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On some features of the effective behaviour of porous solids with J_2- and J_3-dependent yielding matrix behaviour

Sur quelques effets de l'angle de Lode sur les caractéristiques macroscopiques de matériaux en rupture ductile

Ahmed Benallal

LMT, ENS Paris-Saclay/CNRS/Université Paris-Saclay, France

A R T I C L E   I N F O

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A B S T R A C T

Some features od the constitutive behaviour of voided materials taking into account possible effects of the Lode angle in the yielding behaviour of the matrix are discussed. The Gurson approach is used to this end. After providing a parametric representation of the effective behaviour of such materials, some closed-form results are given for pure shear stress states and also at very high stress triaxialities. In the former case corresponding to a zero macroscopic mean stress, the contour of the yield domain in the π-plane has exactly the shape of the yield surface of the matrix in the deviatoric plane, but a size reduced by a factor $1 - f$, with f the porosity of the voided material. In the latter, effective yield stresses for the voided material are slightly different from the Gurson result and found to be set by the yield stress at a microscopic stress Lode angle $\frac{\pi}{3}$ for very high positive triaxiality and by the yield stress at a microscopic stress Lode angle 0 for very high negative triaxiality. This last result is extended for porous materials with yielding depending further on the hydrostatic stress, fully exhibiting the interaction between volumetric and shear interactions on the yielding behaviour of isotropic porous materials. Applications to many usual yielding criteria for the matrix are also provided.

R É S U M É

Les effets du troisième invariant des contraintes sur la surface de charge macroscopique d'un matériau ductile poreux sont analysés dans le cadre de l'approche de Gurson. Ces effets proviennent du processus d'homogénéisation lui-même à cause de l'hétérogénéité des contraintes dans la cellule de Gurson ou encore lorsque le comportement plastique de la matrice dépend du troisième invariant des contraintes. On fournit une représentation paramétrique de la surface de charge valable pour un comportement de la matrice assez général qui permet d'exhiber quelques résultats analytiques, en particulier pour les états de contrainte hydrostatiques et pour des états de cisaillement. Les résultats obtenus pour

E-mail address: benallal@lmt.ens-cachan.fr.

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les chargements hydrostatiques sont étendus au cas où la matrice a un comportement dépendant aussi de la contrainte hydrostatique.

1. Introduction

The paper is concerned with consequences of possible Lode angle effects on the effective properties of plastic porous solids. This is undertaken in the framework of the Gurson approach. Effects of the Lode angle appear at two different levels. Beside the fact that the matrix behaviour can be dependent on the Lode angle, the latter also enters the homogenization process as the stress state in the representative volume cell is heterogeneous. In a recent paper, the author and co-workers [1] assessed the effects of the third stress invariant in the yielding of ductile porous solids arising from the latter effect by considering a von Mises yielding behaviour for the matrix. This was done by simply avoiding the approximation used by Gurson [2] and considering the full expression of the microscopic dissipation. Both effects are considered in this paper, the subject of which is the derivation of macroscopic constitutive equations for voided materials with a matrix yielding behaviour dependent on both the second and third stress invariants.

Many practical situations call for the consistent introduction of the Lode angle (either in plasticity or fracture). Failure under low or negative triaxialities (McClintock [3], Johnson and Cook [4], Bao and Wierzbicki [5], Barsoum and Fakeslog [6]) are possible situations. Shear-dominated stress states such as plugging failure in projectile penetration are other examples [7], and many others can be found in the above references. Nahshon and Hutchinson [8], for instance, amended the Gurson model in a phenomenological way by making the evolution of the porosity also dependent on the third invariant of the stress. The Lode angle effects have also been included and studied in a consistent way by Danas and Ponte Castaneda [9], see also [10], in an alternative approach based on second-order variational homogenization techniques. Other results on the inclusion of Lode angle effects can also be found in [11] and [12].

2. Constitutive relations for the matrix

2.1. Notations

We use the following notations. $\boldsymbol{\sigma}$ and $\dot{\boldsymbol{\epsilon}}$ denote the microscopic stress and the strain rate in the matrix while the macroscopic stress and strain rate are called respectively $\boldsymbol{\Sigma}$ and $\dot{\mathbf{E}}$. The invariants of the microscopic stress tensor are the hydrostatic stress σ_{m}, the von Mises equivalent stress σ_{eq} and the stress Lode angle ω respectively defined by

$$\sigma_{\mathrm{m}} = \frac{1}{3}\sigma_{ii}, \quad \sigma_{\mathrm{eq}} = \sqrt{\frac{3}{2}s_{ij}s_{ij}} \quad \text{and } \omega = \frac{1}{3}\arccos\left(\frac{27}{2}\frac{\det \mathbf{s}}{\sigma_{\mathrm{eq}}^3}\right) \tag{1}$$

where $\mathbf{s}$ is the stress deviator and repeated summation is used. The invariants for the macroscopic stress are defined exactly in the same way and denoted by Σ_{m}, Σ_{eq} and Θ. We also use the invariants of the microscopic strain rate tensor $\dot{\boldsymbol{\epsilon}}$ defined similarly by

$$\dot{\epsilon}_{\mathrm{m}} = \frac{1}{3}\dot{\epsilon}_{ii}, \quad \dot{\epsilon}_{\mathrm{eq}} = \sqrt{\frac{2}{3}\dot{\epsilon}_{ij}\dot{\epsilon}_{ij}} \quad \text{and } \zeta = \frac{1}{3}\arccos\left(\frac{4\det \dot{\boldsymbol{\epsilon}}}{\dot{\epsilon}_{\mathrm{eq}}^3}\right) \tag{2}$$

and those of the macroscopic strain rate $\dot{\mathbf{E}}$ denoted $\dot{E}_{\mathrm{m}}$, $\dot{E}_{\mathrm{eq}}$ and η. We have then

$$\boldsymbol{\Sigma} = \Sigma_{\mathrm{m}}\mathbf{1} + \frac{2}{3}\Sigma_{\mathrm{eq}}\begin{pmatrix} \cos\Theta & 0 & 0 \\ 0 & \cos(\Theta - \frac{2\pi}{3}) & 0 \\ 0 & 0 & \cos(\Theta + \frac{2\pi}{3}) \end{pmatrix} \quad \text{and}$$
$$\dot{\mathbf{E}} = \dot{\mathbf{E}}_{\mathrm{m}}\mathbf{1} + \dot{E}_{\mathrm{eq}}\begin{pmatrix} \cos\eta & 0 & 0 \\ 0 & \cos(\eta - \frac{2\pi}{3}) & 0 \\ 0 & 0 & \cos(\eta + \frac{2\pi}{3}) \end{pmatrix} \tag{3}$$

and exactly the same forms for $\boldsymbol{\sigma}$ and $\dot{\boldsymbol{\epsilon}}$.

