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The legacy of Jean-Jacques Moreau in mechanics

Helicity

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ABSTRACT

This short review is a contribution to an issue of *Comptes Rendus Mécanique* commemorating the scientific work of Jean-Jacques Moreau (1923–2014). His main contribution to fluid mechanics appeared in a brief paper in the *Comptes Rendus à l'Académie des Sciences* in 1961, but was not recognised till much later. It may now be seen as a significant milestone in advancing the theory of ideal fluid flow as described by Euler's equations.

1. Introduction

The singular discovery by Jean-Jacques Moreau of the helicity invariant of the Euler equations [20][1] has its roots in the celebrated laws of vortex motion [9], which imply that in the flow of an ideal fluid with velocity field $\mathbf{v}(\mathbf{x}, t)$, any vortex line can be permanently identified with the set of Lagrangian fluid particles that lie on it, which by definition move with the flow. Moreau's conditions under which the helicity is invariant are precisely the same as those assumed by Helmholtz, namely (i) the fluid is inviscid and governed by the classical Euler equations of fluid mechanics; (ii) the flow is barotropic, i.e. the pressure p is a function of density ρ alone; and (iii) any volume force distribution $\mathbf{F}(\mathbf{x}, t)$ acting on the fluid is conservative, i.e. derivable from a potential: $\mathbf{F} = -\nabla\Phi$. Under these conditions, which also cover the important case of incompressible flow, the vorticity $\boldsymbol{\omega} = \nabla \wedge \mathbf{v}$ satisfies the equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \wedge (\mathbf{v} \wedge \boldsymbol{\omega}) \tag{1}$$

The helicity within any Lagrangian volume V_L (Moreau's 'îlot') is defined by

$$\mathcal{H}\{\boldsymbol{\omega}\} = \int_{V_L} \mathbf{v} \cdot \boldsymbol{\omega} \, \mathrm{d}V \tag{2}$$

and this is conserved, provided the normal component of vorticity $\boldsymbol{\omega} \cdot \mathbf{n}$ on the surface of V_L is zero (a condition that persists for $t > 0$ if it holds at $t = 0$). It is important to note that, in the proof of this invariance, no use is made of the relationship $\boldsymbol{\omega} = \nabla \wedge \mathbf{v}$ between $\boldsymbol{\omega}$ and $\mathbf{v}$, but merely the fact that $\boldsymbol{\omega}$ satisfies (1).

It is amazing that more than 100 years elapsed following Helmholtz's paper before Moreau's discovery. All the more amazing, given the great publicity give to Helmholtz's paper by Tait's translation of it into English [24], Kelvin's enthusiastic

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[1] I was not aware of this extremely succinct paper until 1978 when Moreau sent me a reprint. I believe the first citation of it was in [14]. According to Google Scholar, it has now been cited 145 times, still less that it deserves!

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development of his 'vortex atom' theory [26], and Tait's subsequent extensive investigations of knot structure, and his famous classification of knots of crossing number up to 10 [25].

Kelvin hoped to find stable knotted vortex structures, which might in some way represent the atomic structure of the various elements of the periodic table [11]. These hopes were dashed as it emerged that all non-trivial vortex structures investigated by Kelvin and others were subject to instabilities. It has been shown much more recently [22] that no three-dimensional vortex structure can satisfy Arnold's sufficient condition for stability [1], and this is presumably because all such flows are subject to some type of Kelvin–Helmholtz instability. For this reason, Kelvin's vortex atom theory did not survive, at least in the form in which he presented it. We may note the fine achievement of [10], who have generated knotted and linked vortices in water (visualised by air bubbles), these vortices exhibiting dramatic instabilities and reconnections.

2. The magnetohydrodynamic analogy

Moreau's result is closely related to an earlier result in magnetohydrodynamics [28], although this was not recognised till some time later [12]. The magnetic field $\mathbf{B} = \nabla \wedge \mathbf{A}$ in a perfectly conducting fluid satisfies the induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \wedge (\mathbf{v} \wedge \mathbf{B}) \tag{3}$$

and this implies that

$$\mathcal{H}\{\mathbf{B}\} = \int_{V_L} \mathbf{A} \cdot \mathbf{B} \, \mathrm{d}V = \text{cst.} \tag{4}$$

Woltjer's proof was lacking in certain details, but his conclusion was correct: magnetic helicity is indeed invariant in a perfectly conducting fluid. It was through struggling to understand the meaning of Woltjer's result that I realised that any vector field that is transported according to an equation of type (1) or (3) must have an associated helicity invariant.

