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The legacy of Jean-Jacques Moreau in mechanics

On the self-similar solution to the Euler equations for an incompressible fluid in three dimensions

Solution auto-semblable des équations d'Euler pour un fluide incompressible en trois dimensions

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A R T I C L E   I N F O

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A B S T R A C T

The equations for a self-similar solution to an inviscid incompressible fluid are mapped into an integral equation that hopefully can be solved by iteration. It is argued that the exponents of the similarity are ruled by Kelvin's theorem of conservation of circulation. The end result is an iteration with a nonlinear term entering a kernel given by a 3D integral for a swirling flow, likely within reach of present-day computational power. Because of the slow decay of the similarity solution at large distances, its kinetic energy diverges, and some mathematical results excluding non-trivial solutions of the Euler equations in the self-similar case do not apply.

R É S U M É

L'existence de singularités apparaissant dans l'évolution d'un fluide incompressible au bout d'un temps fini à partir de données initiales lisses et d'énergie finie reste un problème non résolu de la mécanique théorique des fluides, qu'ils soient décrits par les équations d'Euler ou de Navier–Stokes. Cette note examine les équations proposées par Leray en 1934 décrivant une telle singularité de type auto-semblable. Le point de vue adopté consiste à transformer ces équations de Leray en un problème suffisament bien défini pour se prêter à une approche numérique directe, ce qui ne concerne donc a priori que des fonctions lisses, la singularité ayant été éliminée par les transformations de similitude. Cette approche fait apparaître une problème d'équation aux dérivées partielles dans lequel la condition asymptotique contient une fonction libre de l'angle du vecteur position spatiale, soit un vecteur variant sur la surface de la sphère unité. Une fois connue cette fonction, on peut trouver la solution formellement par un développement de Laurent en puissances inverses du rayon. Cette solution n'est pas utilisable pratiquement. En particulier, existent certaines contraintes sur le comportement asymptotique de la solution dues à l'existence de la conservation du flux d'une quantité bien définie à travers toute sphère entourant l'origine. Si ce flux n'est pas nul, la solution diverge à l'origine, ce qui est incompatible avec l'hypothèse d'une solution lisse jusqu'au moment de la singularité en temps. Il faut donc imposer à ce flux d'être nul à travers une grande sphère entourant l'origine, ce

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qui s'exprime par des conditions explicites pesant sur la fonction de l'angle définissant la condition aux limites à grande distance. Le résultat est un schéma pour une solution numérique des équations de similitude de Leray par itération d'une transformation du champ de vitesse dont le point fixe serait la solution recherchée.

1. Introduction

This note is part of a thematic issue of *Comptes rendus Mecanique* in memory of Jean-Jacques Moreau. I use below one of his outstanding results, the conservation of helicity. When, as a student, I read by chance, the note [1] shortly after it was published, it impressed me tremendously by its elegance and the fact that deep new results can be derived from well-known "equations" like the Euler ones for incompressible fluids thanks to the use of profound mathematics and topological methods getting to the core of their meaning and structure. This inspired me for the rest of my scientific life. The piece below tries modestly to follow this lesson.

This Note discusses self-similar solution(s) to the time-dependent Euler equations for an incompressible fluid in 3D. It follows a previous Note [2] on the case of axisymmetric flows and a discussion [3] of the connection between singularities of the velocity field and velocity measurements in a high-speed wind tunnel. The set of equations we start from is

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p \tag{1}$$

and

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

∂_t is for the time derivative, the vector $\mathbf{u}$ is the local value of the fluid velocity, boldface being for vectors, and p is the pressure, a free function allowing one to satisfy the condition of incompressibility (2). Moreover, the nabla sign is for the gradient with respect to coordinate $\mathbf{r}$ and the mass density has been set to 1.

Years ago, Leray wrote [4] the equations for self-similar time-dependent solutions to the above set of equations with the viscous stress added. Our study is mostly limited to the inviscid (or Euler) case. A major question in the search of a self-similar solution to the Euler equations is how to take into account the invariants of those equations. There are two sets of invariants (excluding linear and angular momentum and helicity, considered later). First we have the conservation of energy

$$\mathcal{E} = \frac{1}{2} \int \mathrm{d}\mathbf{r}\, \mathbf{u}^2$$

If this integral converges, it defines a quantity that is constant in the course of time when $\mathbf{u}$ is a solution to the Euler equations. Moreover, there are infinitely many conserved quantities, which are the integrals $\int \mathrm{d}\mathbf{s} \cdot \mathbf{u}$ along any closed line of element $\mathrm{d}\mathbf{s}$, this line being carried by the flow. This Kelvin circulation theorem, a highly non-trivial property, has important consequences in the search for a self-similar solution to the Euler equation: if a closed line is carried by the flow inside the collapsing domain, Kelvin's theorem constrains the possible exponents of the similarity solution and, as we shall see, this constraint is not compatible (for point-like singularities) with the conservation of energy inside this collapsing domain. Consider axially symmetric flows: the circles in planes perpendicular to the axis and with their center on this axis are mapped by the flow in circles with the same geometry. The conservation of circulation on such circles constrains the exponents of the similarity solution.

As did Leray [4], let us suppose there is a singularity in finite time by the evolution of this flow and that this singularity is of the self-similar type. It means that the corresponding solution to the Euler equations is of the type

$$\mathbf{u}(\mathbf{r}, t) = (t^* - t)^{-\alpha} \mathbf{U}(\mathbf{r}(t^* - t)^{-\beta})$$

where t^* is the time of the singularity (set to zero afterwards), where α and β are positive exponents to be found and where $\mathbf{U}(.)$ is to be derived by solving the Euler equations.

That such a velocity field is a solution to Euler's equations implies that $1 = \alpha + \beta$. The conservation of circulation implies $0 = \alpha - \beta$, and therefore $\alpha = \beta = 1/2$. If one imposes instead that the energy in the collapsing domain is conserved, one must satisfy the constraint $-2\alpha + 3\beta = 0$, which yields $\alpha = 3/5$ and $\beta = 2/5$, the same exponents as for the Sedov–Taylor blast, although they correspond to a completely different situation. This shows that no set of singularity exponents can satisfy both constraints of energy conservation and of constant circulation on convected closed curves. We shall choose, for a reason given later, the conservation of circulation. If $\alpha = \beta = 1/2$ (the exponents found by Leray for the case of the Navier–Stokes equations), the Reynolds number is constant inside the collapsing domain, because it is the ratio of a typical value of the circulation to the shear viscosity.

The conservation of the two Noether invariants of angular and linear momenta could lead to other constraints on the exponents. Likewise the energy, they are formally not constant if one takes the scaling exponents $\alpha = \beta = 1/2$. But, contrary to the energy, these invariants can be set to zero for a non-zero velocity field. Another possibility is to take them equal to infinity in the similarity solution. This agrees with the slow decay like $1/r$ of the velocity field at large distances from the singularity point, which makes diverge both momenta. We shall assume that, like the energy, this divergence implies that the scaling exponents of the self-similar solution are not determined by the conservation of those momenta.

As was already noticed by Leray, there is no self-similar non-trivial solution in two space dimensions, and more generally no singular solution from smooth initial data. This is because, in the inviscid case, of the pointwise conservation of vorticity. The one-dimensional case makes sense for compressible fluids only and was shown by Riemann to lead to finite-time singularities in a wide range of initial conditions, but this seems not to lead to any significant insight for the 3D incompressible case.

2. Self-similar equations and their expansion at large distances

Let us introduce as new variable $\mathbf{R} = \mathbf{r}(-t)^{-1/2}$. The Euler equations become a set of equations for $\mathbf{U}(\mathbf{R})$:

$$\frac{1}{2}(\mathbf{U} + \mathbf{R} \cdot \nabla\mathbf{U}) + \mathbf{U} \cdot \nabla\mathbf{U} = -\nabla P \tag{3}$$

$$\nabla \cdot \mathbf{U} = 0 \tag{4}$$

In both equations and later, ∇ is for the gradient with respect to $\mathbf{R}$. We shall call this set the Euler–Leray equations.

The behavior of $\mathbf{U}$ for $\mathbf{R}$ large is derived from the constraint that, as t tends to zero (namely the instant when the velocity field becomes singular at one point), the solution to the similarity equation must become independent of t at large distances, compared to the singular behavior of the diverging solution. This implies that $\mathbf{U}$ decays like $1/R$ in this limit times a vector function of $\hat{\mathbf{R}} = \frac{\mathbf{R}}{R}$. This yields the large distance behavior of the solution to equations (3) and (4): the velocity field decays like $1/R$ at large distances so that the nonlinear term and the pressure gradient are of order R^{-3} and become negligible in this limit. Notice that at the time of the singularity, the velocity field is exactly given by its "asymptotic" dependence like $1/R$ or $1/r$ in the original coordinates times a function of the angle $\hat{\mathbf{R}}$ constrained by the condition of incompressibility.