2.2. Yield function and flow rule

In all the paper, the constitutive behaviour of the matrix of the porous solid is considered as incompressible, isotropic, and rigid-plastic. This section makes it more precise. The yielding of the matrix is described by a yield function f positive and homogeneous of degree one in the stress of the form

$$f(\boldsymbol{\sigma}) - \sigma_0 = \sigma_{\text{eq}} g(\omega) - \sigma_0 \leq 0 \tag{4}$$

where ω is the Lode angle of the microscopic stress tensor defined in (1). The function $g(\omega)$, which describes possible effects of the third invariant of stress on yielding can be normalized in a number of ways (for instance by $g(0) = 1$, in which case σ_0 is the yield limit in uniaxial tension). An important requirement for the yield surface is its convexity. Relation (4) represents a cylindrical surface, the axis of which is the hydrostatic axis in the principal stress frame. Therefore its convexity is governed by the convexity of its cross section in the deviatoric plane as this section is constant along the hydrostatic stress. In a polar reference system in this deviatoric plane, the equation of the yield surface is $\sigma_{\text{eq}}(\omega) = \frac{\sigma_0}{g(\omega)} = \rho(\omega)$. The convexity condition is therefore $\rho^2 + 2(\rho')^2 - \rho\rho'' \geq 0$, and consequently g must satisfy the following condition

$$g(\omega) + g''(\omega) \geq 0 \tag{5}$$

The yield surface is also considered smooth in all the paper for the sake of simplicity. But, as will be seen later, non-smooth usual yield surfaces can be considered by a limiting process. Alternatively, if wished, the non-smoothness can be directly included at the expense of more complicated calculations. With this assumption at hand, using normality as the flow rule, the strain rate is obtained by

$$\dot{\boldsymbol{\epsilon}} = \lambda \frac{\partial f}{\partial \boldsymbol{\sigma}} \tag{6}$$

where

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} = \frac{3}{2} g(\omega) \frac{\mathbf{s}}{\sigma_{\text{eq}}} + \sigma_{\text{eq}} g'(\omega) \frac{\partial \omega}{\partial \boldsymbol{\sigma}} \tag{7}$$

and the gradient of the Lode angle with respect to the stress is given by

$$\frac{\partial \omega}{\partial \boldsymbol{\sigma}} = -\frac{9}{2\sigma_{\text{eq}} \sin 3\omega} \left[\left(\frac{\mathbf{s}.\mathbf{s}}{\sigma_{\text{eq}}^2} - \frac{2}{9}\mathbf{1} \right) - \frac{1}{3} \cos 3\omega \frac{\mathbf{s}}{\sigma_{\text{eq}}} \right] \tag{8}$$

where $\mathbf{1}$ is the second-order unit tensor. We note that the gradient $\frac{\partial \omega}{\partial \boldsymbol{\sigma}}$ as given by (8) is always singular at $\omega = 0$ and $\omega = \frac{\pi}{3}$ (axisymmetric states of stress) so that the gradient of the yield function with respect to the stress is so unless (see (7))

$$g'(0) = g'(\frac{\pi}{3}) = 0 \tag{9}$$

conditions that we will assume throughout the paper. When these conditions are not met, the yield surface has a vertex (or an edge) at these two locations. Due to the isotropy of the matrix behaviour, the yield surface has either a six- or three-fold symmetry, and therefore one can limit the range of ω to $[0, \frac{\pi}{3}]$. To guarantee the smoothness of the yield surface, we assume, beside conditions (9), that the function g is itself continuously differentiable in the whole range $[0, \frac{\pi}{3}]$. Consideration of non-smooth yielding behaviour can be handled directly at the expense of more complex developments, but can also be studied by a limiting process from our results.

2.3. Two examples

Other examples can be considered (see, e.g., [13]); we will illustrate our forthcoming results through two examples. These are:

– the Hosford–Dalgren–Hershey [14,15] yield surface, currently used for a better description of yielding and forming of aluminium alloys. This corresponds, with $m \geq 1$ and with σ_i, $i = 1, 2, 3$, denoting the ordered microscopic principal stresses, to the yield function

$$f(\boldsymbol{\sigma}) = \left[\frac{1}{2} \{(\sigma_1 - \sigma_2)^m + (\sigma_2 - \sigma_3)^m + (\sigma_1 - \sigma_3)^m\} \right]^{\frac{1}{m}} = \sigma_{\text{eq}} g(\omega) \tag{10}$$

which can be written it in the form (4) with the choice

$$g(\omega) = \frac{2}{3} \left[\frac{1}{2} \{(\cos\omega - \cos(\omega - \frac{2\pi}{3}))^m + (\cos(\omega - \frac{2\pi}{3}) - \cos(\omega + \frac{2\pi}{3}))^m + (\cos\omega - \cos(\omega + \frac{2\pi}{3}))^m\} \right]^{\frac{1}{m}} \tag{11}$$

Table 1
Representation of the usual yield criteria by the yield function f given in (13) and (14).

Criterion	k	χ	α	β
von Mises	σ_0	0	0	$\frac{\pi}{6}$
Tresca	σ_0	1	0	$\frac{\pi}{6}$
Drucker–Prager	0	0	0	0
Mohr–Coulomb	$\frac{\sqrt{2}c\cos\psi}{\sqrt{1+3\sin\psi}}$	1	$\frac{\sqrt{2}\sin\psi}{\sqrt{1+3\sin\psi}}$	$\arctan[\sqrt{3}\frac{1-\sin\psi}{3+\sin\psi}]$

It is easily checked now that as required, conditions (9) are satisfied for all finite $m \neq 1$. Moreover, $m = 2$ and $m = 4$ correspond to the von Mises criterion, while the Tresca yield function is obtained in the limiting case $m \to \infty$ or $m \to 1$. Indeed, for this last case corresponding to the Tresca criterion, one gets

$$g(\omega) = \frac{2}{3}(\cos\omega - \cos(\omega - \frac{2\pi}{3})) = \cos(\omega - \frac{\pi}{6}) \tag{12}$$

so that $g'(0) = \frac{1}{2}$ and $g'(\frac{\pi}{3}) = -\frac{1}{2}$ and therefore violating conditions (9);

- the second one (see, e.g., [16]), which is only used at the end of the paper, has the merit of unifying in one simple expression most of the usual yield criteria and allows us to include effects of pressure sensitivity on yielding. The analysis is however limited to pure shear and pure hydrostatic loadings, general loadings for this type of materials being much more complicated and out of the scope of this paper. This corresponds to the following yield function

$$f(\boldsymbol{\sigma}) = \sigma_{\rm eq}(\omega) + 3\,\alpha\,\sigma_{\rm m} \leq k \tag{13}$$

with

$$g(\omega) = \cos\left[\frac{1}{3}\arccos[\chi\cos(3\omega)] - \beta\right] \tag{14}$$

where α, χ and β are constant parameters allowing us to obtain the main usual criteria as shown in Table 1. In this Table, for the von Mises and Tresca criterion, σ_0 is the yield stress in uniaxial tension, while for the Drucker–Prager and Mohr–Coulomb criteria, c and ψ are respectively the cohesion intercept and the internal friction angle.

