C. R. Mecanique 346 (2018) 175–183

Contents lists available at ScienceDirect

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The legacy of Jean-Jacques Moreau in mechanics

Helicity and other conservation laws in perfect fluid motion

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ARTICLE INFO

Article history:
Received 20 May 2017
Accepted 11 October 2017
Available online 21 February 2018

In memoriam Jean-Jacques Moreau

Keywords:
Perfect fluid
Poisson structure
Helicity

ABSTRACT

In this review paper, we discuss helicity from a geometrical point of view and see how it applies to the motion of a perfect fluid. We discuss its relation with the Hamiltonian structure, and then its extension to arbitrary space dimensions. We also comment about the existence of additional conservation laws for the Euler equation, and its unlikely integrability in Liouville's sense.

1. Introduction

The concept of helicity was discovered by Jean-Jacques Moreau [1] in 1961, in the context of incompressible perfect fluids, as a generalization of Kelvin's circulation theorem. It has been later extended to various other mechanical and mathematical situations. There are therefore several ways to present it. I choose here the geometrical point of view, which emphasizes the role of differential forms.

A form γ has a degree $m \le d$, where d is the dimension of its domain. A 0-form is just a function, and a d-form is a volume form; a 1-form can be identified with a vector field. A form of degree m (one speaks of an m-form) has an exterior derivative $\mathrm{d}\gamma$, which is an $(m+1)$-form. If $m=0$, $\mathrm{d}f$ is nothing but the differential of the function f. The operators d have the property that $\mathrm{dd}=0$. The form γ is closed if $\mathrm{d}\gamma=0$; it is exact if there exists an $(m-1)$-form α such that $\gamma=\mathrm{d}\alpha$. An exact form is therefore closed, though the converse is not always true (this depends on the topology of the domain).

Let M be a smooth 3-dimensional oriented manifold without boundary. Let η be an exact 2-form on M, say with compact support; it admits a *potential* α, that is, a 1-form such that $\mathrm{d}\alpha=\eta$. Every other potential is an $\alpha+\beta$, where β is closed. The exterior product $\alpha\wedge\eta$ is a volume form and can be integrated over M. The integral

$$\mathcal{H}[\eta]=\int_M \alpha\wedge\eta$$

does not depend upon the choice of α, because if $\mathrm{d}\beta=0$, then

$$\int_M \beta\wedge\eta=\int_M \beta\wedge(\mathrm{d}\alpha)=\int_M \mathrm{d}(\beta\wedge\alpha)=\int_M 0=0$$

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https://doi.org/10.1016/j.crme.2017.12.001

The integral $\mathcal{H}[\eta]$ is called either the *Hopf invariant* (mostly by geometers) or the *helicity* of η by people working in physics and fluid dynamics.

The helicity is invariant under the action of the group $\mathbf{Diff}_0(M)$ of measure-preserving diffeomorphisms of M, in the sense that, if $\phi \in \mathbf{Diff}_0(M)$, then

$$\mathcal{H}[\phi_*\eta] = \int_M (\phi_*\alpha) \wedge (\phi_*\eta) = \int_M \phi_*(\alpha \wedge \eta) = \int_M \alpha \wedge \eta = \mathcal{H}[\eta]$$

where we have used

$$\mathrm{d}(\phi_*\alpha) = \phi_*\eta, \qquad \phi_*(\alpha \wedge \eta) = \alpha \wedge \eta$$

When $M = \mathbb{R}^3$ or $M = \mathbb{T}^3$ and u is a vector field, identified with the differential form $u \cdot \mathrm{d}x$, the above construction applies to the 2-form $\eta = \mathrm{d}u$. In coordinates, one has

$$\begin{aligned}\eta &= (\partial_1 u_2 - \partial_2 u_1)\mathrm{d}x_1 \wedge \mathrm{d}x_2 + (\partial_2 u_3 - \partial_3 u_2)\mathrm{d}x_2 \wedge \mathrm{d}x_3 + (\partial_3 u_1 - \partial_1 u_3)\mathrm{d}x_3 \wedge \mathrm{d}x_1 \\ &= \sum_{\epsilon(i,j,k)=1} (\mathrm{curl}\, u)_i \mathrm{d}x_j \wedge \mathrm{d}x_k\end{aligned}$$

and

$$u \wedge \eta = (u \cdot \mathrm{curl}\, u)\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$$