Let us put into equation (3) a vector field $\mathbf{U}(\mathbf{R}) = \frac{1}{R}\mathbf{W}_1(\hat{\mathbf{R}})$. For such a field decaying like $1/R$ at large R, the linear term on the left-hand side of equation (3) vanishes, as it should, because it is the vanishing time derivative of the large-distance behavior of the self-similar solution. However, not any field of this type is admissible because it has also to satisfy the incompressibility condition (4).

Let us sketch the Laurent expansion for R large of the solution(s) of equations (3) and (4). At leading order, $\mathbf{U}(\mathbf{R}) \approx \frac{1}{R}\mathbf{W}_1(\hat{\mathbf{R}})$. Therefore it seems natural (and this agrees with the algebra) to look for a solution with the following expansion with a sum running on odd values of the integer n:

$$\mathbf{U}(\mathbf{R}) = \sum_{n=1}^{\infty} \frac{1}{R^n}\mathbf{W}_n(\hat{\mathbf{R}}) \tag{5}$$

Putting this expansion of $\mathbf{U}$ into equation (3), one finds:

$$\frac{1}{2}\sum_{n=3}^{\infty}(1-n)\frac{1}{R^n}\mathbf{W}_n(\hat{\mathbf{R}}) + \mathbf{U} \cdot \nabla\mathbf{U} = -\nabla P \tag{6}$$

This equation is to be solved together with the incompressibility condition (4). This can be done, at least formally, by putting into the nonlinear term $\mathbf{U} \cdot \nabla\mathbf{U}$ the expansion given by equation (5). The writing of (6) order by order with respect to $1/R$ yields two sets of equations. The first set of equations is derived by taking the gradient of equation (6). The algebra is made simpler by writing all terms in this equation as infinite sums of inverse powers of R (k being even):

$$P(\mathbf{R}) = \sum_{k=2}^{\infty} \frac{1}{R^k} P_k(\hat{\mathbf{R}}) \tag{7}$$

and, with m odd integer:

$$\mathbf{U} \cdot \nabla\mathbf{U} = \sum_{m=3}^{\infty} \frac{1}{R^m}\mathbf{T}_m(\hat{\mathbf{R}}) \tag{8}$$

By identification of the coefficients of the same inverse power of R, one derives an equation for $P_k(\hat{\mathbf{R}})$ by taking the divergence of both sides of equation (6):

$$-\nabla^2(R^{-(k-1)}P_{(k-1)}(\hat{\mathbf{R}})) = \nabla \cdot \frac{1}{R^k}\mathbf{T}_k(\hat{\mathbf{R}}) \tag{9}$$

Because the power dependence with respect to R in this equation, it becomes a linear equation on the unit sphere for the unknown function $P_{(k-1)}(\hat{\mathbf{R}})$, which has to be solved at each order of the expansion, assuming $\mathbf{T}_k(\hat{\mathbf{R}})$ known.

The large-distance behavior of $\mathbf{U}$ as $1/R$ puts this kind of solution to the similarity equation outside of the range of applicability of the published mathematical results. According to Chae [5], no non-zero solution exists if the vorticity belongs to a function space L^p with p positive and close to 0. In the present case, the vorticity decays like R^{-2} at R large so that the vorticity field does not belong to L^p for p close to zero.

A general expression of the pressure P is given by the solution to a Poisson equation:

$$\nabla^2 P + \partial_i(U_j \partial_j U_i) = \nabla^2 P + (\partial_i U_j)(\partial_j U_i) = 0 \tag{10}$$

an expression where $\partial_i = \frac{\partial}{\partial R_i}$, the indices i and j being for the Cartesian coordinates with summation on like indices (no confusion should be made between those coordinate indices and the one introduced before denoting the powers of $1/R$). In an unbounded geometry like the one we consider, the Poisson equation for P can be solved as:

$$P(\mathbf{R}) = \frac{1}{4\pi} \int \mathrm{d}\mathbf{R}' \frac{1}{|\mathbf{R}' - \mathbf{R}|} M(\mathbf{R}') \tag{11}$$

where $M(\mathbf{R})$ is the scalar

$$M(\mathbf{R}) = (\partial_i U_j)(\partial_j U_i)$$

The integral in equation (11) converges at large ($\mathbf{R}'$) because $\mathbf{U}$ decays like $1/R$ and M like $1/R^4$ at large R and it converges otherwise if $\mathbf{U}$ is smooth and bounded as a function of $\mathbf{R}$.

Notice that, because of the condition of incompressibility, there is no contribution to the pressure coming from the linear term on the left-hand side of equation (3). There remains to find a recursion formula for $\mathbf{W}_n(\hat{\mathbf{R}})$ for $n > 1$. This is done by writing equation (6) as an equation for $\mathbf{W}_n$ since the other functions of the same order, namely $\mathbf{T}_m(\hat{\mathbf{R}})$ are known either by computing directly the nonlinear term or by solving Poisson equation for the pressure at the right order with respect to $1/R$. The result is:

$$\frac{1}{2}(1-n)\mathbf{W}_n(\hat{\mathbf{R}}) + \mathbf{T}_n(\hat{\mathbf{R}}) = -R^n \nabla(R^{-(n-1)} P_{(n-1)}(\hat{\mathbf{R}}))$$

This expansion in inverse powers of R should be started with the solution to the linear problem, namely by knowing one way or another the vector field $\mathbf{W}_1(\hat{\mathbf{R}})$. This field is constrained by the condition of incompressibility, which reads:

$$\nabla \cdot (\frac{1}{R}\mathbf{W}_1(\hat{\mathbf{R}})) = 0$$

To give an example satisfying this constraint, take, for instance, in Cartesian coordinates, the vector field $\frac{1}{R}W_{1,x} = \frac{y}{R^2}$ and $\frac{1}{R}W_{1,y} = -\frac{x}{R^2}$, where $W_{1,x}$ is the x-component of the vector $\mathbf{W}_1$. This vector field is divergence-free. More generally, such a divergence-less vector field is, in the present case (namely for a vector of modulus depending on the modulus of R like $1/R$), the curl of an arbitrary smooth vector field $\mathbf{A}(\hat{R})$. As shown below, $\mathbf{W}_1$ can be expressed as an integral of a solution over the whole volume. Because the first nonzero $\mathbf{T}_m$ is for $m = 3$, $\mathbf{W}_2 = 0$ and this extends to all even indices for $\mathbf{W}_m$, $\mathbf{T}_m$ and P_m. Only the odd indices yield non-zero Laurent coefficients.

Let us turn to the issue of the multiplicity of solutions to equations (3). Making a huge jump, hopefully not a fatal one, one derives from the formal expansion with respect to $1/R$ found above that, for any choice of $\mathbf{W}_1(\hat{\mathbf{R}})$ compatible with the divergence condition, there is a solution, assuming that the Laurent series with respect to $1/R$ converge. If this is correct (a highly nontrivial "if"), the manifold of solutions is very large because the only constraint on $\mathbf{W}_1(\hat{\mathbf{R}})$ is that $\mathbf{W}_1(\hat{\mathbf{R}})/R$ is divergenceless (the behavior of the solution near $R = 0$ is looked at in section 5).