Here also, one can check for conditions (9). First-order expansions of $g'(\omega)$ in terms of ω around $\omega = 0$ and $\omega = \frac{\pi}{3}$ give, respectively,

$$g'(\omega) = \frac{3\chi\omega\sin\left(\beta - \frac{1}{3}\arccos(\chi)\right)}{\sqrt{1-\chi^2}} + O\left(\omega^3\right) \tag{15}$$

$$g'(\omega) = \frac{3\chi\left(\omega - \frac{\pi}{3}\right)\cos\left(\beta + \frac{1}{3}\arccos(\chi) + \frac{\pi}{6}\right)}{\sqrt{1-\chi^2}} + O\left(\left(\omega - \frac{\pi}{3}\right)^3\right) \tag{16}$$

from which it is clearly seen that, for all situations where $\chi \neq 1$, we have $g'(0) = g'(\frac{\pi}{3}) = 0$ and the yield surface given by (13) and (14) is smooth. Remains the case $\chi = 1$. A first-order expansion in terms of χ around $\chi = 1$ reads

$$g'(\omega) = \sin(\beta - \omega) + (\chi - 1)\left(\frac{1}{3}\cot(3\omega)\cos(\beta - \omega) + \csc^2(3\omega)\sin(\beta - \omega)\right) + O\left((\chi - 1)^2\right) \tag{17}$$

Expansions of the zeroth and first-order terms in (17) in terms of ω around $\omega = 0$ and $\omega = \frac{\pi}{3}$ are obtained respectively as

$$\frac{(\chi - 1)\sin(\beta)}{9\omega^2} + \frac{1}{18}(7\chi + 11)\sin(\beta) - \frac{1}{27}\omega((19\chi + 8)\cos(\beta)) + O\left(\omega^2\right) \tag{18}$$

$$-\frac{(\chi - 1)\cos\left(\beta + \frac{\pi}{6}\right)}{9\left(\omega - \frac{\pi}{3}\right)^2} - \frac{1}{18}(7\chi + 11)\cos\left(\beta + \frac{\pi}{6}\right) - \frac{1}{27}\left(\omega - \frac{\pi}{3}\right)\left((19\chi + 8)\sin\left(\beta + \frac{\pi}{6}\right)\right)$$
$$+O\left(\left(\omega - \frac{\pi}{3}\right)^2\right) \tag{19}$$

and one concludes that when $\chi = 1$, we have

$$g'(0) = \sin\beta \quad \text{and } g'(\frac{\pi}{3}) = -\cos(\beta + \frac{\pi}{6}) \tag{20}$$

For the Tresca yield function that corresponds to $\chi = 1$ and $\beta = \frac{\pi}{6}$ (see Table 1), we recover the results obtained with the Hosford yield expression described above, namely that $g'(0) = \sin\frac{\pi}{6} = \frac{1}{2}$ and $g'(\frac{\pi}{3}) = -\cos(\frac{\pi}{3}) = -\frac{1}{2}$.

For the Mohr–Coulomb criterion corresponding to $\chi = 1$ and $\beta = \arctan[\sqrt{3}\frac{1-\sin\psi}{3+\sin\psi}]$, we have

$$g'(0) = \sin\left(\arctan[\sqrt{3}\frac{1-\sin\psi}{3+\sin\psi}]\right) \quad \text{and } g'(\frac{\pi}{3}) = -\cos\left(\arctan[\sqrt{3}\frac{1-\sin\psi}{3+\sin\psi}] + \frac{\pi}{6}\right) \tag{21}$$

2.4. The dissipation function $\pi(\dot{\boldsymbol{\epsilon}})$

The maximum dissipation function, which we denote by π, is important in the derivation of the effective behaviour, as we will see in the next section. The maximum dissipation is defined by

$$\pi(\dot{\epsilon}) = \sup_{\sigma\in C} \sigma : \dot{\boldsymbol{\epsilon}} \tag{22}$$

where $C = \{\sigma / f(\sigma) \le 0\}$ due to the convexity of the yield function f, the function $\pi(\dot{\boldsymbol{\epsilon}})$ is obtained for every $\dot{\boldsymbol{\epsilon}}$

$$\pi(\dot{\boldsymbol{\epsilon}}) = \begin{cases} \pi_g(\dot{\boldsymbol{\epsilon}}) & \text{if} \quad Tr(\dot{\boldsymbol{\epsilon}}) = 0 \\ +\infty & \text{if} \quad Tr(\dot{\boldsymbol{\epsilon}}) \ne 0 \end{cases} \tag{23}$$

$\pi_g(\dot{\boldsymbol{\epsilon}})$ is obtained by seeking all stress states σ^* lying on the yield surface, i.e. $f(\sigma) = \sigma_0$ and satisfying

$$\dot{\boldsymbol{\epsilon}} = \lambda\frac{\partial f}{\partial\sigma}(\sigma^*) \tag{24}$$

for some positive scalar λ. This leads to

$$\pi(\dot{\boldsymbol{\epsilon}}) = \boldsymbol{\sigma}^* : \dot{\boldsymbol{\epsilon}} \tag{25}$$

For such stress states, we have, using the fact that f is homogeneous of degree one,

$$\boldsymbol{\sigma}^* : \dot{\boldsymbol{\epsilon}} = \lambda\boldsymbol{\sigma}^* : \frac{\partial f}{\partial\boldsymbol{\sigma}} = \lambda f(\boldsymbol{\sigma}^*) = \lambda\sigma_0 \tag{26}$$

The plastic multiplier λ is obtained from the flow rule (6) by

$$\lambda = \frac{\dot{\epsilon}_{\text{eq}}}{\sqrt{\frac{2}{3}\frac{\partial f}{\partial\boldsymbol{\sigma}} : \frac{\partial f}{\partial\boldsymbol{\sigma}}}} \tag{27}$$

To express it fully in terms of the strain rate $\dot{\boldsymbol{\epsilon}}$, the denominator in (27) is written, using relations (7) and (8),

$$\sqrt{\frac{2}{3}\frac{\partial f}{\partial\boldsymbol{\sigma}} : \frac{\partial f}{\partial\boldsymbol{\sigma}}} = \sqrt{g^2(\omega) + (g'(\omega))^2} \tag{28}$$

where we recall that ω is the Lode angle of the microscopic stress $\boldsymbol{\sigma}$. Observe that this expression is therefore dependent only on the Lode angle ω of the microscopic stress tensor.