Therefore, the helicity writes

$$\mathcal{H} = \int_{\mathbb{R}^3(\mathbb{T}^3)} u \cdot \mathrm{curl}\, u \,\mathrm{d}x$$

When $\mathrm{curl}\, u$, called the *vorticity*, is confined in two oriented unknotted flux tubes, and the integral curves of $\mathrm{curl}\, u$ are unlinked closed curves circulating only once parallel to the tubes axes, K. Moffat [2] interprets $\mathcal{H}[\eta]$ as the degree of linkage of the tubes times the product of the circulations in each tube. Additional contributions come when the integral curves are linked, non closed, or when the tubes are knotted, see [3].

V.I. Arnold [28] gave a geometrical interpretation of $\mathcal{H}[\eta]$, which he called *Hopf's asymptotic invariant*. He identified the closed 2-form with a solenoidal vector field X by $\eta = \imath_X vol$; in $\mathbb{R}^3$, this just means $\eta = X_1 \mathrm{d}x_2 \wedge \mathrm{d}x_3 + X_2 \mathrm{d}x_3 \wedge \mathrm{d}x_1 + X_3 \mathrm{d}x_1 \wedge \mathrm{d}x_2$. Let ϕ^t denote the flow associated with X, which is volume-preserving. Then the helicity measures the asymptotic linking of the trajectories $(\phi^t(x))_{t\ge 0}$ as $t \to +\infty$, averaged with respect to x over M.

2. Helicity in fluid dynamics

The velocity field of a perfect incompressible fluid is governed by the Euler equation

$$\partial_t u + (u \cdot \nabla)u + \nabla p = 0, \qquad \mathrm{div}\, u = 0 \tag{1}$$

Let us assume that the physical domain is $\mathbb{R}^d$ or $\mathbb{T}^d$. Although u is naturally a vector field (the velocity field of the fluid), the Euclidean structure allows us to identify it with a differential form of degree one; then the incompressibility condition is interpreted as $\mathrm{d}^* u = 0$. The 2-form $\eta := \mathrm{d}u$ is the vorticity. In three space dimension, we may again identify η with the vector field $\mathrm{curl}\, u$.

It has been known from the mists of time that, when $d = 2$, the scalar vorticity $\eta_s := \partial_1 u_2 - \partial_2 u_1$ is transported by the flow:

$$(\partial_t + u \cdot \nabla)\eta_s = 0$$

This amounts to saying that $\eta_s(t) = \eta_s(0) \circ \phi_t$, where $\eta_s(0)$ is the vorticity at initial time, and $(\phi_t)_{t\ge 0}$ is the flow generated by the vector field u. Since u is the velocity field, $t \mapsto \phi_t(y)$ is the trajectory of the particle that is initially at position y. Because u is solenoidal ($\mathrm{div}\, u = 0$), ϕ_t is measure-preserving: $(\phi_t)_* \mathrm{d}x = \mathrm{d}x$. It follows that, not only $\int_{\mathbb{R}^3} \eta_s \,\mathrm{d}x$ is conserved as time varies, but actually every quantity of the form

$$\int f(\eta_s)\,\mathrm{d}x$$

is a time-invariant of the flow. This remains true even if the physical domain D is bounded with an impermeable boundary, that is, with the boundary condition

$$u \cdot \nu = 0 \qquad (\nu \text{ the outward normal}) \quad \text{on } \partial D$$

These invariants were used by several authors to treat the Cauchy problem. Lichtenstein [4] and Wolibner [5] considered smooth initial data and found that a classical solutions exists globally in time. Yudovich [6] proved the existence and uniqueness of the flow when $\omega(0) \in L^\infty$ and the kinetic energy is finite; see also Chemin's book [7] Chapter 5.

