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The legacy of Jean-Jacques Moreau in mechanics

Recovering convexity in non-associated plasticity

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ABSTRACT

We quickly review two main non-associated plasticity models, the Armstrong–Frederick model of nonlinear kinematic hardening and the Drucker–Prager cap model. Non-associativity is commonly thought to preclude any kind of variational formulation, be it in a Hencky-type (static) setting, or when considering a quasi-static evolution because non-associativity destroys convexity. We demonstrate that such an opinion is misguided: associativity (and convexity) can be restored at the expense of the introduction of state variable-dependent dissipation potentials.

1. Introductory remarks

Although small-strain Prandtl–Von Mises elastoplasticity is hardly a practical, or realistic model for elastoplastic evolutions, it is undoubtedly the theoretical template that any macroscopic departure from elasticity must elaborate upon when some kind of slip-friction mechanism is involved, be it at the mesoscopic scale as is the case for granular materials, or at the microscopic scale as in metals.

For a homogeneous elastoplastic material occupying a volume $\Omega \subset \mathbb{R}^3$, with Hooke's law (elasticity tensor) A subject to a time-dependent loading process with, say, $f(t)$ as body loads, $g(t)$ as surface loads on a part Γ_s of $\partial\Omega$, and $w(t)$ as displacement loads (hard device) on the complementary part Γ_d of $\partial\Omega$, the quasi-static model reads as detailed below.

- Kinematic compatibility: $Eu(t) := 1/2(\nabla u(t) + \nabla u^{\mathrm{T}}(t) = e(t) + p(t)$ in Ω and $u(t) = w(t)$ on Γ_d. Here $u(t)$ is the displacement field at t and $e(t)$ and $p(t)$ (a deviatoric symmetric matrix, that is an element of the space $\mathbb{M}^{3\times 3}_D$ of 3×3-trace free symmetric matrices) are the elastic and plastic strains at t.
- Equilibrium: $\operatorname{div}\sigma(t) + f(t) = 0$ in Ω and $\sigma(t)\nu = g(t)$ on Γ_s, where ν denotes the outer unit normal to $\partial\Omega$. Here $\sigma(t)$ is the Cauchy stress tensor at time t.
- Hooke's law: $\sigma(t) = Ae(t)$ in Ω.
- Stress constraint: $\sigma_D(t) \in K$ [1] and K is the admissible set of stresses (a convex and compact subset of symmetric deviatoric $n \times n$ matrices that contains 0 in its interior).
- Normality rule: $\dot{p}(t) = 0$ if $\sigma_D(t) \in \operatorname{int} K$, while $\dot{p}(t)$ belongs to the normal cone to K at $\sigma_D(t)$ if $\sigma_D(t) \in \partial K$.

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[1] Here and in all that follows the subscript $_D$ affixed to a symmetric matrix denotes the projection of that matrix onto the subspace of trace free matrices, while the dot above any time-dependent quantity refers to the time derivative of that quantity.

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The set of equations described above is actually attuned to the behavior of metals, or more generally, of crystalline solids, because the deviatoric character of the plastic strain is then viewed as the macroscopic manifestation of dislocations. When dealing with materials such as rock, sand or concrete, it is more appropriate to view the set of admissible stresses K as just a convex, possibly unbounded set of symmetric $n \times n$ matrices, because plastic dilations are actually evidenced in such materials.

In any case, this model has been a seemingly inexhaustible source of mechanical as well as mathematical exegeses, starting with the pioneering work of Jean-Jacques Moreau, who recognized and formalized the fundamental role that convexity and duality should play in any rational handling of elastoplasticity.

Unfortunately, in plasticity as elsewhere, complexity soon prevails.