To get a faint idea of a possible convergence of the solution, let us substitute to equation (3) a very simplified form keeping some of its features. Replace the velocity field $\mathbf{U}$ by a scalar function U_{si} of the radius R, and write a simple differential equation for $U_{\mathrm{si}}(R)$ imitating the structure of (3). A simple choice is:

$$U_{\mathrm{si}}(R) + R\frac{\mathrm{d}U_{\mathrm{si}}}{\mathrm{d}R} + U_{\mathrm{si}}(R)\frac{\mathrm{d}U_{\mathrm{si}}}{\mathrm{d}R} = 0$$

Obviously, this equation is very different from the original one, but it keeps the same linear term and also a non-linear term proportional to a product of $U_{\mathrm{si}}(R)$ and of its derivative $\frac{\mathrm{d}U_{\mathrm{si}}}{\mathrm{d}R}$ likewise the term $\mathbf{U} \cdot \nabla \mathbf{U}$ in the original equation. The solution to this differential equation reads:

$$R\, U_{\mathrm{si}}(R) + \frac{1}{2}U_{\mathrm{si}}^2(R) = C$$

where C is a constant. One can write $U_{\mathrm{si}}(R)$ as a root of the second-degree polynomial and expand the result in inverse powers of R thanks to a series converging in a finite domain near $1/R = 0$ on the Riemann sphere. This gives at least an idea of how to handle the convergence of the perturbation series near $1/R = 0$. The function $U_{si}(R)$ defined by the equation above has two possible asymptotic behaviors as R tends to infinity. Either $U_{si}(R) \approx C/R$ or $U_{si}(R) \approx -2R - C/R + ...$ Each behavior is attached to one of the two sheets of the Riemann surface of the functions $U_{si}(R) = -R \pm \sqrt{R^2 + 2C}$. Because R is positive, the Riemann surface associated with the C/R behavior as R tends to plus infinity is the one with the plus sign in front of the square root. If C is positive, the function $U_{si}(R)$ may be continued on the real axis from plus infinity to zero. If, on the contrary, C is negative, this continuation cannot be made because the Riemann surface has two branch points on the real axis at $R = \pm\sqrt{-2C}$, and the solution becomes complex on the cut. This elementary example shows how difficult it is to predict the behavior of the solution resulting from a Laurent expansion near $1/R = 0$. This shows also that, because of the structure of the equation, the function $\mathbf{W}_1(\hat{\mathbf{R}})$ plays the role of an integration constant, so that one expects that it can be freely chosen with the constraint that the divergence of $\mathbf{W}_1(\hat{\mathbf{R}})/R$ is zero with, presumably, some smoothness condition added. The equivalent of the integral relation between $\mathbf{W}_1$ and the whole solution given below in equation (16) becomes in this simple model the integral relation

$$C = \int_0^\infty \mathrm{d}R\, U_{si}(R) \frac{\mathrm{d}U_{si}}{\mathrm{d}R}$$

This is not an added constraint on the solution, but a consequence of its smoothness at $R = 0$ and of its behavior at R large.

3. Schema for a numerical solution

Let us outline a way of approaching the numerical search of a solution to equations (3) and (4). Taking the divergence of equation (3), one finds the Reynolds equation relating the pressure to the velocity field (equation (10)).

From this, one derives an integral equation for a solution $\mathbf{U}(\mathbf{R})$ of the similarity equations. Let us introduce the vector $\mathbf{V}(\mathbf{R})$ with Cartesian components (again no confusion should be made between the index k introduced below for the Cartesian coordinate of vectors and the indices n, m, k used previously for the order in the Taylor expansion at large R):

$$V_k(\mathbf{R}) = \partial_k P + U_j \partial_j U_k \tag{12}$$

Putting there the pressure given by equation (11), one finds $\mathbf{V}$ as a quadratic functional of $\mathbf{U}$. Once the k-component $V_k(\mathbf{R})$ is known, one may obtain U_k by solving equation (3) in spherical coordinates. This yields:

$$U_k(\mathbf{R}) = \frac{1}{R} \int_0^R \mathrm{d}R' (-2V_k(R', \hat{\mathbf{R}})) \tag{13}$$

This gives the idea of a solution to equation (3) by iteration by putting first on the right-hand side of equation (13) a smooth trial vector field $\mathbf{U}(\mathbf{R})$ satisfying the same known constraints as the solution one is looking for (incompressibility and asymptotic behavior), then computing this right-hand side and taking it as initial condition for another iteration, etc. Of course, there is no guarantee that this iteration will converge to a fixed point, but if it does one would have found a numerical solution to the similarity equation. Such a solution, if it exists, is not unique because the set of solutions to (3) has at least a continuous symmetry: if $\mathbf{U}(\mathbf{R})$ is a solution, then $\lambda\mathbf{U}(\lambda\mathbf{R})$, λ arbitrary real constant, is also a solution, the pressure being multiplied by λ^2. Therefore, in a numerical search of the solution(s?) by iteration of equation (13), one has to get rid of a possible idling between different and more or less random values of λ. Even worst, the successive iterates could either grow to infinity or tend quickly to zero: the right-hand side of equation (13) is quadratic with respect to $\mathbf{U}$ so that the iteration resembles the iteration of a quadratic map of real numbers like

$$x^{(n)} = (x^{(n-1)})^2$$

At large n, $x^{(n)}$ tends to zero if the initial value $x^{(1)}$ has modulus less than 1 and diverges to infinity if this modulus is bigger than 1. It is only if $x^{(1)}$ is ± 1 that the iterates remain uniformly bounded without decaying to zero. To avoid either diverging or decaying iterates, one must normalize, for instance to one, the iterate at each step. This normalization cannot be done with the L^2-norm of $\mathbf{U}(\mathbf{R})$: this L^2-norm is formally twice the energy of the solution and it diverges at R large because this solution decays like $1/R$ at R large. This explains why this self-similar solution does not have to conserve the energy, just because its energy diverges and is so undefined. A possible way of normalizing the solution is to take (for instance) its L^4-norm and make this norm constant, for instance equal to 1, from one step of the iteration to the next. Let us sketch the algebra for the iteration in this case. Let write as $\mathbf{U}^{(n)}$ the result of the iteration at the n-th step, $n = 1$ being the initial chosen value of $\mathbf{U}$. The iteration yields $\mathbf{U}^{(n+1)}$ once $\mathbf{U}^{(n)}$ is known. In principle, $\mathbf{U}^{(n+1)}$ is derived by computing the right-hand side of equation (13) with $\mathbf{U} = \mathbf{U}^{(n)}$, and then taking $\mathbf{U} = \mathbf{U}^{(n+1)}$ on the left-hand side. But this does not take into account the normalization of the solution (or equivalently the value of the free dilation parameter λ). To do it, one has to perform another operation before to get $\mathbf{U}^{(n+1)}$, namely chose the value of λ making the L^4-norm of the $(n+1)$-th iterate equal to 1, the prescribed value. Therefore, the result of the first operation (namely the computation of the right-hand side of equation (13)) must be followed by a step of normalization. Let $\bar{\mathbf{U}}^{(n+1)}$ be the unnormalized result of the n-th step of the iteration:

$$\bar{U}_k^{(n+1)}(\mathbf{R}) = \frac{1}{R}\int_0^R \mathrm{d}R'(-2V_k^{(n)}(R', \hat{\mathbf{R}})) \tag{14}$$

where $V_k^{(n)}$ is computed with $\mathbf{U} = \mathbf{U}^{(n)}$. Let $L^4(\mathbf{U})$ be the L^4-norm of a velocity field decaying fastly enough at infinity to make this norm finite. Even if $L^4(\mathbf{U}^{(n)}) = 1$, there is no reason that $L^4(\tilde{\mathbf{U}}^{(n+1)}) = 1$. To impose $L^4(\mathbf{U}^{(n+1)}) = 1$, one must make a dilation transform with $\lambda = \frac{1}{L^4(\bar{\mathbf{U}}^{(n+1)})}$. Therefore, the iteration is followed by another step after the computation of the right-hand side of equation (13), namely the computation of $\mathbf{U}^{(n+1)}$ from $\bar{\mathbf{U}}^{(n+1)}$ by the dilation map:

$$\mathbf{U}^{(n+1)}(\mathbf{R}) = \lambda\bar{\mathbf{U}}^{(n+1)}(\lambda\mathbf{R}) \tag{15}$$

with $\lambda = \frac{1}{L^4(\bar{\mathbf{U}}^{(n+1)})}$. Equations (14) and (15) define the iteration from step n to step $(n+1)$, which keeps constant the L^4-norm of $\mathbf{U}$.

The dilation symmetry allows us to build singular (but not at a single location in space) solutions to Euler's equation of finite energy. Because the velocity of the solution singular at one point only decays like $1/R$ at R large, the energy of this self-similar solution diverges at large distances from one location (in the original space). This divergence can be eliminated by considering a solution made of the addition of two point-wise solutions diverging at very distant points at about the same time. If the solutions at each point are mapped into each other by the dilation transform with $\lambda = -1$ their contributions at distances far bigger than the distance between the two points of singularity cancel at leading order and the velocity field decays at least like $1/R^2$ far from the two points of singularity, which makes the energy convergent.