In three space dimensions, taking the curl of (1) yields instead

$$(\partial_t + u \cdot \nabla)\eta_c = (\eta_c \cdot \nabla)u$$

where η_c stands for the curl of u. This equation, which can be interpreted in terms of a Lie derivative, does not imply the constancy of so many integrals as in 2D. Jean-Jacques Moreau [1] was the first to remark that the helicity $\mathcal{H}[\eta]$ of the vorticity field remains constant as time varies. This constancy, together with that of the kinetic energy and linear/angular momenta

$$\int_{\mathbb{R}^3} \frac{1}{2}|u|^2 \mathrm{d}x, \qquad \int_{\mathbb{R}^3} u\, \mathrm{d}x, \qquad \int_{\mathbb{R}^3} u \wedge x\, \mathrm{d}x$$

rose at least five questions.

1. Are there other invariants of the 3D-Euler equation, in the form of an integral of u and its derivatives?
2. Are these invariants related to the invariance group of the Euler equation, through Noether's Theorem?
3. Can these invariants be used to develop a theory of the Cauchy problem? For instance, is the Euler equation completely integrable in the sense of Liouville?
4. Are there similar invariants for a compressible fluid?
5. Are there similar invariants in other space dimensions? Does the helicity belong to a list parametrized by the dimension?

Question 1 was answered by the negative for integrals depending upon (u, Du) in [8], using brute force. Question 2 was solved by P.J. Olver [9], who derived a Hamiltonian formalism of the Euler equation, and showed that the kinetic energy and linear/angular momenta are associated with the invariance group (isometries, Galilean transformations) of the equation through Noether's Theorem. Even if this extends what was already known for simpler mechanical systems, the proof is much more involved because the Hamiltonian structure is intrinsically non-linear. Olver showed that, on the contrary, the helicity is associated with a degeneracy of the Hamiltonian structure, a fact that we explain in more details below. As for Question 3, we still know only finitely many invariants, and there is little hope that the Euler equation could be completely integrable. Even in two space dimensions, where we are aware of an infinite list of conservation laws, the complete integrability is doubtful, because the invariants, apart from energy and momenta, are still associated with the degeneracy of the symplectic structure [10]. Eventually, V.I. Arnol'd & B. Khesin [11] gave an affirmative answer to Question 4, in the barotropic case where the pressure p is given as a function of the density ρ (see also Khesin & Yu. Chekanov [12]). The system is then

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \qquad \partial_t u + (u \cdot \nabla)u + \frac{1}{\rho}\nabla p(\rho) = 0$$

from which one derives

$$\partial_t (u \cdot \eta) + \operatorname{div}\left((u \cdot \eta)u + (h(\rho) - \frac{1}{2}|u|^2)\eta \right) = 0$$

where $h'(s) = \frac{1}{s}p'(s)$. Integrating over $\mathbb{R}^3$ or $\mathbb{T}^3$, we obtain the constancy of the helicity, provided that u and η are smooth and they decay fast enough at infinity. We warn the reader that once shock waves are present in the flow (the velocity field then admits discontinuities across shock surfaces), the helicity is not constant any more, even though it still makes sense.

When the pressure depends also upon a temperature and the Euler system includes the energy conservation law, the situation is more complex, and there does not seem to be something like a conserved helicity.

2.1. Helicity in other space dimensions

Question 5 was answered by L. Tartar (unpublished) in even space dimension, and by the author [10] in odd dimension.

Even dimensional case. Suppose $d = 2m$. Tartar proved that the density τ of the d-form

$$\eta^{\wedge m} := \underbrace{\eta \wedge \cdots \wedge \eta}_{m \text{ times}} = \tau\, \mathrm{d}x$$

obeys the transport equation

$$(\partial_t + u \cdot \nabla)\tau = 0$$

This is of course reminiscent of the property $\mathcal{L}_u\eta = 0$, where $\mathcal{L}_u$ is the Lie derivative in the direction of u. There follows, as in the 2D-case, that the integrals

$$\int_{\mathbb{R}^{2m}} f\circ\tau\,\mathrm{d}x$$

are invariants of the motion, for every smooth function f. This property follows as well from the fact that $\eta(t) = (\phi_t)_*\eta(0)$, and therefore $\eta(t)^{\wedge m} = (\phi_t)_*\eta(0)^{\wedge m}$, whence $\tau(t) = \tau(0)\circ\phi_t$.