When it comes to metals, it first rears its head in the form of hardening, the realization that the convex K is not a fixed set, but rather that it evolves with the history of the loading process. For metals, it usually grows "larger", making the metal "harder". From a mathematical standpoint, the resulting complexification is paradoxically beneficial because, be it for isotropic or kinematic hardening (for which the convex K actually translates with the plastic strain), the functional analytic environment of the new model is more classical. There is no need to resort to spaces where the symmetrized gradient is a measure and the evolution runs in classical Sobolev spaces; see, e.g., [1, Section 4] in the case of linear isotropic hardening.

The introduction of hardening is soon seen to be insufficient to fully explain well-documented effects like the Bauschinger effect, which shows a reduction in admissible compressive strength after a material is pulled in tension. Indeed linear kinematic hardening fails to produce the observed smooth transition from elastic unloading to plastic hardening in the compressive part of the hysteresis diagram [2, Figure 2(a)]. To account for such a behavior, P.J. Armstrong and C.O. Frederick [3] proposed a kinematic hardening model for which the normality rule does not hold. That model, later modified and popularized most notably by J.-L. Chaboche (see, e.g., [4]), has become the fulcrum of a slew of evermore sophisticated models that attempt to account for the many quirks of metal behavior under cyclic loading.

When it comes to dirtier materials such as soils or concrete, it shows up in the misfit between the angle of friction φ, which determines the shape of the set of admissible stresses – a cone with the hydrostatic stress as the axis, a positive hydrostatic stress as the summit (corresponding to maximal three-axial traction) and aperture $\arctan(\sin\varphi)$ – and the angle of dilatancy ψ, which corresponds to the aperture of a cone of the same type with the direction of the plastic flow[2] as the normal. It is usually such that $\varphi > \psi$ [5], and then the normality rule no longer holds. Such is the case for the Drucker–Prager-, or Mohr–Coulomb-type models that are widely used in soil or rock mechanics [6,7].

As noted, the striking feature shared by the Armstrong–Frederick-type models and the Drucker–Prager- or Mohr–Coulomb-type models is the failure of normality. In the terminology of elastoplasticity, they are *non-associated* models. The associated mathematical pathology is a lack of convexity. Indeed, classically, that is in the case of perfect plasticity, the enforcement of the Clausius–Duhem inequality – itself a rewriting of the second law of thermodynamics – imposes that

$$\sigma_{(D)}(t) \cdot \dot{p}(t) \geq 0 \tag{1.1}$$

throughout the evolution. This is precisely what normality achieves, since the normality rule equivalently reads as $\dot{p}(t) \in \partial I_K(\sigma_{(D)}(t))$, where I_K is the indicatrix function of the set K, so that, by elementary convex analysis and since $0 \in \operatorname{int} K$, (1.1) holds.[3] Alternatively, one may view normality as equivalent to Drucker–Ilyushin's postulate [8,9].[4] In the absence of normality, inequality (1.1) comes as a restriction on the possible K's and flow rules. Note that, since $\varphi > \psi$ Clausius–Duhem is automatically satisfied in the Drucker–Prager-, or Mohr–Coulomb-type models, at least when $\psi \geq 0$.[5]

In any case, popular wisdom has it that non-associated flow rules result in a system of equations that cannot fit within any kind of variational framework. Such a vague statement cannot be accepted at face value. It could be interpreted as the inability to derive the analogue of a Hencky plasticity-type variational formulation. In a more modern setting, it could also be seen as the inability to formulate a variational evolution similar to that in [10]. We will not go down that route and refer the interested reader to [10] for an exposition of the variational evolution in perfect elastoplasticity and to [11–13] for its extension to the non-associative framework.

Rather, we will simply demonstrate here that non-associated flow rules can be rendered associative at the expense of the addition of a state variable in the dissipation potential. Once this is done, it should be intuitively clear that both Hencky and evolutionary plasticity can be framed variationally. Of course, the mathematical hurdles might prove overwhelming, but this is not the story that we wish to tell here.