There is an obvious generalisation [14] which is perhaps worth restating here: for any two solenoidal vector fields $\mathbf{M}(\mathbf{x}, t) = \nabla \wedge \mathbf{P}$ and $\mathbf{N}(\mathbf{x}, t) = \nabla \wedge \mathbf{Q}$ satisfying boundary conditions $\mathbf{n} \cdot \mathbf{M} = \mathbf{n} \cdot \mathbf{N} = 0$ on the surface of a Lagrangian volume V_L, we may define the 'mutual helicity' of the fields $\{\mathbf{M}, \mathbf{N}\}$ by

$$\mathcal{H}\{\mathbf{M}, \mathbf{N}\} = \int_{V_L} \mathbf{Q} \cdot \nabla \wedge \mathbf{P} \, \mathrm{d}V = \int_{V_L} \mathbf{P} \cdot \nabla \wedge \mathbf{Q} \, \mathrm{d}V \tag{5}$$

If these fields both satisfy the 'transport equations'

$$\frac{\partial \mathbf{M}}{\partial t} = \nabla \wedge (\mathbf{v} \wedge \mathbf{M}), \quad \frac{\partial \mathbf{N}}{\partial t} = \nabla \wedge (\mathbf{v} \wedge \mathbf{N}) \tag{6}$$

then it is straightforward to prove that $\mathcal{H}\{\mathbf{M}, \mathbf{N}\} = \text{cst.}$, and the interpretation is that this represents the net mutual linkage within V_L of the fields $\mathbf{M}$ and $\mathbf{N}$.

3. Relaxation under the constraint of conserved helicity

The conserved magnetic helicity places a lower bound on the magnetic energy $M\{\mathbf{B}^2\} = \frac{1}{2} \int \mathbf{B}^2 \, \mathrm{d}V$ of a localised magnetic field, as recognised by [2]. Suppose that the magnetic field is located in a fixed domain $\mathcal{D}$ of length-scale ℓ, with $\mathbf{n} \cdot \mathbf{B} = \mathbf{n} \cdot \mathbf{v} = 0$ on the boundary $\partial \mathcal{D}$. Then first, by the Schwarz inequality,

$$|\mathcal{H}\{\mathbf{B}\}| \le \left[\int \mathbf{A}^2 \, \mathrm{d}V \int \mathbf{B}^2 \, \mathrm{d}V \right]^{\frac{1}{2}} \tag{7}$$

the integrals being over $\mathcal{D}$. Second, we have a Poincaré inequality

$$\int \mathbf{B}^2 \, \mathrm{d}V \ge q^2 \int \mathbf{A}^2 \, \mathrm{d}V \tag{8}$$

where $q = \mathcal{O}(\ell^{-1})$ is a positive constant determined by the geometry of $\mathcal{D}$. Hence, combining these inequalities, we have the 'Arnold inequality'

$$\int \mathbf{B}^2 \, \mathrm{d}V \ge q |\mathcal{H}\{\mathbf{B}\}| \tag{9}$$

This simply means that, if $|\mathcal{H}\{\mathbf{B}\}| \ne 0$, then under any distortion of the field B by a fluid flow in $\mathcal{D}$, the magnetic energy $M\{\mathbf{B}\}$ has a positive lower bound. This result was generalised by [7] to cover fields whose helicity is zero, but which nevertheless have nontrivial topology, in the sense that there exist closed field lines which cannot be shrunk to a point in $\mathcal{D}$ without cutting other field lines. We assume such nontrivial topology in what follows.