The odd-dimensional case. When $d = 2m+1$ instead, $u\wedge(\eta^{\wedge m})$ is a volume form. We proved in [10] that the generalized helicity

$$\int_{\mathbb{R}^{2m+1}} u\wedge(\eta^{\wedge m})$$

is an invariant of the motion.

An interpretation using transport is less clear than in the even-dimensional case, because $u(t)$ is not convected by the flow ϕ_t.

3. The Hamiltonian point of view

In order to explain the occurrence of all these conservation laws, we make a digression and discuss the Hamiltonian structure of the incompressible Euler equation.

In a Brave New World, the phase space of a Hamiltonian system is a symplectic manifold M. The symplectic structure is defined by a closed, non-degenerate differential form Ω of degree 2 over M. The argument of Ω is a pair of vector fields $X, Y\in TM$. The Hamiltonian of the system is a given function[1] $H\in\mathcal{C}^\infty(M)$. The evolution of the state is governed by the differential equation

$$\Omega\left(\frac{\mathrm{d}x}{\mathrm{d}t},\cdot\right) = \mathrm{d}H \tag{2}$$

where $\mathrm{d}H$, a 1-form, denotes the differential of H. This equation stands in the cotangent bundle T^*M. Notice that

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \mathrm{d}H\cdot\frac{\mathrm{d}x}{\mathrm{d}t} = \Omega\left(\frac{\mathrm{d}x}{\mathrm{d}t},\frac{\mathrm{d}x}{\mathrm{d}t}\right) = 0$$

so that $t\mapsto H(x(t))$ is a constant of the motion.

Because of the non-degeneracy, the map $X\mapsto\lambda_X := \Omega(X,\cdot)$ is an isomorphism between TM and T^*M, and one may define a skew-symmetric operation over T^*M by

$$[\lambda_X,\lambda_Y] = \Omega(X,Y)$$

The space $\mathcal{C}^\infty(M)$ inherits another skew-symmetric operation

$$\{f,g\} = [\mathrm{d}f,\mathrm{d}g]$$

which turns out to be a Poisson structure. Namely, the bracket satisfies

$$\{f,g\} = -\{g,f\},\qquad \{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\} = 0 \tag{3}$$

together with

$$\{fg,h\} = f\{g,h\}+g\{f,h\} \tag{4}$$

The conditions (3) mean that $\mathcal{C}^\infty(M)$, endowed with the bracket, is a Lie algebra, whereas (4) means that $\{\cdot,g\}$ is a vector field. This field X_g turns out to be $\lambda^{-1}(\mathrm{d}g)$; in other words, we have $\lambda_{X_g} = \mathrm{d}g$.

The Hamiltonian system rewrites therefore

$$\frac{\mathrm{d}x}{\mathrm{d}t} = X_H(x) \tag{5}$$

[1] We consider only the autonomous case, where H does not depend explicitly on the time variable.

One verifies easily the relation

$$\{f,g\} = \mathrm{d}f \cdot X_g \tag{6}$$

and in particular

$$\frac{\mathrm{d}}{\mathrm{d}t} f(x(t)) = \{f, H\}$$

When $f = H$, this gives again the constancy of $t \mapsto H(x(t))$. Those functions f for which $\{f, H\} \equiv 0$ are said to be in *involution* with H; they are invariants of the motion.

3.1. Hamiltonian systems without symplectic structure

The above formalism has been generalized in the following way. Only the Poisson structure is retained, but the symplectic one is dropped. This suggests that the bracket may be degenerate. One then speaks of a Poisson manifold M. Given a function $g \in \mathcal{C}^\infty(M)$, and because of (4), a vector field X_g can be defined through (6), and the ODE (5) still makes sense. The fields X_g form a Lie algebra for the bracket $[X_f, X_g] = \{f, g\}$. The manifold M is partitioned into regularly immersed symplectic manifolds, which are called symplectic leaves, whose dimension may vary from a leaf to another one. Each leaf N is tangent to the vector fields X_g and is minimal for this property. The leaf N can be given a symplectic form Ω_N, which is the restriction to T_N of $[\cdot,\cdot]$,

$$\Omega_N(X_f, X_g) = \{f, g\}, \qquad \forall f, g \in \mathcal{C}^\infty(M)$$

If M is foliated by the symplectic leaves of constant codimension p, one may view the Poisson structure as a smooth family of symplectic manifolds, which depends upon p parameters. This is seen as a degeneracy of the symplectic structure.