The first section will be devoted to the Armstrong–Frederick model introduced in [3], while the second section will be focussing on the Drucker–Prager and Mohr–Coulomb models [7]. The corresponding detailed works are [13] and [12]. Other non-associated models could be treated similarly. In particular, the reader is referred to [14,11,15] for the case of Cam–Clay plasticity.

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2 $\sin\psi$ is also the ratio of the volumetric to axial strain rates in a triaxial test [5].

3 Notationwise, if $x \in \mathbb{R}^N \mapsto g(x) \in \mathbb{R} \cup \infty$ is a convex function, we denote throughout by $\partial g(x)$ the sub-differential set of g at x.

4 As an aside, while normality is equivalent to Ilyushin's principle, it is strictly more restrictive than Clausius–Duhem's inequality, as demonstrated in [9].

5 The case $\psi < 0$, while worth investigating if thinking of, e.g., loose sands, is not within the purview of the present study, because it leads to softening. Softening, which forces a drastic departure from convexity, lies beyond the reach of the method expounded in this work.

2. Non-linear kinematic hardening – the Armstrong–Frederick model

We first recall, for the reader's ease of reading, the main ingredients of the Armstrong–Frederick model. As in classical perfect elastoplasticity, the plastic strain p belongs to $\mathbb{M}^{3\times3}_D$, the set of symmetric trace-free matrices, and the constitutive equation, which relates the stress tensor σ to the elastic part e of the strain, is Hooke's law.

An additional kinematic hardening variable $\alpha \in \mathbb{M}^{3\times3}_D$ is also introduced. The back stress $\chi \in \mathbb{M}^{3\times3}_D$ is then related to that variable and to the plastic strain through

$$\chi = B(p-\alpha), \tag{2.1}$$

where B acts on deviatoric matrices.

We then introduce the internal energy

$$W(e, p-\alpha) := 1/2 Ae\cdot e + 1/2 B(p-\alpha)\cdot(p-\alpha).$$

Viewing W as a function of Eu, p, α it also reads as

$$\hat{W}(Eu, p, \alpha) = W(Eu-p, p-\alpha).$$

The thermodynamic force $-\dfrac{\partial \hat{W}}{\partial p} = \sigma_D - \chi$ associated with p is constrained to remain in the following subset K of the set $\mathbb{M}^{3\times3}_D$:

$$\sigma_D - \chi \in K := \{\tau \in \mathbb{M}^{3\times3}_D : f(\tau) \le 0\}, \tag{2.2}$$

where $f : \mathbb{M}^{3\times3}_D \to \mathbb{R}$ is a yield function. It is assumed that

$$\begin{cases} f(\tau) = \tilde{f}(\tau) - \sigma_c,\ \sigma_c > 0 \\ \tilde{f} \ge 0 \text{ is Lipschitz, convex and positively one-homogeneous.} \end{cases}$$

In particular, K is compact, convex, and $0 \in \operatorname{int} K$.

Now, the thermodynamic force associated with α is

$$-\frac{\partial \hat{W}}{\partial \alpha} = \chi$$

given by (2.1).

The Armstrong–Frederick hardening model is characterized by the following flow rule:

$$\begin{cases} \dot{p} = \lambda\nu,\ \ \nu \in \partial I_K(\sigma_D - \chi) \\ \dot{\alpha} = \lambda\chi \end{cases} \tag{2.3}$$

with $\lambda \ge 0$, $\lambda = 0$ if $f(\sigma_D - \chi) < 0$. Here I_K is the indicatrix function of the set K defined in (2.2).

