t, r are unknown vector functions and numbers while $e^j_{(0)}$ and $e^j_{(3)}$ are the unit vectors of the y^j_p- and z^j-axes. A reason to introduce the algebraic unknowns t and r into the Ansatz (14) and to impose the orthogonality conditions

$$I^t(w^3_2) := (w^3_2, 1)_{\Sigma^3} = 0, \quad I^r(w^3_2) := (w^3_2, y^2_1)_{\Sigma^3} - (w^3_1, y^2_2)_{\Sigma^2} = 0 \tag{16}$$

will become clear in the next sections.

To extend the asymptotic expansion (13) by the terms $hU^1_{(j)}$ and $hU^2_{(j)}$, one needs to impose the differential equations for the vector function $w^j = (w^j_\bullet, w^j_3)$

$$-\mu \Delta_y w^j_\bullet - (\lambda_\bullet + \mu)\nabla_y \nabla_y \cdot w^j_\bullet = g^j_\bullet \text{ in } \Sigma^j \tag{17}$$

$$\frac{\mu}{3}\frac{\lambda+\mu}{\lambda+2\mu}\Delta^2_y w^j_3 = g^j_3 \text{ in } \Sigma^j \tag{18}$$

where the right-hand sides $g^j_\bullet = 0$ and $g^j_3 = f_{(j)} \cdot e^j_{(3)}$ are taken from the representations of the force

$$f^{\mathrm{h}}(x) = f_{(j)}(y^j) + \widetilde{f}^{\mathrm{h}}_{(j)}(x) \text{ in } \Omega^{\mathrm{h}}_j, \ j = 2, 3 \tag{19}$$

We have introduced $g^j_\bullet$ into (17) and $\widetilde{f}^{\mathrm{h}}_{(j)}$ into (19) for particular purposes in Section 5. Clearly, the longitudinal projections $f_{(j)} \cdot e^j_{(p)}$, $p = 1, 2$, do not affect the system (17), (18). We however will use the principle force applied to Ω^{h}_3 in the x_3-direction and the principal torque about the x_2-axis, namely,

$$F = \int_{\Sigma^j} f_{(j)}(y) \cdot e^j_{(2)} \mathrm{d}y, \quad M = \int_{\Sigma^j} \left(y^3_1 f_{(j)}(y) \cdot e^j_{(2)} - y^3_2 f_{(j)}(y) \cdot e^j_{(1)}\right) \mathrm{d}y \tag{20}$$

A simple example of the force (19) is the gravity force $f(x) = f^0 e^0$ where $f^0 \in \mathbb{R}$ and e^0 is a unit vector. Changing the position of the junction Ξ^{h} with respect to e^0 yields various values of F and M in (20).

5. Boundary layer and transmission conditions

The Dirichlet conditions

$$w^j = 0, \quad \partial_2 w_3 = 0 \text{ at } \gamma^0 = \{x : x_1 \in (0, \ell_1), x_2 = \ell_2\} \tag{21}$$

are associated with the clamped side of the plate Ω^{h}_2. There exist quite many approaches to establish the following boundary conditions corresponding to the traction-free lateral sides:

$$s^{(\nu)}(w^j_\bullet) = 0, \quad S_k w^j_3 = 0 \text{ on } \partial\Sigma^j \setminus (\Upsilon \cup \gamma^0) \quad j, k = 2, 3 \tag{22}$$

Here, $\gamma^0 = \{x : x_1 \in (0, \ell_1), x_2 = \ell_2, x_3 = 0\}$, $\Upsilon = \{x : x_1 \in (0, \ell_1), x_2 = x_3 = 0\}$, $\nu = (\nu_1, \nu_2)$ is the unit vector of the outward normal, $\tau = (-\nu_2, \nu_1)$ is a tangent vector on $\partial\Sigma^j$, $s^{(\nu)}_p = \nu_1 s^j_{p1} + \nu_2 s^j_{p2}$,

$$\begin{gathered} s^j_{pq}(w^j_\bullet) = \mu(\partial_p w^j_q + \partial_q w^j_p) + \delta_{p,q}\lambda_\bullet(\partial_1 w^j_1 + \partial_2 w^j_2), \quad \partial_p = \partial/\partial y^j_p \\ S_3 = \frac{\mu}{3}\frac{\lambda+\mu}{\lambda+2\mu}\frac{\partial}{\partial\nu}\left(\frac{\partial^2}{\partial\nu^2} + \frac{3\lambda+4\mu}{2(\lambda+\mu)}\frac{\partial^2}{\partial\tau^2}\right), \quad S_2 = \frac{\mu}{3}\frac{\lambda+\mu}{\lambda+2\mu}\left(\frac{\partial^2}{\partial\nu^2} + \frac{\lambda}{2(\lambda+\mu)}\frac{\partial^2}{\partial\tau^2}\right) \end{gathered} \tag{23}$$