3.2. Invariants and symplectic leaves

A trajectory $x(t)$ of a Hamiltonian system stays on the symplectic leaf passing through $x(0)$. Therefore, if $f : M \mapsto \mathbb{R}$ is a smooth function, constant on each symplectic leaf, $t \mapsto f(x(t))$ is a constant. This is reminiscent of the fact that $\{f, g\} = 0$ for every other function g (that is, $X_f = 0$), and in particular $\{f, H\} = 0$; such functions are called *Casimir invariants*. Their constancy has nothing to do with the choice of the Hamiltonian H; it holds true for every Hamiltonian flow constructed over the same Poisson structure.

On the other hand, a Hamiltonian system of practical interest is often invariant under the action of a Lie group G_H, which acts upon the time variable as well as the unknown. In the best situation where we deal with a symplectic structure, there does not exist non-trivial Casimirs, and Noether's Theorem establishes a linear correspondence between the Lie algebra $\mathfrak{g}_H$ and the vector space $\mathcal{I}_H$ of functionals that are invariant along the flow. For instance, the time translation is associated with the conservation of the Hamiltonian itself. In particular, the dimensions of $\mathcal{I}_H$ and $\mathfrak{g}_H$ coincide.

If instead only a Poisson structure is available, Noether's Theorem applies only to the subgroup $G_H^s \subset G_H$ of those transformations that preserve the symplectic leaves. This can be understood by saying that the Hamiltonian system decouples into independent Hamiltonian systems on each leaf, and Noether's Theorem applies only to the latter. Besides, every leaf-wise constant function is an invariant of the flow, of the Casimir form; it does not result from the application of Noether's Theorem. We may only say that the dimension of $\mathfrak{g}_H^s$ is less than that of $\mathcal{I}_H$.

3.3. The situation for the Euler equation

The 3D Euler equation is written in terms of the curl of the velocity, which we denote ω from now on[2]:

$$\frac{\partial \omega}{\partial t} = (\omega \cdot \nabla)u - (u \cdot \nabla)\omega \tag{7}$$

The velocity field is determined by ω through the elliptic system

$$\operatorname{curl} u = \omega, \qquad \operatorname{div} u = 0$$

It can be recovered by the Biot–Savart formula

$$u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \omega(y) \times \frac{x-y}{|x-y|^3} \, \mathrm{d}y$$

[2] After all, we have only a Poisson manifold and do not need any more the Greek letter *omega* for a symplectic structure.

When $\Omega = \mathbb{R}^3$, Olver [13] found a way to rewrite it as a Hamiltonian system within the Poisson formalism. The Hamiltonian is

$$H[\omega] = \frac{1}{2} \int_{\Omega} |u|^2 \mathrm{d}x \qquad \text{(the kinetic energy)}$$

Using the scalar product in $L^2(\mathbb{R}^3)$, the differential $\mathrm{d}H$ can be identified with the variational derivative

$$\frac{\delta H}{\delta \omega} = \psi$$

where the stream vector ψ is the solution to the elliptic system

$$\operatorname{div} \psi = 0, \qquad \operatorname{curl} \psi = u, \qquad \text{in other words } \Delta \psi = -\omega$$

Then (7) can be recast as

$$\frac{\partial \omega}{\partial t} = \mathcal{D}(\omega) \frac{\delta H}{\delta \omega}, \qquad \mathcal{D}(\omega) z := (\omega \cdot \nabla) \operatorname{curl} z - (\operatorname{curl} z \cdot \nabla) \omega$$

As suggested by the notation, the coefficients of the bracket

$$\{f, g\} := \int \frac{\delta f}{\delta \omega} \mathcal{D}(\omega) \frac{\delta g}{\delta \omega} \, \mathrm{d}x$$

do depend upon ω, a situation sometimes referred to as a *nonlinear* Poisson structure. The fact that the bracket satisfies the closure requirements follows from a theoretical analysis initiated by Manin [14] and completed by Gel'fand & Dorfman [15] and Olver [9].