If we define in turn[6]

$$\mathcal{A}(\sigma_D, \chi) := \left\{ \lambda \begin{bmatrix} \nu \\ \hline \chi \end{bmatrix} : \nu \in \partial I_K(\sigma_D - \chi),\ \lambda \ge 0,\ \lambda = 0 \text{ if } f(\sigma_D - \chi) < 0 \right\},$$

then (2.3) reads as

$$\begin{bmatrix} \dot{p} \\ \hline \dot{\alpha} \end{bmatrix}(x,t) \in \mathcal{A}(\sigma_D(x,t), \chi(x,t)). \tag{2.4}$$

In the possibly most familiar setting of a Von Mises criterion, that is when $f(\tau) := |\tau| - \sigma_c$,[7] then (2.4) reads as

$$\dot{p} \parallel (\sigma_D - \chi), \quad \dot{\chi} + |\dot{p}| B\chi = B\dot{p} \tag{2.5}$$

which is the classical flow rule for the Armstrong–Frederick model.[8]

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6 In this section, $\begin{bmatrix} \cdots \\ \hline \cdots \end{bmatrix}$ denotes an element of $(\mathbb{M}^{3\times3}_D)^2$.

7 The usual expression is $\sqrt{2/3}\sigma_c$ where σ_c is the yield stress; we drop the coefficient $\sqrt{2/3}$ in all that follows for simplicity.

8 Remark that, in its full generality, that model allows for a law of the form $\dot{\chi} + |\dot{p}| B\chi = C\dot{p}$ with $B \ne C$ in lieu of the second equation in (2.5). This is not covered by our analysis.

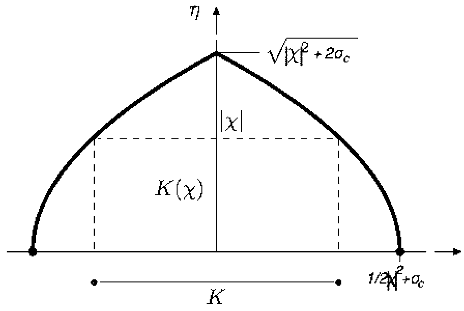

Fig. 1. The set $K(\chi)$ for $f(\tau)=|\tau|-\sigma_c$. The horizontal axis represents $\mathbb{M}_D^{3\times3}$.

So the local relations that preside over the model are

$$\left.\begin{array}{ll} \sigma(x,t)=Ae(x,t),\ \chi(x,t)=B(p(x,t)-\alpha(x,t)) & \text{(constitutive relations)} \\ \sigma_D(x,t)-\chi(x,t)\in K & \text{(stress constraint)} \\ \begin{bmatrix}\dot p \\ \dot\alpha\end{bmatrix}(x,t)\in\mathcal{A}(\sigma_D(x,t),\chi(x,t)) & \text{(flow rule).} \end{array}\right\} \tag{2.6}$$

The system (2.6) is not associative, since the flow rule is non-associated because of the second component of (2.4).

We now propose to rewrite the stress constraint and the flow rule in an associative way. To this effect, we introduce, for any $\chi\in\mathbb{M}_D^{3\times3}$, the set

$$K(\chi):=\left\{\begin{bmatrix}\tau \\ \eta\end{bmatrix}\in\mathbb{M}_D^{3\times3}\times\mathbb{M}_D^{3\times3}: f(\tau)+\frac{1}{2}|\eta|^2\le\frac{1}{2}|\chi|^2\right\} \tag{2.7}$$

(see Fig. 1). The following statements are immediate:

(1) the stress constraint $f(\sigma_D-\chi)\le 0$ is satisfied iff $\begin{bmatrix}\sigma_D-\chi \\ \chi\end{bmatrix}\in K(\chi)$;

(2) $\sigma_D-\chi\in\partial K$ iff $\begin{bmatrix}\sigma_D-\chi \\ \chi\end{bmatrix}\in\partial K(\chi)$ (the topological boundary of $K(\chi)$);

(3) the flow rule is satisfied iff $\begin{bmatrix}\dot p \\ \dot\alpha\end{bmatrix}\in\partial I_{K(\chi)}\begin{bmatrix}\sigma_D-\chi \\ \chi\end{bmatrix}$, where $I_{K(\chi)}$ is the indicatrix function of the set $K(\chi)$.