It follows from equation (13) that the vector $\mathbf{W}_1(\hat{\mathbf{R}})$, namely the coefficient of $1/R$ in the leading-order term of the expansion of $\mathbf{U}$ at R large, is related to an integral of $\mathbf{V}$ as:

$$\mathbf{W}_1(\hat{\mathbf{R}}) = \int_0^\infty \mathrm{d}R'(-2\mathbf{V}(R', \hat{\mathbf{R}})) \tag{16}$$

This can be used to write an equation transformed of (13) in such a way that $\mathbf{W}_1$ appears explicitly by a simple algebraic manipulation that yields:

$$\mathbf{U}(\mathbf{R}) = \frac{1}{R}\left(\mathbf{W}_1(\hat{\mathbf{R}}) - \int_R^\infty \mathrm{d}R'(-2\mathbf{V}(R', \hat{\mathbf{R}}))\right) \tag{17}$$

an iteration based on this equation with a prescribed $\mathbf{W}_1$ would allow us to impose at every step the asymptotic behavior $\frac{1}{R}W_{1,k}(\hat{\mathbf{R}})$ for $U_k(\mathbf{R})$. Let us outline a possible way of doing this iteration. Each of the two possible integral relations (13) and (17) leads to obvious but different difficulties. Equation (13) does not yield a simple way to impose the boundary condition at infinity $\mathbf{U}(\mathbf{R}) \approx \frac{1}{R}\mathbf{W}_1(\hat{\mathbf{R}})$ with a prescribed $\mathbf{W}_1$. On the other hand, equation (17) leads to difficulties if the field $\mathbf{V}$ does not satisfy the condition (16): if it does not, the right-hand side diverges like $1/R$ near $R = 0$. This gives the idea to use both (13) and (17) by introducing a somewhat arbitrary smooth function $N(R)$ such that $N(R = 0) = 0$, $0 < N(R) < 1$ (this last condition is not essential). A possible choice for $N(R)$ is $N(R) = \frac{R^2}{(R^2+R_0^2)}$. If one multiplies equation (13) by $(1 - N(R))$, equation (17) by $N(R)$ and adds the results one finds a new possible relation whose solution is a fixed point of an iterative solution to the Euler–Leray equations. The iteration now is free of the two defects just pointed out: the boundary condition at R very large is explicitly in the iteration and the range of values of R close to zero is eliminated out of the right-hand side of (17) because if it is multiplied by $N(R)$, a function tending to zero as R tends to zero. The net result will be:

$$\mathbf{U}(\mathbf{R}) = \frac{N(R)}{R}\left(\mathbf{W}_1(\hat{\mathbf{R}}) - \int_R^\infty \mathrm{d}R'(-2\mathbf{V}(R', \hat{\mathbf{R}}))\right) + \frac{(1 - N(R))}{R}\int_0^R \mathrm{d}R'(-2\mathbf{V}(R', \hat{\mathbf{R}})) \tag{18}$$

It is also of interest to notice that the equations are obviously consistent with the axisymmetric geometry of a so-called swirling flow that reduces the number of components of $\mathbf{R}$ from three to two.

A successful numerical search of a fixed point of the iteration is strongly linked to the dimensionality of space where the integral on the right-hand side of equation (13) or (18) has to be performed: the bigger this dimension, likely the more difficult it is to find a fixed point. This depends on which contribution to $\mathbf{V}$ one considers. The contribution $U_j\partial_j U_k$ is just integrated once over the length of $\mathbf{R}$, which yields a one-dimensional integral. Given the velocity field $\mathbf{U}$, the contribution of the pressure gradient requires a priori integration on four dimensions: three-dimensional integration to obtain the pressure from the velocity field, plus one-dimensional integration in equation (13) or (18).

Finding an iterative solution to equation (13) or (18) is a non-trivial endeavor. From the point of view of a mathematical approach to this question, it should be observed that the number of derivatives and of integrations on the right-hand side of the equations is the same. Therefore, qualitatively, the functions obtained after each iteration step have no obvious tendency to have more and more small scale oscillations, as it would perhaps be the case if the order of derivation increases in the iteration. Moreover, because the iteration involves quadratic quantities, it is likely that, if the initial data are analytic with respect to the angle θ in the axisymmetric case, taking the square and a derivative with respect to this angle will maintain the analyticity of the function of this function of the angle (namely keep the width of the band of analyticity near the real axis) and shall not a priori increase the small-scale oscillations of the functions. In this respect, the choice of an analytic function $\mathbf{W}_1(\theta)$ will yield a smooth solution to the iteration, whereas a non-analytic $\mathbf{W}_1(\theta)$ could yield more and more irregular iterates as the iteration proceeds. In this respect, equation (17) implies that $\mathbf{U}(\mathbf{R})$ and $\mathbf{W}_1(\theta)$ should share closely related properties of analyticity as functions of θ.

4. Axisymmetric geometries

In order to make the problem more explicit, let us write equation (13) in an explicit form for an axisymmetric geometry. In order to allow the integration over the radius R (crucial for getting an iterative schema), it is more convenient to use spherical coordinates. Therefore, the space coordinates are R (the radius), the latitude angle θ in-between 0 and π and the longitude, or azimuthal, angle ϕ in-between 0 and 2π (the algebra below was done thanks to the formulae at the end of section 15 of the Landau–Lifschitz paper in *Fluid Mechanics* [6]). Although the velocity field does not depend on ϕ, it may have a non-zero azimuthal component U_ϕ. The vertical coordinate is $R\cos(\theta)$, and the radius in the horizontal plane is $R\sin(\theta)$. We need to know the scalar M, which is the divergence of the vector $K_i = U_j \partial_j U_i$. Let us write the components of $\mathbf{K}$ in spherical coordinates, the spherical components of the velocity field are $\mathbf{U} = (U_r, U_\theta, U_\phi)$ and the ones of $\mathbf{K}$ read:

$$K_r = \mathbf{U} \cdot \nabla U_r - \frac{U_\theta^2 + U_\phi^2}{R}$$

$$K_\theta = \mathbf{U} \cdot \nabla U_\theta + \frac{U_\theta U_r}{R} - \frac{U_\phi^2 \cot^2(\theta)}{R}$$

and

$$K_\phi = \mathbf{U} \cdot \nabla U_\phi + \frac{U_\phi U_r}{R} + \frac{U_\theta U_\phi \cot(\theta)}{R}$$

The symbol $\mathbf{U} \cdot \nabla G(R,\theta)$ has the meaning:

$$\mathbf{U} \cdot \nabla G = U_r \partial_R G + \frac{U_\theta}{R} \partial_\theta G$$

and $\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$.

Knowing $\mathbf{K}$, one can obtain $M = (\partial_i U_j)(\partial_j U_i)$ as a quadratic function of $\mathbf{U}$ and of its derivatives:

$$M(R,\theta) = (\partial_i U_j)(\partial_j U_i) = \nabla \cdot \mathbf{K} = \frac{\partial_R (R^2 K_r)}{R^2} + \frac{\partial_\theta (\sin(\theta) K_\theta)}{R \sin(\theta)} \tag{19}$$

Notice that K_θ as given above is well defined for $\theta = 0$ and π because both U_ϕ and U_θ vanish proportionally to $\sin^2(\theta)$ near $\theta = 0$ and π. The vector $\mathbf{V}$ can now be computed thanks to equation (12),

$$\mathbf{V} = \nabla P + \mathbf{K}$$

where P is given by equation (11). In this equation, the integration element $\mathrm{d}\mathbf{R}'$ is the standard volume element in spherical coordinates $(\mathrm{d}R' R'^2 \mathrm{d}\theta' \sin(\theta') \mathrm{d}\phi')$, and the integration over ϕ' can be done analytically because the only place where the integrand depends on ϕ' is in the denominator

$$|\mathbf{R} - \mathbf{R}'|^{-1} = (R^2 + R'^2 - 2RR'(\sin(\theta')\sin(\theta)\cos(\phi' - \phi) + \cos(\theta)\cos(\theta'))^{-1/2}$$

Therefore, the calculation of the pressure P can be reduced to a two-dimensional integral over R' and θ':

$$P(R,\theta) = \frac{1}{4\pi} \int_0^\infty \mathrm{d}R' R'^2 \int_0^\pi \mathrm{d}\theta' \sin(\theta') \frac{M(R',\theta')}{\left(R^2 + R'^2 - 2RR'\cos(\theta)\cos(\theta')\right)^{1/2}} J(\kappa) \tag{20}$$

where the function $J(\kappa)$ is defined by:

$$J(\kappa)=\int_0^{2\pi}\mathrm{d}\phi\frac{1}{(1-\kappa\cos(\phi))^{1/2}}$$

It can be expressed by means of complete elliptic integrals. The quantity denoted as κ in equation (20) is given by

$$\kappa=\frac{2RR'\sin(\theta)\sin(\theta')}{R^2+R'^2-2RR'\cos(\theta)\cos(\theta')}$$

Let us recall that the purpose of this calculation is to make as explicit as possible the expression of the right-hand side of equation (13).