We refer, e.g., to the papers [9,10], where formulas (21) and (22) are derived on the basis of the boundary layer phenomena, see also [11, Ch. 16] for general elliptic problems. A similar phenomenon occurs in the vicinity of the coupling bar Θ^{h} and is described by means of solutions to the elasticity problem in the union $\theta \cup \varpi_2 \cup \varpi_3$ of the core θ and two semi-infinite strips $\varpi_j = \{\xi \in \mathbb{R}^2 : \xi_j > 0, |\xi_{5-j}| < 1/2\}$ written in the stretched coordinates ξ from (3). Solvability results and decay properties of solutions in the latter problem are described in [12, §6] with the help of algebraic tools, so that providing the exponential decay of the boundary layer term yields the family of stable transmission conditions

$$w^2_1 = w^1_3, \quad w^2_3 = t + x_1 r, \, w^3_3 = 0, \quad \partial w^2_3/\partial y^2_2 = -\partial w^3_3/\partial y^3_2 \text{ on } \Upsilon \tag{24}$$

and the family of intrinsic transmission conditions

$$s^2_{12}(w^2) = -s^3_{12}(w^3), \quad s^2_{22}(w^2) = 0, \quad s^3_{22}(w^3) = 0, \quad S_2(w^2) = S_2(w^3) \text{ on } \Upsilon \tag{25}$$

The constants t and r come into (24) from the asymptotic ansatz (13)–(15) and their arbitrariness is taken into account by imposing the integral conditions with the data (20), namely

$$\int_0^{\ell_1} S_3(\nabla_y) w_3^2(y_1^2, 0)\,\mathrm{d}y_1^2 = F, \quad \int_0^{\ell_1} y_2 S_3(\nabla_y) w_3^2(y_1^2, 0)\,\mathrm{d}y_1^2 = M \tag{26}$$

6. The variational model of the folded plates

The presented transmission conditions, in particular, imply the continuity of the bending angle and momentum about the line segment as well as of the longitudinal displacement and force at Υ. The bend w_3^3 vanishes on Υ, but the bend w_2^3 on Υ is equal to some rigid motion of Σ^1 as was indicated in (14). In this way, the variational formulation of the problem (17), (18), (21), (22) and (24)–(26) reads:

$$\sum_{j=2,3} E(w^j, v^j; \Sigma^j) = \sum_{j=2,3} (g^j, v^j)_\Sigma + Ft(v^2) + Mr(v^2) \quad \forall (v^1, v^2) \in \mathcal{H} \tag{27}$$

The Hilbert space $\mathcal{H}$ consists of vector functions $w^j \in H^1(\Sigma^j)^2 \times H^2(\Sigma^j)$ that satisfy the orthogonality conditions (16) and the stable conditions (21), (24) with any numbers $t = t(w^2), r = r(w^2)$. This problem has a unique solution for arbitrary data $F, M \in \mathbb{R}$ and $g^j \in L^2(\Sigma^j)^3$ under the compatibility conditions $I^t(g_2^3) = 0$, $I^r(g_2^3) = 0$. If $g_\bullet^j \in H^1(\Sigma^j)^2$, the weak solution $(w^1, w^2) \in \mathcal{H}$ belongs to the smaller space $H^{2+\delta}(\Sigma^j)^2 \times H^{3+\delta}(\Sigma^j)$ with some $\delta > 0$. The additional smoothness property allows us to prove the error estimate for the model with the bound $ch^{\min\{1,\delta+1/2\}}$ in the norms on the left of (8) and (12). This estimate involves cumbersome structures, and we here present only strong convergence results:

$$h^2 u_{(2)}^{\mathrm{h}}(y^2, h\zeta^2) \quad \to \quad (0, 0, w_3^2(y^2)) \text{ in } L^2(\Omega_2^1)$$

$$h^2 u_{(3)}^{\mathrm{h}}(y^3, h\zeta^3) \quad \to \quad (-y_2^3 r(w^3), t(w^3) - y_1^3 r(w^3), w_3^3(y^3)) \text{ in } L^2(\Omega_3^1)$$

7. Final remarks

We derive the transmission conditions by means of constructing the boundary layers, which requires additional smoothness properties of the solution to the limit problem (27). These properties are verified in the case of the perpendicular isotropic plates Ω_2^h and Ω_3^h. However, for anisotropic plates joined at an arbitrary angle $\alpha \in (0, \pi)$, the solution singularities at the endpoints of the segment Υ are not known yet, although the described asymptotic procedure provides a limit problem of types (17), (18), (21), (22), (24), (25). The inconvenient singularities can be compensated by the three-dimensional boundary layers near the corners of the plates (1), like it has been done in the papers [13,14].

Other configurations of thin plates in elastic junctions did not appear yet in the literature on asymptotic analyses, although asymptotically sharp anisotropic weighted Korn inequalities for elastic plate junctions are presented in [3,4].

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