The symplectic leaves. The degeneracy of the symplectic structure is clear once we observe that the vector fields $X_f = \mathcal{D}(\omega) \frac{\delta f}{\delta \omega}$ preserve the orbits of the group Diff_0 of measure preserving diffeomorphisms with compact support over $\mathbb{R}^3$, acting on the space of 2-differential forms. Indeed, if ϕ is the flow of the vector field $v := \operatorname{curl} \frac{\delta f}{\delta \omega}$, and if ω satisfies $\frac{\mathrm{d}\omega}{\mathrm{d}t} = X_{f(t)}$, then $\omega(t) = (\phi_t)^* \omega(0)$.

Noether's Theorem applies therefore to those transformations that preserve these orbits, namely the translation invariance with respect to either time or space variables, and the rotations. They yield the usual conservation laws for energy, linear momentum, and angular momentum.

Besides, every level function of the orbits is a conserved quantity, but would be so for any other choice of the Hamiltonian H. This is the case for the total helicity, as shown in [9]. Recently, Kudryavtseva [16] considered those level functions F that are continuously differentiable in the $\mathcal{C}^1$-topology, and whose differential can be written as

$$\mathrm{d}F(\omega) \cdot \alpha = \int_{\mathbb{R}^3} Z \wedge \alpha \, \mathrm{d}x$$

She proved that the only such function is, up to constant factors, the total helicity. Her result was made a little more general in [17]. This shows that the result in [8], established only in the context of the Euler equation, actually holds true for every Hamiltonian system over the same Poisson structure.

3.4. Other space dimensions

Olver's formalism was written down in every space dimension, in terms of the vorticity $\eta = du$ and a Poisson structure. The Hamiltonian is always the kinetic energy,

$$H[\eta] = \frac{1}{2} \int_{\mathbb{R}^2} |u|^2 \mathrm{d}x$$

In two space dimensions, the operator $\mathcal{D}(\eta_s)$ is

$$\mu \longmapsto \partial(\eta_s, \mu) := \partial_1 \eta_s \partial_2 \mu - \partial_2 \eta_s \partial_1 \mu$$

As mentioned above, the invariants

$$\int_{\mathbb{R}^2} f \circ \eta_s \, \mathrm{d}x$$

are Casimirs. Notice that $\mathcal{D}(\eta_s) f \circ \eta_s \equiv 0$.

Likewise, the invariants discovered by Tartar (even-dimensional case) or the author (odd-dimensional case), mentioned in Section 2.1, are Casimirs associated with the Poisson structure. In particular, they cannot help us to find a complete integrability in the sense of Liouville.

3.5. Integrability?

A Hamiltonian system, defined over a symplectic manifold, is integrable in the sense of Liouville, if it can be recast as a system of the form

$$\frac{\mathrm{d}\vec{I}}{\mathrm{d}t}=0, \qquad \frac{\mathrm{d}\vec{\theta}}{\mathrm{d}t}=F(\vec{I})$$

where the variable $\vec{\theta}$ takes values in a torus $\mathbb{R}^m/\mathbb{Z}^m$. The bounded trajectories are quasi-periodic. The components of $\vec{I}$ are invariants (the action variables) and $\vec{\theta}$ is the set of angle variables. The actions turn out to be in involution; typically, there must be N independent invariants in involution if the system has $2N$ degrees of freedom. One among the action variables is the Hamiltonian H itself.

To understand whether the Euler equation is integrable, we must consider its restriction to a symplectic leaf. Then the Casimirs become trivial invariants and do not count as action variables. Because a typical leaf is infinite dimensional, we should need infinitely many independent conservation laws in involution. However, we know only finitely many of them (see [8]), namely the energy and the momenta. Integrability is thus doubtful.