Let us consider the three components of $\mathbf{U}$ in spherical coordinates and write equation (13) fully explicitly. The component U_ϕ is found quite easily because of the lack of contribution of the pressure gradient. It reads:

$$U_\phi=-\frac{2}{R}\int_0^R\mathrm{d}R'\left(\mathbf{U}\cdot\nabla U_\phi+\frac{U_\phi U_r+U_\theta U_\phi\cot(\theta)}{R'}\right)\tag{21}$$

Notice that the argument θ is the same on the left- and right-hand sides of this equation. The radial component U_r can be written as:

$$U_r=-\frac{2}{R}(P(R,\theta)-P(R=0))-\frac{2}{R}\int_0^R\mathrm{d}R'\left(\mathbf{U}\cdot\nabla U_r-\frac{U_\theta^2+U_\phi^2}{R}\right)\tag{22}$$

where $P(R,\theta)$ is given by a two-dimensional integral written explicitly in equation (20) although

$$P(R=0)=\frac{1}{2}\int_0^\infty\mathrm{d}R'R'\int_0^\pi\mathrm{d}\theta'\sin(\theta')M(R',\theta')$$

Moreover, as in the case of U_ϕ, the angle θ is the same on the left and in the second integral on the right, whereas $P(R,\theta)$ is computed by an integral on the angle θ'.

Lastly there remains to consider the component U_θ.

$$U_\theta=-\frac{2}{R}\int_0^R\frac{\mathrm{d}R'}{R'}\partial_\theta P(R',\theta)-\frac{2}{R}\int_0^R\mathrm{d}R'\left(\mathbf{U}\cdot\nabla U_\theta+\frac{U_\theta U_r}{R}-\frac{U_\phi^2\cot^2(\theta)}{R}\right)\tag{23}$$

The first term on the right-hand side of equation (23) is the only one in the three equations for the iteration requiring a triple integration, because the pressure itself requires a double integration. Moreover, this integral converges for R' small because the pressure becomes independent on θ in the limit $R'=0$.

A core issue when dealing with solutions to the Euler equations for incompressible fluids is the smoothness (or lack of) of the solutions. Without pretending to bring anything new to this very difficult problem, we shall make a few remarks on this question in the present setting. The "physical" reason for this difficulty lies in the property that steady solutions to Euler's equation may have what is called a tangential discontinuity, namely a solution such that – for instance – (in Cartesian coordinates), the component u_x of the velocity can be any function, smooth or not of the variable y, including a function showing a finite jump at $y=0$, like for instance $u_x(y)=H(y)$ where $H(.)$ is Heaviside function, zero for $y<0$ and equal to one for $0<y$. This tangential discontinuity is a steady solution if the other Cartesian components of the velocity are zero for $y=0$. Let us see if solutions to equation (3) could have such a tangential discontinuity. Somehow, and very roughly speaking, the integral equations (21), (22) and (23) have a structure going against a tangential discontinuity because such a discontinuity will yield no contribution to the $\mathbf{U}\cdot\nabla\mathbf{U}$ because it is zero for such a tangential discontinuity. The other contributions are likely regularized by the integrations, either in the equations as they stand or when computing the integrals giving the pressure. Hopefully, a numerical search of solutions to this iteration will bring some light to this question.

5. Invariants of the Euler dynamics and the Euler–Leray equations

Let us look at the relationship between the invariants of the Euler equations and the solutions to the Euler–Leray equations. There are a number of invariants (namely of time independent quantities) defined by space integrals of functions of the velocity field and its time derivatives. Those invariants are the energy, the linear and angular momentum and the helicity [1]. The Euler equation can be put in a form emphasizing that it is a conservation relation:

$$\partial_t u_i+\partial_j\Pi_{ij}=0\tag{24}$$

where $\Pi_{ij} = u_i u_j + p\delta_{ij}$, δ_{ij} being the Kronecker discrete delta. The Euler–Leray equations (3) and (4) can also be written in a conservative form:

$$\partial_j \left(\frac{1}{2}(R_j U_i - 2R_i U_j) + U_i U_j + P\delta_{ij} \right) = 0 \tag{25}$$

This property of the Euler–Leray equations puts a constraint on the asymptotic behavior of their solutions and their behavior at $R = 0$. Let us introduce the non-symmetric tensor

$$\Sigma_{ij} = \frac{1}{2}(R_j U_i - 2R_i U_j) + U_i U_j + P\delta_{ij}$$

From equation (25), the flux of Σ_{ij} across a sphere with its center at $R = 0$ is independent of the radius of the sphere. Assuming the pressure to be smooth at $R = 0$, the only possibility for this flux to be constant across a small sphere surrounding the origin $R = 0$ is to have U diverging like $1/R$ near the origin, which yields a constant non-zero contribution arising from both terms proportional to RU and $U_i U_j$ in Σ. Of course, we do not look for solutions with such a diverging U at $R = 0$, because this would make the self-similar solution singular at all times, whereas we are looking for a solution becoming singular at time zero and which is smooth otherwise.

In the case considered by Leray, namely the self-similar solutions to the Navier–Stokes equations (with viscosity), one can also put the equation for the momentum in the divergence form

$$\partial_j \Sigma_{ij}^{\mathrm{NS}} = 0$$

with

$$\Sigma_{ij}^{\mathrm{NS}} = \frac{1}{2}(R_j U_i - 2R_i U_j) + U_i U_j + P\delta_{ij} - \nu \partial_j U_i$$

ν being the kinematic viscosity.

This puts a constraint on the behavior of Σ_{ij} at large R and lastly on the "initial" condition $\mathbf{W}_1$, which must be such that the flux of Σ_{ij} across a large sphere is zero. The leading-order contribution (for R large) to this flux comes from the term $\frac{1}{2}(R_j U_i - 2R_i U_j)$ in Σ_{ij}. It yields a contribution to the flux formally of order R^2 at large R, which must be made equal to zero. This is done by taking $\mathbf{W}_1$ such that

$$\int \mathrm{d}\hat{R} \left(2\hat{R}_i \hat{R}_j W_{1,j} - W_{1,i} \right) = 0 \tag{26}$$

In this compact expression, $\hat{R}_i$ is for the i-component of $\frac{\mathbf{R}}{R}$ and the integration is carried out on the surface of the unit sphere. Notice that this makes a set of three conditions, one for each of the possible values of the index i. There does not seem to be an impossibility of satisfying those constraints. For instance, one can take an angular dependence of $\mathbf{W}_1$ such it has no contribution from low-order spherical harmonics. Because the product $\hat{R}_i \hat{R}_j$ can be written with low-order spherical harmonics, the integral in (26) vanishes because of the orthogonality of the spherical harmonics of different indices.

There is also a finite (independent of R) contribution to the flux across a large sphere coming from the non-linear term, from the pressure, and from the first correction to the contribution of the term linear with respect to U in Σ_{ij}. Such contributions to a constant flux come from the first correction to the part of Σ_{ij} that is linear with respect to U. We just canceled the largest contribution at R large (of order R^2) to the flux of Σ_{ij}. The next-order contribution to the flux across a large sphere coming from the term of order R^{-3} in U at R large is a constant and the same is true of the terms coming from the nonlinear term $U_i U_j$ in Σ_{ij} and from the pressure P, which both decay like $1/R^2$ at R large. This makes a host of contributions to a constant flux of Σ_{ij} across a large sphere, the result being to be set to zero to get rid of a divergence of U at small R by the conservation of the flux of Σ_{ij} across any sphere surrounding the center, small or large. The calculation of this constant contribution to the flux of Σ_{ij} is quite cumbersome and is left to a more detailed publication. One of the reasons is that the pressure contributes too and is to be computed at order $1/R^2$. This requires to solve a linear partial differential equation on the unit sphere, derived from Poisson's equation for functions behaving with a power law as functions of R. For $\mathbf{W}_1$ given, this is in principle straightforward, but very cumbersome, the end result being a set of conditions given by the cancellation of integrals on the unit sphere of functions quadratic with respect to $\mathbf{W}_1$.