Reduction of field energy occurs through contraction of lines of force in response to Maxwell tension in the lines of force ('B-lines') (a manifestation of the Lorentz force $\mathbf{F} = \mathbf{j} \wedge \mathbf{B}$, with $\mathbf{j} = \nabla \wedge \mathbf{B}$). Indeed, from (3), we have

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -\int_{\mathcal{D}} \mathbf{v} \cdot (\mathbf{j} \wedge \mathbf{B})\, \mathrm{d}V \tag{10}$$

Here, for simplicity, we assume incompressible flow, and we adopt a simple Darcy-type model for the dynamics:

$$k\mathbf{v} = \mathbf{j} \wedge \mathbf{B} - \nabla p \tag{11}$$

where k is a positive constant, and the pressure p is chosen to maintain the incompressibility condition $\nabla \cdot \mathbf{v} = 0$. In this way, $\mathbf{v}$ is, on average, parallel to $\mathbf{j} \wedge \mathbf{B}$, thus responding in a natural way to the Lorentz force. We then have

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -k \int_{\mathcal{D}} \mathbf{v}^2\, \mathrm{d}V + \int_{\partial\mathcal{D}} (\mathbf{v} \cdot \mathbf{n}) p\, \mathrm{d}S = -k \int_{\mathcal{D}} \mathbf{v}^2\, \mathrm{d}V \tag{12}$$

Thus, on this model, the energy decreases monotonically, and, being bounded below, must tend to a constant as $t \to \infty$. In the limit, excluding abnormal singularities of $\mathbf{v}$, we must have $\mathbf{v} \equiv 0$, and so, from (11), a magnetostatic situation is established:

$$\mathbf{j} \wedge \mathbf{B} - \nabla p = 0 \tag{13}$$

The remarkable thing here is that we can start with an arbitrary field topology at time $t = 0$, e.g. a topology containing magnetic flux tubes in arbitrarily knotted and linked configurations. Such knots and links are conserved in the relaxation process governed by (3) and (11), and we may conclude that a field exists that contains these knots and links and also satisfies the magnetostatic equation (13). As regards knots, a relationship between helicity and the twist and writhe of a knotted ribbon bounded by two adjacent B-lines was established by [18] (see also [3]); this provides an essential bridge between fluid dynamics and knot theory, of a kind that Kelvin was seeking.

Even more remarkable is the conclusion when we translate to the context of Euler flow, via the exact analogy $\mathbf{B} \longleftrightarrow \mathbf{u}$, $\mathbf{j} \longleftrightarrow \boldsymbol{\omega} = \nabla \wedge \mathbf{u}$, $p \longleftrightarrow -h$, whereby (13) translates to the equation for steady Euler flow:

$$\boldsymbol{\omega} \wedge \mathbf{u} + \nabla h = 0 \tag{14}$$

where h represents the Bernoulli function. On this basis, we may conclude, as argued by [15], that a steady Euler flow exists having arbitrarily knotted stream-tube- (rather than vortex-tube-) topology.

This argument is admittedly heuristic, but the conclusion is supported by [5], who have recently proved the existence of Beltrami flows (for which $\boldsymbol{\omega} = K\mathbf{u}$) containing arbitrarily knotted and linked stream-vortex tubes. Their fields $\mathbf{u}$ fall off to zero at ∞, but the energy integral is divergent. Nevertheless, the proof of this theorem is an outstanding achievement.

4. Helicity in the dynamo context

Considering again the energy equation (10), it is evident that if a velocity field $\mathbf{v}$ can be contrived in such a way as to be on average *opposed* to the Lorentz force, then magnetic energy will *increase*. This brings us into the domain of dynamo theory, a huge subject in which again helicity plays a key role [13]. It is in fact the chirality (i.e. lack of reflexional symmetry) of a turbulent velocity field that is relevant in dynamo theory, helicity being the simplest measure of such chirality.

An important conclusion that can be drawn from numerous studies over the past half-century is that helicity is strongly conducive to dynamo action. In fact, if the mean helicity $\mathcal{H}\{\mathbf{v}\} = <\mathbf{v} \cdot \boldsymbol{\omega}>$ is nonzero in turbulent flow of fluid of finite conductivity, and if the fluid domain is large enough, then the medium is, in nearly all circumstances, unstable to the growth of a large-scale magnetic field. This growth is exponential until the Lorentz force is strong enough to react back upon the flow, at which stage saturation is achieved. Even if the helicity level is weak and the conductivity low, this result still holds under the crucial condition that the fluid domain is large enough, a condition that is clearly satisfied in planetary, stellar and galactic contexts, though difficult to achieve in the laboratory [19]. This is a prime example of the emergence of large-scale order (i.e. the mean magnetic field) emerging from small scale chaos (the chirally turbulent velocity field).