We show now that ω is in fact dependent only on the Lode angle ζ of the microscopic strain rate $\dot{\boldsymbol{\epsilon}}$ defined in (2). To this end, we have

$$\cos 3\zeta = 4\frac{\det\dot{\boldsymbol{\epsilon}}}{\dot{\epsilon}^3_{\text{eq}}} = \frac{\left[g(\omega)(g^2(\omega) - 3(g'(\omega))^2\right]\cos 3\omega + \left[g'(\omega)((g'(\omega))^2 - 3g^2(\omega))\right]\sin 3\omega}{\left[g^2(\omega) + (g'(\omega))^2\right]^{\frac{3}{2}}} \tag{29}$$

leading to

$$\cos 3\zeta = \cos 3(\omega - \alpha) \quad \text{and therefore } \zeta = \omega - \varpi \tag{30}$$

where the angle ϖ is given by

$$\tan 3\varpi = \frac{g'(\omega)((g'(\omega))^2 - 3g^2(\omega))}{g(\omega)(g^2(\omega) - 3(g'(\omega))^2)} \tag{31}$$

We also emphasize, at this stage, that the angle ϖ in (31) is the angle between the strain rate and the stress deviators in the deviatoric plane. Use of the formulae $\cos 3\varpi = 4\cos^3\varpi - 3\cos\varpi$ and $\sin 3\varpi = -4\sin^3\varpi + 3\sin\varpi$, one finds that

$$\sin\varpi = -\frac{g'(\omega)}{\sqrt{g^2(\omega) + (g'(\omega))^2}} \quad \text{and } \cos\varpi = -\frac{g(\omega)}{\sqrt{g^2(\omega) + (g'(\omega))^2}} \tag{32}$$

and allows us through (1) to get

$$\zeta(\omega) = \omega - \alpha(\omega) = \omega + \arctan \frac{g'(\omega)}{g(\omega)} \tag{33}$$

This gives an implicit relation between ζ and ω, the solution to which gives the relation $\omega(\zeta)$. An important issue here is seen from the derivative of $\zeta(\omega)$, given by

$$\zeta'(\omega) = \frac{g(\omega)\left(g''(\omega) + g(\omega)\right)}{g'(\omega)^2 + g(\omega)^2} \tag{34}$$

and which is always positive thanks to the convexity of the yield surface and to relation (5). The function $\zeta(\omega)$ is therefore invertible, and one can consequently define $\omega(\zeta)$ without any ambiguity. $\omega(\zeta)$ is then an increasing function of ζ. Note for further reference that the derivative $\omega'(\zeta)$, obtained as the derivative of an inverse function, is given by

$$\omega'(\zeta) = \frac{1}{\zeta'(\omega)} = \frac{1}{\zeta'(\omega(\zeta))} \tag{35}$$

and that the function $\omega(\zeta)$ is obtained from $\zeta(\omega)$ by a symmetry with respect to the bisector in the plane (ζ, ω). This concludes the determination of the function π, and from (25), (26), (27), (28) we have

$$\pi(\dot{\boldsymbol{\epsilon}}) = \sigma_0 \frac{\dot{\epsilon}_{eq}}{\sqrt{g^2(\omega(\zeta)) + (g'(\omega(\zeta)))^2}} = \sigma_0 \dot{\epsilon}_{eq} G(\zeta) \tag{36}$$

where it is understood that $\zeta(\dot{\boldsymbol{\epsilon}})$ is dependent only on the strain rate $\dot{\boldsymbol{\epsilon}}$.

Let us note that when $\boldsymbol{\sigma}^*$ is axisymmetric ($\omega = 0$ or $\omega = \pi/3$), so is $\dot{\boldsymbol{\epsilon}}$. Therefore, by construction of the function $\omega(\zeta)$, we have

$$\omega(0) = 0 \quad \text{and } \omega(\pi/3) = \pi/3 \tag{37}$$

The function $G(\zeta)$ defined in (36) plays the same role as $g(\omega)$ and satisfies, using relations (36), (37) and (9)

$$G'(0) = G'(\pi/3) = 0 \tag{38}$$

The dissipation $\pi(\dot{\boldsymbol{\epsilon}})$ is convex as (using exactly the same arguments as for the yield function f)

$$G(\zeta) + G''(\zeta) = \frac{\left(g'(\omega(\zeta))^2 + g(\omega(\zeta))^2\right)^{3/2}}{g(\omega(\zeta))^3 \left(g''(\omega(\zeta)) + g(\omega(\zeta))\right)} \tag{39}$$

is positive. It is also homogeneous of degree one in the strain rate $\dot{\boldsymbol{\epsilon}}$. As the function g, G is homogeneous of degree zero.

3. Constitutive relations for voided materials with Lode-angle-dependent matrix behaviour

In his work, Gurson [2] proposed an approximate yield criterion for voided materials using a limit analysis approach of a hollow sphere cell with radius b and a void with radius a. More specifically, he used a simple incompresssible rigid-plastic constitutive behaviour for the matrix satisfying the von Mises criterion. To obtain the approximate yield criterion, Gurson used in the upper bound theorem of limit analysis a particular trial field $\dot{\boldsymbol{\epsilon}}(\dot{\mathbf{E}})$ satisfying compatibility and boundary conditions corresponding to prescribed macroscopic rates of deformation $\dot{\mathbf{E}}$ at the boundary of the hollow sphere. By bounding the macroscopic dissipation from above, Gurson was able to calculate upper bounds to the macroscopic stresses required to sustain the plastic flow, and these upper bound macroscopic stresses for the considered cell geometry and for a range of macroscopic deformation rates allow us to construct an upper bound yield locus for the porous material. These stresses are defined by

$$\boldsymbol{\Sigma} = \frac{\partial \Pi}{\partial \dot{\mathbf{E}}} \tag{40}$$

with

$$\Pi(\dot{\mathbf{E}}) = \frac{1}{V} \int_V \pi(\dot{\boldsymbol{\epsilon}}(\dot{\mathbf{E}}))\, \mathrm{d}V \tag{41}$$

where $\pi(\dot{\boldsymbol{\epsilon}})$ is the microscopic dissipation defined in section 2. The same approach with the same trial velocity field is followed here, but with the matrix behaviour governed by the constitutive equations described in section 2.