This constraint can be written by using the expansion in inverse powers of P and $\mathbf{U}$ given formally in section 2. Let $\Sigma_{ij,2}$ be the coefficient of R^{-2} in the Laurent expansion of Σ_{ij} in inverse powers of R. The tensor $\Sigma_{ij,2}$ is a function of $\hat{R}$. The condition of vanishing flux across a sphere at large distance reads:

$$\int \mathrm{d}\hat{R} \hat{R}_i \Sigma_{ij,2} = 0 \tag{27}$$

From its definition, the tensor $\Sigma_{ij,2}$ is related to $\mathbf{U}_1$, $\mathbf{U}_3$ and P_2 as follows:

$$\Sigma_{ij,2} = \frac{1}{2}(\hat{R}_i U_{j,3} - 2\hat{R}_j U_{i,3}) + U_{i,1} U_{j,1} + P_2 \delta_{ij})$$

where $U_{j,1} = W_{1,j}$, where P_2 is the solution to the Poisson equation (9) at order $k = 3$.

It is of interest also to know if the other conservation relations (energy, angular momentum, and helicity) can be put in the same divergence form as the conservation of linear momentum for the Euler–Leray equations. The case of the conservation of energy can be easily dealt with because the conservation of energy is derived by taking the scalar product of the original Euler equation (1) with $\mathbf{u}$ and writing $\mathbf{u} \cdot \nabla p$ and $\mathbf{u} \cdot (\mathbf{u} \cdot \nabla \mathbf{u})$ as a divergence. Doing the same with the Euler–Leray equation (namely taking its scalar product with $\mathbf{U}$), one is left with a piece $\mathbf{U} \cdot \nabla P + \mathbf{U} \cdot (\mathbf{U} \cdot \nabla \mathbf{U})$ that can be written, like it can in the original Euler equation, like a divergence, plus the piece $\frac{\mathbf{U}}{2} \cdot (\mathbf{U} + \mathbf{R} \cdot \nabla \mathbf{U})$. This expression cannot be written in a divergence form. If it could, it should be the divergence of a vector of Cartesian components Q_k, which is necessarily of the form $Q_k = aR_k U_i U_i + bR_i U_i U_k$, a and b being numerical coefficients. The divergence of this vector $\partial_k Q_k$ should be equal to $U_i(U_i + R_j \nabla_j U_j)$, which is impossible because one has three incompatible conditions $3a + b = 1$, $b = 0$ and $2a = 1$ to satisfy in order to identify the divergence of $\mathbf{Q}$ with $\frac{\mathbf{U}}{2} \cdot (\mathbf{U} + \mathbf{R} \cdot \nabla \mathbf{U})$.

One can do something quite similar in the case of angular momentum and of helicity. The conservation of angular momentum is derived by multiplying Euler's equation (1) by $\mathbf{R} \times \nabla$ and showing that the result can be written as

$$\partial_t (\mathbf{u} \times \mathbf{r})_j + \partial_i \Theta_{ij} = 0$$

This works for the term $(\mathbf{u} \cdot \nabla \mathbf{u} + \nabla p)$ in equation (1). It shows that the contribution of the same term in the Euler–Leray equation can also be written in a divergence form. There remains, as for the energy, to look at the part coming from the part of the Euler–Leray equation that is linear with respect to U. Therefore, to write that the angular momentum can be written in a divergent form for this equation, one should write $\mathbf{R} \times (\mathbf{U} + \mathbf{R} \cdot \nabla \mathbf{U})$ as a divergence form, namely like $\partial_j \Theta'_{ij}$. Because this divergence has to be linear with respect to $\mathbf{R}$ and $\mathbf{U}$, the tensor Θ' must be quadratic with respect to $\mathbf{R}$ and linear with respect to $\mathbf{U}$. The only possibility compatible with the invariance with respect to the frame is $\Theta'_{ij} = R_i (\mathbf{R} \times \mathbf{U})_j$. This expression of the tensor Θ' cannot yield the sought expression for $\partial_j \Theta'_{ij}$ because there remains after the derivative ∂_j is taken the combination $e_{ikl} R_j R_l \partial_i U_k$, e_{ikl} being the fully antisymmetric Levi-Civita tensor. This second-order tensor is not in the expression of $\mathbf{R} \times (\mathbf{U} + \mathbf{R} \cdot \nabla \mathbf{U})$. Therefore the conservation of angular momentum cannot be written in a divergence form from the Leray–Euler equations. It can be shown that this is related also to the property that the tensor Σ_{ij} is not symmetric.

The conservation of helicity by solutions to the Euler equations [1] yields a divergence equation for the Euler–Leray equation. This was pointed out to this author by an anonymous referee and the derivation given below is due to Jean Ginibre.

Let us introduce the vorticity $\omega = \nabla \times \mathbf{u}$. The helicity is the pseudo scalar $\omega \cdot \mathbf{u}$, and its space integral is a constant in time if $\mathbf{u}$ is a solution to Euler's equation in 3D. The flux of helicity is $\mathbf{u}(\omega \cdot \mathbf{u})$. Therefore, we look for the divergence of this flux, namely $\nabla \cdot (\mathbf{u}(\omega \cdot \mathbf{u}))$. After some algebra, one finds:

$$\nabla \cdot (\mathbf{u}(\ \omega \cdot \mathbf{u})) = 2((\omega \cdot \mathbf{K} + \mathbf{u} \cdot (\nabla \times \mathbf{K}))$$

where $\mathbf{K} = (\mathbf{u} \cdot \nabla)\mathbf{u}$. Let us take now $\mathbf{v} = \mathbf{u} + \mathbf{R}/2$. If $\mathbf{u}$ is a solution to the Euler–Leray equations, then $(\mathbf{v} \cdot \nabla)\mathbf{v}) + \nabla P = 0$ where P is the pressure, as it appears in the Euler–Leray equations. From this, one derives

$$\nabla \cdot (\mathbf{v}(\omega \cdot \mathbf{v})) = -2(\omega \cdot \nabla P) = -2\nabla(\omega P)$$

whence

$$\nabla \cdot (\mathbf{v}(\omega \cdot \mathbf{v}) + 2\omega P - \frac{1}{4}(x^2 + y^2 + z^2)\omega) = 0$$

Returning to the divergenceless field $\mathbf{u}$, one finds:

$$\nabla \cdot (\mathbf{u}(\omega \cdot \mathbf{u}) + 2\omega P) = 0$$

This is the divergence condition derived from the conservation of helicity. In numerical computations, it is easier to impose that the fluxes of momentum (the tensor denoted Σ_{ij} and the just found flux of helicity) across a sphere surrounding the origin are zero by symmetry. This can be done by taking a velocity field that is odd by the inversion symmetry $(x, y, z) \to (-x, -y, -z)$. In this case, both fluxes across a large sphere vanish.

The same trick can be used for the other conservation relation: by imposing a symmetry to the flux at large distance, one can make its total contribution to the flux across a large sphere equal to zero. There is a special case for the obvious conservation (of mass in physical term) derived from the incompressibility condition. Indeed the flux of the velocity across a large sphere must be equal to zero. If one chooses to eliminate the flux of Σ_{ij} across such a large sphere by taking a velocity field that is odd under the inversion symmetry, this does not insure that the flux of mass is zero, because this flux is, like the velocity field, odd with respect to the inversion. However, one can still cancel this flux by taking the x-component of the velocity field at large distance behaving like $\frac{x(y^2 - z^2)}{R^4}$, and the other component derived by permutation. This makes an odd velocity field at large distance, but its flux across a large sphere is zero because the contribution of this x-component is the integral of $x^2(y^2 - z^2)$ on a large sphere, which vanishes because of a compensation between the contribution of $x^2 y^2$ and of $x^2 z^2$. The some happens for the other Cartesian components of the velocity field. To summarize, one can cancel all contributions of fluxes across a large sphere by choosing convenient boundary conditions at large distances for the velocity field.