Note that doing so introduces a quadratic term in the definition of the convex of plasticity and that it is essential that the convex be defined by adding (up to a constant) a one-homogeneous function ($\tilde f$) to a quadratic function of χ. So do not, in the Von Mises case for example, replace $f(\tau)=|\tau|-\sigma_c$ by $\hat f(\tau)=|\tau|^2-\sigma_c^2$, even if $f(\tau)\le 0$ iff $\hat f(\tau)\le 0$.

The associated dissipation potential is the Legendre transform of the indicatrix function $I_{K(\chi)}$, that is,

$$H(\chi,\dot p,\dot\alpha):=\sup\left\{\begin{bmatrix}\dot p \\ \dot\alpha\end{bmatrix}.\begin{bmatrix}\tau \\ \eta\end{bmatrix}:\begin{bmatrix}\tau \\ \eta\end{bmatrix}\in K(\chi)\right\},$$

and, as such, it is one-homogeneous in $\begin{bmatrix}\dot p \\ \dot\alpha\end{bmatrix}$. So, equivalently, item (3) above can be written as

$$\begin{bmatrix}\sigma_D-\chi \\ \chi\end{bmatrix}\in\partial H\left(\chi,\begin{bmatrix}\dot p \\ \dot\alpha\end{bmatrix}\right),$$

where $\partial H\left(\chi,\begin{bmatrix}\dot p \\ \dot\alpha\end{bmatrix}\right)$ is the sub-differential of $H(\chi,\cdot)$ at the point $\begin{bmatrix}\dot p \\ \dot\alpha\end{bmatrix}$.

Thus, we are led to investigating the following local relations:

$$\left.\begin{array}{l} \sigma(x,t)=Ae(x,t),\ \chi(x,t)=B(p(x,t)-\alpha(x,t)) \\ \begin{bmatrix}\sigma_D-\chi \\ \chi\end{bmatrix}(x,t)\in\partial H\left(\chi(x,t),\begin{bmatrix}\dot p(x,t) \\ \dot\alpha(x,t)\end{bmatrix}\right). \end{array}\right\} \tag{2.8}$$

Note that there is no need to write the stress constraint, because it is implicitly contained in system (2.8). Indeed, by Hahn–Banach's theorem,

$$\left[\frac{\sigma_D - \chi}{\chi}\right](x,t) \in K(\chi(x,t))$$

is equivalent to

$$\left[\frac{\sigma_D - \chi}{\chi}\right](x,t) \in \partial H\left(\chi(x,t), \left[\frac{0}{0}\right]\right).$$

But, since H is one-homogeneous,

$$\partial H\left(\chi(x,t), \left[\frac{\dot p(x,t)}{\dot\alpha(x,t)}\right]\right) \subset \partial H\left(\chi(x,t), \left[\frac{0}{0}\right]\right).$$

In conclusion, we have rendered the model associative at the expense of having an explicit dependence of the dissipation potential upon the back-stress χ.[9] Of course, this is just the start of a long story because one should then prove existence of an evolution. Doing so is a rather complicated process which does require some amount of tweaking [13]... In other words, we have succeeded in imparting a variational structure to the Armstrong–Frederick model. For example, the equivalent of Hencky's plasticity would consist in looking to minimize

$$\int_\Omega \{1/2A(Eu-p)\cdot(Eu-p) + 1/2B(p-\alpha)\cdot(p-\alpha) + H\left(B(p-\alpha), \left[\frac{p}{\alpha}\right]\right) - f\cdot u\}\,\mathrm{d}x$$

among all triplets (u, p, α) with u satisfying the proper boundary conditions and p, α $\mathbb{M}^{3\times3}_D$-valued.

3. The Drucker–Prager and Mohr–Coulomb cap models

Once again, we recall, for the reader's convenience, the main ingredients of the Drucker–Prager model.