3.5.1. The case of vortex dynamics

The situation is not even better for the vortex dynamics in 2D, where the vorticity is a singular measure,

$$\omega(\cdot,t)=\sum_{j=1}^{N}\omega_j\delta_{x_j(t)}, \qquad x_j(t)\in\mathbb{R}^2$$

The strengths ω_j are non-zero constants (they define the $2N$-dimensional leaf), and the evolution is governed by (see [18])

$$\frac{\mathrm{d}x_j}{\mathrm{d}t}=\frac{1}{2\pi}\sum_{k\neq j}\omega_k\frac{(x_k-x_j)^{\perp}}{|x_k-x_j|^2}, \qquad \forall j=1,\dots,N$$

This system can be recast in the Hamiltonian form

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\mathcal{D}\nabla K, \qquad \mathcal{D}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\otimes\mathrm{diag}(\omega_j^{-1};\ j=1,\dots,N) \tag{8}$$

where the Hamiltonian is

$$K[x_1,\dots,x_N]=\sum_{1\le j\le k\le N}\omega_j\omega_k\log|x_k-x_j|$$

The non-trivial invariants of (8) are K itself, together with

$$\sum_j\omega_jx_j \quad (\text{a vector}), \qquad \sum_j\omega_j|x_j|^2$$

When $N\geq 5$, there seems to be too few invariants for integrability in the sense of Liouville. For $N=4$, there are enough many invariants, but they are not in involution. Actually, Ziglin [19] proved that, if the ω_j $(1\le j\le 3)$ are close to each other, and ω_4 is much smaller than the other three, then the four-vortex dynamics does not admit an extra analytic first integral. Therefore, it is widely believed that vortex dynamics with $N\geq 4$ is not integrable. Only when $N=2$ or 3, vortex dynamics is Liouville integrable.

Remarks. The system (8), which governs the evolution of finitely many discrete vortices is written in Lagrangian variables (the unknowns x_j follow particle trajectories) instead of the Eulerian formulation (1). This has two advantages:

(1) the Poisson structure of 2D-vortex dynamics is *linear*: the coefficients of $\mathcal{D}$ do not depend upon the state $x\in\mathbb{R}^{2N}$;
(2) one gets rid of the degeneracy; since $\mathcal{D}$ is invertible, we have a genuine symplectic structure.

3.5.2. Time asymptotics

If the Euler equation (1) were completely integrable, and if the solutions were globally defined and bounded, we should expect them to behave as almost-periodic functions of time. This seems to be incompatible with the following observations.

Suppose, on the one hand, that the domain $D\subset\mathbb{R}^3$, rotationally invariant about an axis, is bounded laterally by two cylinders of radii $r_{\pm}$, that is $D=A\times S^1$ where A is a bounded subset of $(0,+\infty)_r\times\mathbb{R}_z$. Then most rotationally invariant flows with swirl display the following behaviour: the quantities

$$\int_A \beta f(ru_\theta)\,\mathrm{d}r\,\mathrm{d}z, \qquad u_\theta := u\cdot\vec{e}_\theta,\ \beta := \operatorname{curl} u\cdot\vec{e}_\theta$$

grow linearly in time; see [20,21].

On the other hand, J. Bedrossian and N. Masmoudi [22] showed that the two-dimensional Euler equation in $\mathbb{T}\times\mathbb{R}$ satisfies a *viscous damping*, a phenomenon analogous to the well-known Landau damping of the Vlasov equation. As a consequence, the global solution converges towards a planar Couette flow as $t\to\pm\infty$. The convergence of the vorticity is weak, while that of the velocity is strong. This phenomenon was predicted by L. Kelvin [23] and W. Orr [24]. Once again, such a convergence is opposite to an almost-periodic behaviour.

3.6. The Euler equation in bounded domains

The situation is more complicated when the domain D is bounded. Up to our knowledge, the Hamiltonian structure has not been yet clarified in full generality. The only case where the formulation can be adapted from that in the whole space, *mutatis mutandis*, is when $d=2$ and D is simply connected.

Because the vorticity is still transported by the flow of the fluid, $\omega(t)$ remains within the orbit of $\omega(0)$ under the action of $\mathrm{Diff}_0(D)$. Therefore the functionals I_f remain conserved if $d=2p$ is even. In the odd-dimensional case, the total/generalized helicity is not a constant of the motion, because the velocity field u is not transported by the flow and, therefore, $u\wedge(\wedge^p\omega)$ at time t differs from $(\phi_t)_*(u_0\wedge(\wedge^p\omega_0))$.