The condition that the conductivity be *finite* is important. The case of infinite conductivity is quite singular. In this situation, stretching of B-lines certainly leads to rapid growth of mean magnetic energy. However, this is not dynamo action as normally understood, because ohmic field dissipation is entirely absent. Moreover, the field structure tends to become more and more complex as the stretching proceeds.

In dynamo theory, when the velocity field $\mathbf{v}$ is steady, it is normal to consider growing magnetic modes of the form $\mathbf{B}(\mathbf{x}, t) = \mathfrak{Re}[\hat{\mathbf{B}}(\mathbf{x}) \exp p\,t]$, and much interest attaches to the behaviour of the growth rate p as a function of magnetic

Reynolds number $R_m = V\ell/\eta$, where V and ℓ are velocity and length scales of the flow, and η is the magnetic resistivity of the fluid. On dimensional grounds, $p = (V/\ell)\mathcal{F}(R_m)$, for some function $\mathcal{F}(R_m)$ determined in principle by the flow. We have a dynamo if $\mathfrak{Re}(p) > 0$, and this is described as 'slow' or 'fast' [27] according as $\mathfrak{Re}[\mathcal{F}(R_m)] \to 0$ or cst. (> 0) as $R_m \to \infty$. Computations suggest that nearly all dynamos are, in this sense, slow.

In the infinite conductivity situation ($\eta = 0$, $R_m = \infty$), the invariant magnetic helicity again exerts a powerful constraint: for a 'normal mode' of the above form, this helicity is proportional to $\exp[2\,\mathfrak{Re}(p)]$, but this is incompatible with invariant helicity unless this helicity is zero. We may conclude that fast dynamo action, at least for analytic magnetic fields, is impossible if the magnetic helicity of the mode $\hat{\mathbf{B}}(\mathbf{x})$ is non-zero. This result remains true even if the magnetic helicity is zero, as shown by [17] who inferred that a fast dynamo magnetic field, if it exists at all, must have a very singular structure, apparently non-differentiable wherever it is non-zero. This conclusion is supported by many subsequent investigations (see, for example, [4]), which show that the transverse scale of any magnetic field growing exponentially by dynamo action is of order $R_m^{-\frac{1}{2}}$ as $R_m \to \infty$. A good example is given by the helical dynamo of [21], for which the normal modes are increasingly concentrated near the cylindrical surface of discontinuity of the assumed velocity field as $R_m \to \infty$ [8].

5. Conclusion

Jean-Jacques Moreau showed extraordinary insight in his 1961 paper proving the conservation of helicity. He was not aware of the parallel result of [28], but in some ways Moreau's result has greater significance, in that it provided a hitherto undiscovered invariant of the nonlinear Euler equations of ideal fluid motion.

In the above discussion, I have focussed on the crucial role played by helicity, whether kinetic or magnetic, in two complementary areas. The first concerns relaxation of a magnetic field to minimum energy magnetostatic equilibrium states, and the existence of such states containing arbitrarily knotted and linked structures. Here, topology is conserved during the relaxation process, as reflected in the conservation of helicity. The problem has important applications in regard to magnetic structures in the solar corona; also in the physics of thermonuclear devices such as the tokamak. It also has strong links with the theory of 'tight knots' which has attracted much attention in recent years [23], largely stimulated by problems in molecular biology [6].

The second major area in which helicity plays a central role is dynamo theory, the theory of spontaneous generation of magnetic fields by the (generally turbulent) flow of a conducting fluid. Although turbulence still presents many challenges for theoreticians, one of the great triumphs of the past half-century has been the discovery that chirality provides the key to understanding how this spontaneous growth can occur; and mean helicity provides the simplest indicator of this chirality. In large rotating self-gravitating bodies, helicity is generated by turbulent convection strongly influenced by rotation ('magnetostrophic turbulence' as described by [16]), and, in conjunction with finite conductivity, is responsible for magnetic field generation in many planetary, stellar and galactic systems.

In all, Moreau's "constantes d'un îlot tourbillonnaire" has had wider application at a fundamental level than he could have possibly anticipated!

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