In contrast with the case presented in the previous section, there is no additional internal variable here. The (Cauchy) stress is constrained to remain in a closed convex set K of the set $\mathbb{M}^{3\times3}_{\text{sym}}$ of 3×3 symmetric matrices:

$$\sigma \in K := \{\tau \in \mathbb{M}^{3\times3}_{\text{sym}} : f(\tau) \le 0\},$$

where $f : \mathbb{M}^{3\times3}_{\text{sym}} \to \mathbb{R}$ is continuous and convex, which implies that K is closed and convex, and, assuming further that $f(0) < 0$, $0 \in \operatorname{int} K$.

The behavior of the plastic strain is governed by a non-associative flow rule. Specifically, denoting by $\dot p$ the time derivative of p,

$$\dot p \in \mathcal{A}\sigma, \tag{3.1}$$

where, according to [17], $\mathcal{A}\sigma = \{0\}$ if $\sigma \in \operatorname{int} K$, while $\mathcal{A}\sigma$ is the exterior normal cone at σ to the boundary of the sub-level set $\{\tau \in \mathbb{M}^{3\times3}_{\text{sym}} : g(\tau) \le g(\sigma)\}$ of g if $\sigma \in \partial K$. Note that if $\sigma \in \partial K$ is not a minimum point of g, then we have from [18, Corollary 23.7.1]

$$\mathcal{A}\sigma = \{\lambda\xi : \lambda \ge 0 \text{ and } \xi \in \partial g(\sigma)\}.$$

In the previous expression, $g : \mathbb{M}^{3\times3}_{\text{sym}} \to \mathbb{R}$ is the plastic potential, a continuous and convex function.

This time, the local relations that preside over the model are

$$\left.\begin{array}{ll} \sigma(x,t) = Ae(x,t) & \text{(Hooke's law)} \\ \sigma(x,t) \in K & \text{(stress constraint)} \\ \dot p(x,t) \in \mathcal{A}\sigma(x,t) & \text{(flow rule)}. \end{array}\right\} \tag{3.2}$$

[9] In all fairness, we are not first in re-associating the Armstrong–Frederick model in this manner. Unbeknownst to us, a similar "trick" had been used (in a viscoplastic context) to derive an associated viscoplastic version of Armstrong–Frederick in [16, Section IV.3]. We thank P. Suquet for pointing this out to us.

Here f and g are of the form

$$\begin{aligned} f(\sigma) &= \kappa(\sigma_D) + \mathrm{tr}\sigma \ \sin\varphi - 2c\cos\varphi, \\ g(\sigma) &= \kappa(\sigma_D) + \mathrm{tr}\sigma \ \sin\psi - 2c\cos\psi, \end{aligned} \tag{3.3}$$

where the parameters φ (the friction angle), ψ (the dilatancy angle), and c (the cohesion) satisfy $0 < \psi < \varphi < \frac{\pi}{2}$, $c > 0$, and $\kappa : \mathbb{M}^{3\times3}_D \to [0, +\infty)$ is a convex, positively 1-homogeneous function on $\mathbb{M}^{3\times3}_D$ such that $\kappa(0) = 0$. The Drucker–Prager model corresponds to

$$\kappa(\sigma^D) = \sqrt{\frac{1}{6}\sum_{i<j}(\sigma^D_i - \sigma^D_j)^2} = \frac{1}{\sqrt{2}}|\sigma^D|,$$

while the Mohr–Coulomb model corresponds to

$$\kappa(\sigma^D) = \max_{i,j}\{\sigma^D_i - \sigma^D_j\},$$

where σ^D_i, $i = 1, 2, 3$, are the ordered eigenvalues of σ^D.

The system (3.2) is not associative, since the flow rule is non-associated because of the flow rule (3.1). Further, this model allows for infinitely large hydrostatic compressions. It is known that pressure-dependent models such as those considered here are bad predictors of the volumetric strain response. To remedy this, a model that closes the yield surface was first proposed in [19]. This is the cap model as developed in, e.g., [6,7]. It consists in cutting the domain K by a new surface (the cap) so as to obtain a bounded set while keeping an associative flow rule on that new surface.[10]

We do not introduce the cap at this juncture but rather proceed toward a reformulation of this problem in a more suitable form. Eventually, we aim at producing an associated system. The process is more involved than in the Armstrong–Frederick case.