6. Limit cases

Even though the equations we are dealing with result from a number of transformations, it could be worthwhile to continue along the same path, namely to try to find situations where the equations can be still made simpler and hopefully solved one way or another. The two simplifications we have found rely on assumptions about the geometry of the solution we are looking for. This makes sense because, at least heuristically, we have shown that there should be a large manifold of solutions so that some of them may involve special limits making simpler the equations. We looked at axisymmetric solutions such that the velocity field is concentrated near $\theta = 0$. In spherical coordinates, because of factors depending on the angle θ through circular functions like $\sin(\theta)$ or $\cot(\theta)$ either diverging or tending to zero as θ tends to zero, the equation for the iteration becomes simpler and can be analyzed by asymptotic methods in the limit where the velocity field is concentrated in a narrow angular domain near $\theta = 0$ and $\theta = \pi$. The width of this angular domain is a small number called δ. This amounts to assume that, in the manifold of solutions to the iteration, a continuum exists with this kind of dependence with respect to θ. Such solutions would correspond to a choice of $\mathbf{W}_1(\hat{\mathbf{R}})$ like

$$\mathbf{W}_1(\hat{\mathbf{R}}) = (w_{1r}, w_{1\theta}, w_{1\phi})$$

with the radial component of $\mathbf{W}_1$ given by

$$w_{1r}(\theta) = f(\theta/\delta)$$

where $f(\cdot)$ is a numerical function like (for instance) $f(v) = \mathrm{e}^{-v^2}$. The θ-component of $\mathbf{W}_1$ is

$$w_{1\theta}(\theta) = -\theta f(\theta/\delta)$$

The next step is to find the kind of approximation that can be made in this limit δ small. It amounts to write that a given quantity Q has a scaling law with respect to δ, that is that it can be written like $Q = \delta^{\gamma}\tilde{Q}$, where $\tilde{Q}$ is of order one as θ/δ is finite, whereas γ is an exponent depending on Q. Exponents like γ are derived from the equations by imposing that, once the right set of exponents is found, the equations can be written as numerical equations without small or large parameter.

The most obvious exponent is attached to θ: assuming that θ is of order δ is equivalent to say that $\theta = \tilde{\theta}\delta$ with $\tilde{\theta}$ of order one (the neighborhood of $\theta = \pi$ can be dealt with thanks to symmetries). The next step in the search of the exponents γ is less obvious, and relies on trial-and-error. The values we have found are as follows. First one finds for R and U_r the scaling exponent $\gamma = 0$. There remains to find the exponents associated with U_θ and U_ϕ. The exponent associated with U_θ is $\gamma = 1$, as it results from the incompressibility condition. The one associated with U_ϕ is derived by looking at equation (23): if one wants to have interaction between U_θ and U_ϕ, one needs to have in this equation the last term $\frac{U_\phi^2 \cot^2(\theta)}{R}$ of the same order of magnitude as the others, namely of the same order as U_θ that is δ. Because $\cot^2(\theta)$ is of order $1/\delta^2$, this imposes that the exponent γ for U_ϕ is 3/2.

Summarizing, one finds the following set of equations for $\hat{U}_r$, $\hat{U}_\theta$ and $\hat{U}_\phi$:

$$\tilde{U}_\phi = -\frac{2}{R}\int_0^R \frac{\mathrm{d}R'}{R'}\left(R'\tilde{\mathbf{U}}\cdot\nabla\tilde{U}_\phi + \tilde{U}_\phi\hat{U}_r + \frac{\tilde{U}_\theta\tilde{U}_\phi}{\tilde{\theta}}\right) \tag{28}$$

$$\tilde{U}_r = -\frac{2}{R}(\tilde{P}(R,\tilde{\theta}) - \tilde{P}(R=0)) - \frac{2}{R}\int_0^R \mathrm{d}R'\,\tilde{\mathbf{U}}\cdot\nabla\tilde{U}_r \tag{29}$$

and

$$\tilde{U}_\theta = -\frac{2}{R}\int_0^R \mathrm{d}R'\frac{\partial_\theta\tilde{P}(R',\tilde{\theta})}{R'} - \frac{2}{R}\int_0^R \mathrm{d}R'\left(\tilde{\mathbf{U}}\cdot\nabla\tilde{U}_\theta + \frac{\tilde{U}_\theta\tilde{U}_r}{R'} - \frac{\tilde{U}_\phi^2}{\tilde{\theta}^2 R'}\right) \tag{30}$$

In the equations above,

$$\tilde{\mathbf{U}}\cdot\nabla\tilde{U} = \tilde{U}_r\partial_R\tilde{U} + \frac{1}{R}\tilde{U}_\theta\partial_{\tilde{\theta}}\tilde{U}$$

There remains to find how the pressure $P(R,\theta)$ depends on θ for θ small, in order to find how pressure enters the scaled equations. Because of the symmetry around the axis, the pressure is there (namely for $\theta = 0$) at an extremum. Therefore, the difference $(\hat{P}(R,\hat{\theta}) - \hat{P}(R=0))$ is a quadratic function of θ near $\theta = 0$, so that the first term on the right-hand side of equation (29) is, up to a small correction of order δ^2, independent of θ and equal to $-\frac{2}{R}(\tilde{P}(R,\tilde{\theta}=0) - \tilde{P}(R=0))$, a function of R only. Therefore, at leading order, equation (29) becomes:

$$\tilde{U}_r = -\frac{2}{R}(\hat{P}(R,\tilde{\theta}=0) - \tilde{P}(R=0)) - \frac{2}{R}\int_0^R \mathrm{d}R' \, \tilde{\mathbf{U}} \cdot \nabla \tilde{U}_r \tag{31}$$

The term depending on the pressure on the right-hand side of (30) is of order δ because P is at a quadratic extremum at $\theta = 0$, so that its first derivative with respect to θ is linear with respect to θ and therefore of order δ, as all terms in (30).

In order to take into account the condition of incompressibility, one may introduce a stream function $\Psi(R,\theta)$ such as

$$U_r = -\frac{1}{R^2 \sin(\theta)} \partial_\theta \Psi(R,\theta) \approx -\frac{1}{R^2\theta} \partial_\theta \Psi(R,\theta)$$

and

$$U_\theta = \frac{1}{R \sin(\theta)} \partial_R \Psi(R,\theta) \approx \frac{1}{R\theta} \partial_R \Psi(R,\theta)$$

This stream function may help to build a divergenceless smooth velocity field with the requested asymptotic properties at large R for the initial step of an iterative solution.

Another possible simplification of the equations is found in the limit of the "large quantum numbers" of the solution, namely when the angular dependence of $\mathbf{W}_1(\hat{\mathbf{R}})$ is such that there are many oscillations of this function of the angle on the unit sphere. If one decomposes this angular dependence in spherical harmonics, something we shall not do explicitly because it would add very little to the practical calculations, only the high index spherical harmonics would have a non-zero amplitude. Because of the incompressibility condition, both the dependence with respect to θ and ϕ must be short ranged on the surface of the unit sphere. Let σ be this range (a small dimensionless number). As we did in the derivation of the equations in the limit δ small for a swirling flow (the parameter δ playing the same role as the present σ), we have to find first the equations valid at leading order in the limit σ small. This is done again by trial-and-error. The end result is a set of equations derived from the full set given initially, but with various terms absent because of their dependence with respect to σ, making them of a higher order with respect to σ. The order of magnitude with respect to σ (small) of the various quantities involved is as follows:

- $\theta \sim \sigma$ and $\phi \sim \sigma$. This must not be understood as meaning that the full range of variation of those two angles is small, of order σ, but that any function of θ and ϕ is a rapidly varying function of both angles, namely a function of θ/σ and ϕ/σ, whereas the range of variation of the two angles is finite, $(0, +\pi)$ for θ and $(0, 2\pi)$ for ϕ;
- $U_\theta \sim U_\phi \sim \sigma$;
- $R \sim \sigma^0$;
- $U_r \sim P \sim \sigma^2$.

Keeping only the leading order terms in the Euler–Leray equations written in spherical coordinates, one finds:

$$U_\theta + R\partial_R U_\theta + \mathbf{U} \cdot \nabla U_\theta = -\frac{1}{R}\partial_\theta P \tag{32}$$

$$U_\phi + R\partial_R U_\phi + \mathbf{U} \cdot \nabla U_\phi = -\frac{1}{R\sin(\theta)}\partial_\phi P \tag{33}$$

$$\partial_\theta(\sin(\theta)U_\theta) + \partial_\phi U_\phi = 0 \tag{34}$$

In the first two equations, the symbol $\mathbf{U} \cdot \nabla f(\theta, \phi)$ is such that:

$$\mathbf{U} \cdot \nabla f = \frac{U_\theta}{R}\partial_\theta f + \frac{U_\phi}{R\sin(\theta)}\partial_\phi f$$

Because the vector field $\mathbf{U}$ changes rapidly as a function of θ and ϕ, one can consider in the equation above $\sin(\theta)$ as a constant compared to the fast variations of $\mathbf{U}$. This allows us to manipulate the equations by taking $\sin(\theta)$ as a constant whilst keeping the validity of those equations. This allows us to eliminate the function $\sin(\theta)$ from the equations by rescaling the various functions and their argument. As one can check, this is done by taking as new variable ϕ the rescaled $\phi_{\mathrm{res}} = \phi/\sin^2(\theta)$ and $U_{\phi,res} = U_\phi \sin(\theta)$, whilst the other functions U_θ and P are left unchanged. Doing this, one gets rid of the factors depending explicitly on θ in the equations. They become the Euler–Leray equations of a 2D flow in a flat domain in the plane $(\theta, \phi_{\mathrm{res}})$.