When D is not simply connected, for instance if it is a solid torus, then the velocity field is not uniquely determined by the vorticity. In the simplest example of a planar domain with N holes, the boundary consists of $N+1$ closed loops Γ_j. Kelvin's Theorem tells us that the circulations

$$c_j = \int_{\Gamma_j} u\cdot\tau\,\mathrm{d}s$$

(τ the unit tangent vector, oriented clockwise) are constants of the motion. However, these quantities are not determined by the vorticity; we only have the constraint

$$\sum_{j=0}^{N} c_j = \int_D \omega\,\mathrm{d}x \tag{9}$$

If the 2D-Euler equation can be recast as a Hamiltonian system, we anticipate that the c_j's will be constant on symplectic leaves; in other words, we do not expect that they be associated with a symmetry group of the equation. We actually expect that the symplectic leaves are exactly given by an orbit under $\mathrm{Diff}_0(D)$ and constants $c_1,\dots,c_N$ (the constraint (9) giving the remaining constant c_0).

3.7. The compressible Euler system

Arnold and Khesin [11] explain (see Chapter VI.2) that the Poisson structure used to write the incompressible Euler equation (1) as a Hamiltonian system is that of the Lie algebra of the group $\mathrm{Diff}_0(\mathbb{R}^3)$ of measure-preserving diffeomorphisms, say with compact support. The formalism therefore extends to many physically relevant equations where a finite- or infinite-dimensional Lie group plays a central role. For instance, the Euler system governing a compressible barotropic fluid

$$\partial_t\rho + \operatorname{div}(\rho u) = 0$$
$$\partial_t(\rho u) + \operatorname{div}(\rho u\otimes u) + \nabla p(\rho) = 0$$

is a Hamiltonian system, associated with the semidirect product of $\mathrm{Diff}(\mathbb{R}^d)$ and $\mathcal{C}^\infty(\mathbb{R}^d)$. The Hamiltonian is still the mechanical energy, which decomposes as the sum of the kinetic and internal parts:

$$H(\rho,\rho u) = \int_{\mathbb{R}^3}\left(\frac{1}{2}\rho|u|^2 + \Phi(\rho)\right)\mathrm{d}x$$

Here the function Φ is related to the pressure law by $\rho\Phi'(\rho)-\Phi(\rho)=p(\rho)$.

Again, the Poisson structure admits Casimirs, which are of the form

$$\int_{\mathbb{R}^{2m+1}} u\wedge(\mathrm{d}u)^m \qquad \text{or} \qquad \int_{\mathbb{R}^{2m}} f\left(\frac{(\mathrm{d}u)^m}{\rho\mathrm{d}x}\right)\rho\,\mathrm{d}x$$

according to the parity of the space dimension.

We proved in [25] that when[3] $p(\rho) = A\rho^{1+2/d}$, the compressible Euler system admits an additional conservation law

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\left(\frac{1}{2}\rho|tu-x|^2+t^2\Phi(\rho)\right)\mathrm{d}x=0$$

That one is associated with an extra dimension of the symmetry group of the system, according to Noether's Theorem. The additional symmetry resembles the so-called conformal symmetry of the nonlinear Schrödinger equation with critical exponent; in particular, it acts on the time variable as a homography. This symmetry was used in order to construct global smooth solutions to the Cauchy problem, for some initial data with finite mass and energy.

Such a global existence is the result of a competition between dispersion and nonlinearity, where a smallness assumption lets the dispersion win. This phenomenon actually occurs for a more general polytropic equation of state $p(\rho) = A\rho^\gamma$ with $\gamma > 1$, as shown by M. Grassin in her thesis [26]. When the space dimension is even, these solutions can even be *eternal*, meaning that they are defined for all $t \in \mathbb{R}$. On the contrary, when d is odd, we showed recently [27] that eternal smooth solutions whose gas is compactly supported may not exist: a singularity must develop at the boundary with vacuum.

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[3] This equation of state is that of a *monoatomic* gas.