3.1. Stress invariants and some general consequences

When the macroscopic dissipation $\Pi(\dot{\mathbf{E}})$ (41) depends on the three invariants $\dot{E}_{\mathrm{m}}$, $\dot{E}_{\mathrm{eq}}$, $\det \dot{\mathbf{E}}'$ of the macroscopic strain rate $\dot{\mathbf{E}}$, we have shown (see [1]) that relation (40) leads to

$$\Sigma_{\mathrm{m}} = \frac{1}{3}\frac{\partial \Pi}{\partial \dot{E}_{\mathrm{m}}} \tag{42}$$

$$\mathbf{\Sigma}' = \frac{2}{3}\frac{\partial \Pi}{\partial \dot{E}_{\mathrm{eq}}}\frac{\dot{\mathbf{E}}'}{\dot{E}_{\mathrm{eq}}} - \frac{4}{3\dot{\mathbf{E}}_{\mathrm{eq}}\sin 3\eta}\frac{\partial \Pi}{\partial \eta}\left[\left(\frac{\dot{\mathbf{E}}\cdot\dot{\mathbf{E}}}{\dot{E}^2_{\mathrm{eq}}} - \frac{1}{2}\mathbf{1}\right) - \frac{1}{2}\cos 3\eta \frac{\dot{\mathbf{E}}}{\dot{E}_{\mathrm{eq}}}\right] \tag{43}$$

We note here that (43) leads to a singularity for $\eta = 0$ and $\eta = \frac{\pi}{3}$ unless $\frac{\partial \Pi}{\partial \eta}|_{\eta=0} = \frac{\partial \Pi}{\partial \eta}|_{\eta=\frac{\pi}{3}} = 0$. This is actually the case with the assumptions adopted here, the microscopic dissipation $\pi(\dot{\boldsymbol{\epsilon}})$ being non singular, the trial field being linear in terms of the macroscopic strain rate, so that the stress is continuous and the macroscopic dissipation is a smooth function at $\eta = 0$ and $\eta = \frac{\pi}{3}$. This gives incidentally $\eta = 0 \to \Theta = 0$ and $\eta = \frac{\pi}{3} \to \Theta = \frac{\pi}{3}$.

The second of these relations gives

$$\Sigma_{\mathrm{eq}} = \sqrt{\frac{3}{2}\mathbf{\Sigma}' : \mathbf{\Sigma}'} = \sqrt{U^2 + V^2} \tag{44}$$

$$\det \mathbf{\Sigma}' = \frac{2}{27}\left[U\left(U^2 - 3V^2\right)\cos 3\eta - V\left(V^2 - 3U^2\right)\sin 3\eta\right] \tag{45}$$

where we have set

$$U = \frac{\partial \Pi}{\partial \dot{E}_{\mathrm{eq}}} \quad \text{and } V = \frac{1}{\dot{E}_{\mathrm{eq}}}\frac{\partial \Pi}{\partial \eta} \tag{46}$$

Alternatively, (44) and (45) allow us to calculate the Lode angle Θ of the macroscopic stress $\mathbf{\Sigma}$ as

$$\cos 3\Theta = \frac{U\left(U^2 - 3V^2\right)\cos 3\eta - V\left(V^2 - 3U^2\right)\sin 3\eta}{\left(U^2 + V^2\right)^{3/2}} = \cos 3(\eta + \Xi) \tag{47}$$

using the angle Ξ defined by

$$\tan 3\Xi = \frac{V\left(V^2 - 3U^2\right)}{U\left(U^2 - 3V^2\right)} \quad \text{or } \tan \Xi = \frac{V}{U} = \frac{\frac{1}{\dot{E}_{\mathrm{eq}}}\frac{\partial \Pi}{\partial \eta}}{\frac{\partial \Pi}{\partial \dot{E}_{\mathrm{eq}}}} \tag{48}$$

leading to

$$\Theta = \eta + \Xi = \eta + \arctan\frac{U}{V} \tag{49}$$

Ξ is then the angle between the macroscopic stress deviator $\mathbf{S}$ and the macroscopic strain rate deviator $\mathbf{E}'$ in the deviatoric plane (the same plane is used to represent both quantities).

3.2. The trial field and the macroscopic dissipation

The strain rate associated with the trial velocity field used by Rice and Tracey [17] and Gurson [2] is given by

$$\dot{\boldsymbol{\epsilon}} = \dot{\mathbf{E}}' + \lambda \dot{E}_m(\mathbf{1} - 3\,\mathbf{e}_r \otimes \mathbf{e}_r) \tag{50}$$

where $\dot{\mathbf{E}}'$ and $\dot{E}_m$ are the deviatoric and volumetric components of the macroscopic strain rate while $\mathbf{e}_r$ is the unit vector in the radial direction, $\mathbf{1}$ the second order unit tensor and $\lambda = \frac{b^3}{r^3}$. For this field, we have

$$\dot{\epsilon}_{\mathrm{eq}} = \dot{E}_{\mathrm{eq}}\sqrt{1 - 4\mu\lambda H + 4H^2\lambda^2} \tag{51}$$

and its associated Lode angle ζ reads (using relation (2))

$$\cos 3\zeta = \frac{4\det \dot{\boldsymbol{\epsilon}}}{\dot{\epsilon}^3_{\mathrm{eq}}} = \frac{\cos 3\eta + 6(1-2\delta)\lambda H + 12\mu\lambda^2H^2 - 8\lambda^3H^3}{(1 - 4\mu\lambda H + 4H^2\lambda^2)^{\frac{3}{2}}} \tag{52}$$

where we have introduced the ratio H of the volumetric to the effective macroscopic strain rates (strain rate triaxiality)

$$H = \frac{\dot{E}_{\mathrm{m}}}{\dot{E}_{\mathrm{eq}}} \tag{53}$$

and the parameters μ and δ are given by

$$\mu = \frac{\mathbf{e}_r^{\mathrm{T}} \cdot \dot{\mathbf{E}}' \cdot \mathbf{e}_r}{\dot{E}_{\mathrm{eq}}} = \frac{\dot{E}_{rr}}{\dot{E}_{\mathrm{eq}}} = \frac{1}{2}\left[-\sqrt{3}\sin\eta\cos 2\theta\sin^2\phi + \frac{1}{2}\cos\eta(3\cos 2\phi + 1)\right] \tag{54}$$

$$\delta = \frac{\mathbf{e}_r^{\mathrm{T}} \cdot (\dot{\mathbf{E}}')^2 \cdot \mathbf{e}_r}{\dot{E}_{\mathrm{eq}}} = \frac{1}{8}\left(-2\sqrt{3}\sin(2\eta)\cos(2\theta)\sin^2(\phi) + \cos(2\eta)(3\cos(2\phi)+1) + 4\right) \tag{55}$$