7. Self-similar problem in Lagrange coordinates

Another point of interest is the underlying structure of the Euler–Leray equations for an inviscid incompressible flow. As is well known, the Euler equations can be derived from a Lagrange variational principle, that is, their solution makes an extremum of a functional. As is well known too, this requires the use of Lagrange variables that yield very clumsy equations because, in particular, of the incompressibility condition. Nevertheless, it is always interesting and sometimes helpful to connect a problem with its variational formulation.

Let ξ be the position of a fluid particle at time t. This is a function of time t and of the position at $t = t_0$ fixed $\mathbf{x} = \xi(t = t_0)$, the trajectory being defined for a given Eulerian velocity field $\mathbf{u}$ by the equation:

$$\partial_t \xi = \mathbf{u}(\xi, t) \tag{35}$$

The density of kinetic energy in the Lagrange functional is $\rho \frac{(\partial_t \xi)^2}{2}$, where ρ is the constant mass density of the fluid, which we set to one. The other contribution to the Lagrange function (namely the integrand of the Lagrange functional) is to express the constraint of incompressibility. This constraint is expressed thanks to the 3×3 Jacobi determinant $\mathrm{Det}(\partial_k \xi_l)$ made of the first derivatives of ξ with respect to the coordinate of the initial position $\mathbf{x}$ (the symbol $(\partial_k \xi_l)$ in the argument of the determinant is to recall that it is made of the first derivatives of ξ, no particular meaning must be given to the coordinate index k and l there). The incompressibility is equivalent to impose that this determinant is always and everywhere equal to 1. The Lagrange condition that this determinant takes the value 1 is imposed by adding to the Lagrange function a contribution $q(\mathbf{x}, t)\mathrm{Det}(\partial_k \xi_l)$, where $q(\mathbf{x})$ is the Lagrange multiplier. Therefore, the Lagrange functional to be minimized by the equation of motion reads:

$$\mathcal{S} = \int \mathrm{d}t \int \mathrm{d}\mathbf{x}\, \mathcal{L} \tag{36}$$

The integrand, or Lagrange function, reads:

$$\mathcal{L} = \frac{(\partial_t \xi)^2}{2} + q(\mathbf{x}, t)\mathrm{Det}(\partial_k \xi_l) \tag{37}$$

The equation of motion is derived by imposing that the variation of $\mathcal{S}$ by a small arbitrary change of ξ cancel at the linear order with respect to this variation. Because of the occurrence of the cubic (in 3D) function $\mathrm{Det}(\partial_k \xi_l)$ in the Lagrange function, the Lagrange dynamical equations are quite complicated and not very helpful as far as the search of explicit solutions is concerned. Nevertheless, it s not necessary to write them in full to obtain the Lagrange function from which the Euler–Leray equation can be derived. Let us write the dynamical Lagrange equation as:

$$\partial_t^2 \xi_i + \partial_j \left(q(\mathbf{x}, t) \frac{\delta \mathrm{Det}(\partial_k \xi_l)}{\delta(\partial_j \xi_i)} \right) = 0 \tag{38}$$

an equation where the summation on like indices runs on the indices i and j and where $\frac{\delta \mathrm{Det}(\partial_k \xi_l)}{\delta(\partial_j \xi_i)}$ is for the derivative of the determinant with respect to its entry $(\partial_j \xi_i)$. Notice that the second term on the left-hand side is quadratic with respect to the derivatives $(\partial_k \xi_l)$. Moreover, one has to add to this equation the incompressibility condition

$$\mathrm{Det}(\partial_k \xi_l) = 1$$

a condition satisfied thanks to the Lagrange multiplier $q(\mathbf{x})$. The Lagrange equation (38) is often written by introducing the pressure $p = \left(q(\mathbf{x}, t) \frac{\delta \mathrm{Det}(\partial_k \xi_l)}{\delta(\partial_j \xi_i)} \right)$, a change of function hiding the connection between the added term and the fact that it is a Lagrange condition, whereas the formal simplification due to this introduction of the pressure leaves the nonlinearity of the incompressibility condition anyway. Taking the curl of equation (38) allows us to recover the Cauchy invariants because the second derivative with respect to time of the curl of ξ is zero.

Self-similar solutions are such $\xi = (-t)^{1/2}\Xi(X)$ with $X = x(-t)^{-1/2}$ so that $\mathrm{Det}(\partial_k \xi_l)$ is independent of time. Therefore, the Lagrange function becomes, after the Jacobian of the transformation is incorporated:

$$\mathcal{L} = (-t)^{1/2} \left(\frac{1}{2}(\Xi - X\partial_X \Xi)^2 + Q(\mathbf{X})\mathrm{Det}(\partial_k \Xi_l) \right)$$

Putting this into the Lagrange functional one finds that, by variation on functions depending on the rescaled position $\mathbf{X}$, the equation for the self-similar problem reads:

$$(2 - 3X\partial_X - X^2 \partial_X^2)\Xi_i + \partial_j \left(Q(\mathbf{X}) \frac{\delta \mathrm{Det}(\partial_k \Xi_l)}{\delta(\partial_j \Xi_i)} \right) = 0 \tag{39}$$

with the incompressibility condition

$$\mathrm{Det}(\partial_k \Xi_l) = 1$$

All this being derived from the condition that the Lagrange functional

$$\int \mathrm{d}\mathbf{X} \left(\frac{1}{2} (\Xi - X \partial_X \Xi)^2 + Q(\mathbf{X}) \mathrm{Det}(\partial_k \Xi_l) \right)$$

is stationary under small changes of Ξ, constrained by the incompressibility condition.

8. Conclusion and perspectives

Even though the existence of solution(s) to the Euler–Leray equations has been the subject of many studies over the years, it does not seem to have been attacked along the path suggested above, namely by trying to formulate it in a way amenable to direct numerical studies. Such direct numerical studies short-cut in principle one of the major difficulties in detecting the occurrence of singularities in direct solutions to the time-dependent Euler equations: such singularities, if they ever exist, show up at more or less random locations in space and time and are very rapidly varying there, which makes a very challenging numerical problem because of the finite resolution to any numerical schema. Things are made even worst for functions changing very rapidly at the scale of the grid. In this respect, the simplification brought by Leray's assumption of self-similarity is very helpful, because it maps this problem of singular solutions to the Euler equations into the search of a smooth non-trivial solution to a set of equations with space variables only. Of course, a serious drawback remains, namely how such a self-similar solution can be connected to a "general" (or "generic") solution to the Euler equation far from the singularity, and which is a priori different than the self-similar solution. One expects that self-similar solutions accommodate a wide enough range of solutions at large distances to be matchable with a big enough class of asymptotics to be relevant. Another relevant topic, perhaps closer to real life situations, is the effect of an added viscosity. Many formal calculations done in this Note can be extended without difficulty to the original Leray equations with the viscosity. This includes the existence of a divergence form of the self-similar equation, and so a constraint for the equivalent of $\mathbf{W}_1$, since the viscosity does not change formally the algebra there. The main effect of the viscosity is to introduce a dimensionless number, the ratio of the circulation to the viscosity, this parameter being a kind of Reynolds number of the collapsing solution. Besides those remarks, we can say very little on the case with viscosity. One could only conjecture that if no solution exists without viscosity, none exists with viscosity.

As last remarks, let us notice that, if a singularity of Euler–Leray (or Navier–Stokes Leray) equations exists, physics tells us that it has to be "regularized" one way or another. If the singularity exists for Euler, but not Navier–Stokes equations, it is natural to expect that it is regularized by viscosity. Another possibility of regularization is by the compressibility effects: supposing the velocity to diverge at singularity, this implies immediately that the Mach number becomes large there, therefore that the assumption of incompressibility, implying a small Mach number, breaks down. What compressibility does to singularities of the self similar type is likely a non-trivial problem. However, there is a possible physical effect of even weak compressible effects: the large acceleration in the collapsing domain should be a source of sound waves that could perhaps be recorded as a specific source of noise in the high-Reynolds-number turbulent flows.

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