3.3. Parametric representation of the yield criterion

With the representative cell of Gurson [2], i.e. a hollow sphere cell with external radius b and a void with radius a, denoting the porosity by $f = \frac{a^3}{b^3}$, one uses a spherical coordinate spherical system (r, θ, ϕ) and the variable change $\lambda = \frac{b^3}{r^3} = \frac{1}{x}$, with $\mathrm{d}\Omega = \sin\varphi \mathrm{d}\theta \mathrm{d}\varphi$, so that any integral over the volume of the cell V is obtained by:

$$\frac{1}{V}\int_V (.)\mathrm{d}V = \frac{1}{4\pi}\int_1^{\frac{1}{f}}\int_\Omega (.)\frac{\mathrm{d}\lambda\,\mathrm{d}\Omega}{\lambda^2} \tag{56}$$

Substituting the expression of the microscopic dissipation π given by (36) in the macroscopic dissipation $\Pi(\dot{\mathbf{E}})$, and after some algebraic manipulations, relations (42) and (46) become

$$\begin{aligned}
\frac{\Sigma_{\mathrm{m}}}{\sigma_0} = {} & \frac{1}{6\pi}\int_1^{\frac{1}{f}}\int_\Omega \frac{G(\zeta)[2\lambda H - \mu]}{\sqrt{1 - 4\mu\lambda H + 4H^2\lambda^2}}\frac{\mathrm{d}\lambda\,\mathrm{d}\Omega}{\lambda} + {}\\
& \frac{1}{6\pi}\int_1^{\frac{1}{f}}\int_\Omega \frac{1}{\sin 3\zeta}\frac{g'(\omega(\zeta))G(\zeta)}{g((\zeta))}\left[\frac{1 - 2\delta + 4\lambda H\mu - 4\lambda^2 H^2}{1 - 4\mu\lambda H + 4H^2\lambda^2} - {}\right.\\
& \left.\frac{(\cos 3\eta + 6(1-2\delta)\lambda H + 12\mu\lambda^2 H^2 - 8\lambda^3 H^3)(2\lambda H - \mu)}{(1 - 4\mu\lambda H + 4H^2\lambda^2)^2}\right]\frac{\mathrm{d}\lambda\,\mathrm{d}\Omega}{\lambda}
\end{aligned} \tag{57}$$

$$\begin{aligned}
\frac{U}{\sigma_0} = {} & \frac{1}{4\pi}\int_1^{\frac{1}{f}}\int_\Omega \frac{G(\zeta)[1 - 2\mu\lambda H]}{\sqrt{1 - 4\mu\lambda H + 4H^2\lambda^2}}\frac{\mathrm{d}\lambda\,\mathrm{d}\Omega}{\lambda^2} + {}\\
& \frac{\sigma_0}{4\pi}\int_1^{\frac{1}{f}}\int_\Omega \frac{1}{\sin 3\zeta}\frac{g'(\omega(\zeta))G(\zeta)}{g(\omega(\zeta))}\left[\frac{\cos 3\eta - 4H\lambda(1-2\delta) + 4\mu\lambda^2 H^2}{1 - 4\mu\lambda H + 4H^2\lambda^2} - {}\right.\\
& \left.\frac{(\cos 3\eta + 6(1-2\delta)\lambda H + 12\mu\lambda^2 H^2 - 8\lambda^3 H^3)(1 - 2\lambda H\mu)}{(1 - 4\mu\lambda H + 4H^2\lambda^2)^2}\right]\frac{\mathrm{d}\lambda\,\mathrm{d}\Omega}{\lambda^2}
\end{aligned} \tag{58}$$

$$\begin{aligned}
\frac{V}{\sigma_0} = {} & \frac{\sigma_0 H}{2\pi}\int_1^{\frac{1}{f}}\int_\Omega \frac{G(\zeta)\left[-\frac{\partial\mu}{\partial\eta}\right]}{\sqrt{1 - 4\mu\lambda H + 4H^2\lambda^2}}\frac{\mathrm{d}\lambda\,\mathrm{d}\Omega}{\lambda} + {}\\
& \frac{\sigma_0}{4\pi}\int_1^{\frac{1}{f}}\int_\Omega \frac{1}{\sin 3\zeta}\frac{g'(\omega(\zeta))}{g(\omega(\zeta))}G(\zeta)\left[\frac{-\sin 3\eta - 4H\lambda(\frac{\partial\delta}{\partial\eta}) + 4H^2\lambda^2(\frac{\partial\mu}{\partial\eta})}{1 - 4\mu\lambda H + 4H^2\lambda^2} + {}\right.\\
& \left.\frac{2H\lambda(\cos 3\eta + 6(1-2\delta)\lambda H + 12\mu\lambda^2 H^2 - 8\lambda^3 H^3)\frac{\partial\mu}{\partial\eta}}{(1 - 4\mu\lambda H + 4H^2\lambda^2)^2}\right]\frac{\mathrm{d}\lambda\,\mathrm{d}\Omega}{\lambda^2}
\end{aligned} \tag{59}$$

In the absence of Lode angle effects in the yield behaviour of the matrix and after integration with respect to λ, the results obtained in [1] are recovered.

4. Closed-form results

Though implicit, the parametric representation given above allows us to obtain some exact results. These are summarized below, in particular for hydrostatic and shear loadings.

4.1. Shear loadings

When $H \to 0$, it is easily checked that $\cos 3\zeta \to \cos 3\eta$ so that $\zeta = \eta$. From (57), we have also

$$\frac{\Sigma_m}{\sigma_0} = \frac{1}{6\pi}\int_1^{\frac{1}{f}}\int_\Omega \frac{-\mu G(\eta)}{\lambda}\mathrm{d}\lambda\,\mathrm{d}\Omega + \frac{1}{6\pi}\int_1^{\frac{1}{f}}\int_\Omega \frac{g'(\omega(\eta))G(\eta)}{\sin 3\eta\, g(\omega(\eta))}\left[(1-2\delta)+\mu\cos 3\eta\right]\frac{\mathrm{d}\lambda\,\mathrm{d}\Omega}{\lambda} \tag{60}$$

and these terms vanish as $\int_\Omega \mu\,\mathrm{d}\Omega = \int_\Omega (1-2\delta)\,\mathrm{d}\Omega = 0$. We further have from (58)

$$\frac{U}{\sigma_0} = \frac{1}{4\pi}\int_1^{\frac{1}{f}}\int_\Omega G(\eta)\frac{\mathrm{d}\lambda\,\mathrm{d}\Omega}{\lambda^2} = (1-f)G(\eta) \tag{61}$$

while from (59)

$$\frac{V}{\sigma_0} = -\frac{1}{4\pi}\int_1^{\frac{1}{f}}\int_\Omega \frac{g'(\omega(\eta))G(\eta)}{g(\omega(\eta))}\mathrm{d}x\,\mathrm{d}\Omega = -(1-f)\frac{g'(\omega(\eta))G(\eta)}{g(\omega(\eta))} \tag{62}$$

One then gets the effective stress as

$$\frac{\Sigma_{eq}}{\sigma_0} = \sqrt{\left(\frac{U}{\sigma_0}\right)^2 + \left(\frac{V}{\sigma_0}\right)^2} = \frac{(1-f)}{g(\omega(\eta))} \tag{63}$$

Now using (49), we get with the help of (61) and (62)

$$\Theta = \eta + \arctan\frac{V}{U} = \eta + \arctan\frac{g'(\omega(\eta))}{g(\omega(\eta))} \tag{64}$$

and finally from (33)

$$\Theta = \omega(\eta) \tag{65}$$

Inserting this result in (63), one finally obtains

$$\Sigma_{eq}\, g(\Theta) = (1-f)\sigma_0 \tag{66}$$

which is the equation of the section of the effective yield surface in the π-plane which is, up to the size reduction factor $1-f$, exactly the equation of the yield surface of the matrix in the same plane. The situation is similar to that in the Gurson model and actually holds for any matrix yield surface.

4.2. Pure hydrostatic loadings

When $H \to +\infty$, i.e. for purely hydrostatic macroscopic strain rates, the macroscopic stress is also purely hydrostatic. Indeed, from (52), $\cos 3\zeta \to -1$ so that $\zeta \to \frac{\pi}{3}$ and therefore from (37) $\omega(\zeta) \to \frac{\pi}{3}$ and $g(\omega(\zeta)) \to g(\frac{\pi}{3})$. From (57), one gets:

$$\frac{\Sigma_m}{\sigma_0} \to \frac{1}{6\pi}\int_1^{\frac{1}{f}}\int_\Omega \frac{G(\pi/3)\mathrm{d}\lambda\,\mathrm{d}\Omega}{\lambda} = \frac{1}{6\pi}\int_1^{\frac{1}{f}}\int_\Omega \frac{\mathrm{d}\lambda\,\mathrm{d}\Omega}{\lambda\sqrt{g'(\frac{\pi}{3})^2 + g(\frac{\pi}{3})^2}} = \frac{-\frac{2}{3}\ln f}{\sqrt{g'(\frac{\pi}{3})^2 + g(\frac{\pi}{3})^2}} = \frac{-\frac{2}{3}\ln f}{g(\frac{\pi}{3})} \tag{67}$$

taking into account condition (9). A similar result is obtained when $H \to -\infty$. In this case, $\cos 3\zeta \to 1$ so that $\zeta \to 0$ and therefore from (37) $\omega(\zeta) \to 0$ and $g(\omega(\zeta)) \to g(0)$. One finds:

$$\frac{\Sigma_m}{\sigma_0} \to \frac{1}{6\pi}\int_1^{\frac{1}{f}}\int_\Omega \frac{G(0)\mathrm{d}\lambda\,\mathrm{d}\Omega}{\lambda} = \frac{\frac{2}{3}\ln f}{\sqrt{g'(0)^2 + g(0)^2}} = \frac{\frac{2}{3}\ln f}{g(0)} \tag{68}$$

We also have for $H \to \infty$, using (58) and (59)

$$U = \frac{1}{4\pi}\int_1^{\frac{1}{f}}\int_\Omega \frac{G(\pi/3)(-\mu)\mathrm{d}\lambda\,\mathrm{d}\Omega}{\lambda^2} \to 0 \tag{69}$$

$$V = \frac{1}{4\pi} \int_{1}^{\frac{1}{f}} \int_{\Omega} \frac{G(\pi/3)\left(\frac{-\partial\mu}{\partial\eta}\right)}{\lambda} \mathrm{d}\lambda\, \mathrm{d}\Omega \to 0 \tag{70}$$

as $\int_\Omega \mu\, \mathrm{d}\Omega = \int_\Omega \left(\frac{-\partial\mu}{\partial\eta}\right) \mathrm{d}\Omega = 0$. The same result is obtained when $H \to -\infty$.

In both cases, the result slightly differs from that given by Gurson [2] by the term appearing in the denominator, due to the dependence of the yield surface of the matrix on the third invariant of stress. Observe here that for high positive stress triaxialities ($H \to \infty$), the macroscopic yield stress is set by the microscopic yield stress in compression while for high negative stress triaxialities ($H \to -\infty$), the macroscopic yield stress is set by the microscopic yield stress in tension. This result is a Lode angle effect and can be directly recovered by solving the problem of a hollow sphere constituted of a matrix with yielding behaviour as used here and subjected to a uniform macroscopic pressure Σ_m at its external boundary $r = b$. For this situation, the microscopic stress state has the form

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{rr} & 0 & 0 \\ 0 & \sigma_{\theta\theta} & 0 \\ 0 & 0 & \sigma_{\phi\phi} = \sigma_{\theta\theta} \end{bmatrix} \to \mathbf{s} = \frac{2}{3}(\sigma_{rr} - \sigma_{\theta\theta}) \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \tag{71}$$

so that the Lode angle is constant and takes either the values 0 or $\frac{\pi}{3}$ depending on the sign of $\sigma_{rr} - \sigma_{\theta\theta}$. The equilibrium equations read in this case

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{2}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0 \tag{72}$$

while the boundary conditions at the inner and outer surfaces of the sphere take the form

$$\sigma_{rr}(b) = \Sigma_\mathrm{m}\,\mathrm{d} \text{ and } \sigma_{rr}(a) = 0 \tag{73}$$

For positive Σ_m, the radial stress is increasing from 0 at $r = a$ to $\Sigma_\mathrm{m} > 0$ at $r = b$, so that from the equilibrium equation we have

$$\frac{\partial \sigma_{rr}}{\partial r} = -\frac{2}{r}(\sigma_{rr} - \sigma_{\theta\theta}) \geq 0 \tag{74}$$

implying that $(\sigma_{rr} - \sigma_{\theta\theta}) \leq 0$ and from (71) the corresponding Lode angle is $\frac{\pi}{3}$. For negative Σ_m, the radial stress σ_{rr} is decreasing from 0 to $\Sigma_\mathrm{m} < 0$, and a similar reasoning leads to $(\sigma_{rr} - \sigma_{\theta\theta}) \geq 0$ and to a Lode angle equal to 0. The two situations can be summarized by

$$\sigma_{rr} - \sigma_{\theta\theta} = -|\sigma_{rr} - \sigma_{\theta\theta}|\,\mathrm{sgn}(\Sigma_\mathrm{m}) = -\sigma_\mathrm{eq}\,\mathrm{sgn}(\Sigma_\mathrm{m})\,\mathrm{d} \quad \text{and } \omega = \frac{1}{3}\arccos[-\mathrm{sgn}(\Sigma_\mathrm{m})] \tag{75}$$

with $\mathrm{sgn}(x)$ denoting the sign of x. At the plastic limit load, the plastic zone reaches the external surface of the sphere, the yield condition reads everywhere in the cell

$$\sigma_\mathrm{eq} g(\omega) = |\sigma_{rr} - \sigma_{\theta\theta}| g(\omega) = \sigma_0 \tag{76}$$

so that the equilibrium equation becomes, on using (75) and (76)

$$\frac{\partial \sigma_{rr}}{\partial r} = -\frac{2}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = \frac{2}{r}\frac{\mathrm{sgn}(\Sigma_\mathrm{m})\sigma_0}{g(\frac{1}{3}\arccos[-\mathrm{sgn}(\Sigma_\mathrm{m})])} \tag{77}$$

Integration with respect to r between a and b gives

$$\Sigma_\mathrm{m} = 2\mathrm{sgn}(\Sigma_\mathrm{m}) \ln\left(\frac{b}{a}\right) \frac{\sigma_0}{g(\frac{1}{3}\arccos[-\mathrm{sgn}(\Sigma_\mathrm{m})])} = -\frac{2}{3}\mathrm{sgn}(\Sigma_\mathrm{m}) \ln f \frac{\sigma_0}{g(\frac{1}{3}\arccos[-\mathrm{sgn}(\Sigma_\mathrm{m})])} \tag{78}$$

which is the result given above. Indeed, for positive Σ_m, we have (67), while for negative Σ_m, we get (68).

4.3. Extension to materials with matrix yielding dependent on the three stress invariants

It is interesting now to extend the results of this section to porous solids with yielding of the matrix dependent on the three stress invariants. We stress it here that the objective here is not to derive full constitutive equations for such materials as this is much more complicated than the analysis provided above. Rather we will limit the analysis only to pure hydrostatic loadings and provide extension of the results obtained in section 4.2. This will in particular unify the results for the usual yield criteria when they are applied to the yield functions (13) and (14). We consider therefore in the following a porous solid whose yielding is given by the yield function (13). For the hydrostatic loadings case, we still have relation (75) for the difference $\sigma_{rr} - \sigma_{\theta\theta}$ stress, while the hydrostatic stress reads

$$\sigma_{\rm m} = \frac{1}{3}(\sigma_{rr} + 2\sigma_{\theta\theta}) \tag{79}$$

Equilibrium is still given by (72), while yielding reads now

$$\sigma_{\rm eq} g\,(\frac{1}{3}\arccos[-\operatorname{sgn}(\Sigma_{\rm m})]) + 3\alpha\sigma_{\rm m} = -\operatorname{sgn}(\Sigma_{\rm m})(\sigma_{rr} - \sigma_{\theta\theta})\, g\,(\frac{1}{3}\arccos[-\operatorname{sgn}(\Sigma_{\rm m})]) + \alpha(\sigma_{rr} + 2\sigma_{\theta\theta}) = \sigma_0 \tag{80}$$

From the latter relation, one can compute the hoop stress $\sigma_{\theta\theta}$ in terms of the radial stress σ_{rr} as

$$\sigma_{\theta\theta} = \frac{\sigma_0 - (\alpha - \operatorname{sgn}(\Sigma_{\rm m})\, g\,(\frac{1}{3}\arccos[-\operatorname{sgn}(\Sigma_{\rm m})]))\,\sigma_{rr}}{2\,\alpha + \operatorname{sgn}(\Sigma_{\rm m}) g(\frac{1}{3}\arccos[-\operatorname{sgn}(\Sigma_{\rm m})])} \tag{81}$$

Substitution of this last expression in the equilibrium equation (72) and solving for σ_{rr} with the boundary condition $\sigma_{rr}(a) = 0$ leads to

$$\sigma_{rr}(r) = \frac{\sigma_0 \left(1 - \left(\frac{a}{r}\right)^{\frac{6\alpha}{2\alpha + \operatorname{sgn}(\Sigma_{\rm m}) g(\frac{1}{3}\arccos[-\operatorname{sgn}(\Sigma_{\rm m})])}}\right)}{3\alpha} \tag{82}$$

from which one obtains the limit load yield point when $H \to \pm\infty$ by he condition $\sigma_{rr}(b) = \Sigma_{\rm m}$. This gives

$$\frac{\Sigma_{\rm m}}{\sigma_0} = \frac{1 - f^{\gamma}}{3\alpha} \tag{83}$$

with

$$\gamma = \frac{2\alpha}{2\alpha + \operatorname{sgn}(\Sigma_{\rm m}) g(\frac{1}{3}\arccos[-\operatorname{sgn}(\Sigma_{\rm m})])} \tag{84}$$

For positive mean stresses, $\gamma = \frac{2\alpha}{2\alpha + g(\frac{\pi}{3})}$, while for negative mean stresses $\gamma = \frac{2\alpha}{2\alpha - g(0)}$. From equation (14), we have

$$g(0) = \cos\left[\frac{1}{3}\arccos[\chi] - \beta\right], \quad g(\frac{\pi}{3}) = \cos\left[\frac{1}{3}\arccos[-\chi] - \beta\right] \tag{85}$$

and this gives the macroscopic yield stresses for all materials with this type of yielding. Further, a limit process (as $\chi \to 1$) allows us to obtain these macroscopic yield stresses for the non-smooth cases such as the Tresca criterion, the Mohr–Coulomb criterion, but also for a fully triangular yield shape.

A first order expansion of (1) with respect to α gives

$$\frac{\Sigma_{\rm m}}{\sigma_0} = -\frac{2\log(f)}{3g(\frac{1}{3}\arccos[-\operatorname{sgn}(\Sigma_{\rm m})])} - \frac{2\alpha((\log(f) - 2)\log(f))}{3g(\frac{1}{3}\arccos[-\operatorname{sgn}(\Sigma_{\rm m})])^2} + O\left(\alpha^2\right) \tag{86}$$

and shows that the result is consistent with the one given above in the limit $\alpha \to 0$. It is also consistent with the results provided in [18], [19], and [20]. The result (84) is given in exactly in the same form by [18], but the content of the coefficient γ is however different, including beside the coefficient α extra terms coming from Lode angle effects. In the revision process, we have learned that results in this direction are also provided in [21] and [22].

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