First, we demonstrate below that f and g defined in (3.3) can be taken to be distance functions without altering the formulation. To this effect, we set

$$G := \{\sigma \in \mathbb{M}^{3\times3}_{\mathrm{sym}} : g(\sigma) \le 0\}, \quad \hat{g}(\sigma) := \mathrm{dist}(\sigma, \partial G),$$

where $\mathrm{dist}(\cdot, \partial G)$ denotes the signed distance from ∂G in the $(\mathrm{tr}\,\sigma, \sigma^D)$-plane. Then

$$g = \sqrt{1 + \sin^2\psi}\,\hat{g}$$

on a neighborhood of K, since, in the $(\mathrm{tr}\,\sigma, \sigma^D)$-plane, the set K is a cone with vertex $\bar{\sigma}^m_K := (2c\cot\varphi, 0)$ and aperture $\arctan(\sin\varphi)$, while the 0-level set of g is a cone with vertex $\bar{\sigma}^m_G := (2c\cot\psi, 0)$ and aperture $\arctan(\sin\psi)$, and further $2c\cot\psi > 2c\cot\varphi$ by the assumptions on φ and ψ. Therefore,

$$\bigcup_{\lambda\ge0}\lambda\,\partial g(\sigma) = \bigcup_{\lambda\ge0}\lambda\,\partial\hat{g}(\sigma)$$

for every $\sigma \in \partial K$ and we can replace g by $\hat{g}$ without changing the problem. Analogously, if we introduce

$$\hat{f}(\sigma) = \mathrm{dist}(\sigma, \partial K),$$

K coincides with the set $\{\sigma \in \mathbb{M}^{3\times3}_{\mathrm{sym}} : \hat{f}(\sigma) \le 0\}$, so that we can replace f by $\hat{f}$ without changing the problem. Also note that the functions $\hat{f}$ and $\hat{g}$ are still convex, since the signed distance from the boundary of a convex set is a convex function (see, e.g., [20, Chapter 7, Theorem 10.1]). We thus replace f and g with $\hat{f}$ and $\hat{g}$ and keep the notation f and g.

In a second step, we introduce the simple cap(s) $\mathrm{tr}\,\tau = -R$, $R > 0$, resulting in two sets

$$\hat{K} := K \cap \{\tau \in \mathbb{M}^{3\times3}_{\mathrm{sym}}; \mathrm{tr}\,\tau \ge -R\}, \ \hat{G} := G \cap \{\tau \in \mathbb{M}^{3\times3}_{\mathrm{sym}}; \mathrm{tr}\,\tau \ge -R\} \tag{3.4}$$

(see Fig. 2) and the functions

$$\hat{f} := \mathrm{dist}(\cdot, \partial\hat{K}), \quad \hat{g} := \mathrm{dist}(\cdot, \partial\hat{G}). \tag{3.5}$$

We define as cap model the system with yield surface given by $\hat{f}$ and flow rule given by $\hat{g}$. The set $\{\sigma \in \mathbb{M}^{3\times3}_{\mathrm{sym}} : \hat{f}(\sigma) \le 0\}$ coincides with $\hat{K}$. Moreover, $\partial\hat{g}$ coincides with ∂g on $\partial\hat{K} \cap \partial K$, except for a small region close to the cap, and it is pointing in the same directions as $\partial\hat{f}$ on $\partial\hat{K} \cap \partial\hat{G} \cap \{\sigma^m = -R\}$.

[10] In truth, there are subtleties in the choice of the correct cap that we will gleefully ignore in the sequel, because they have a very minor impact on the resulting model.

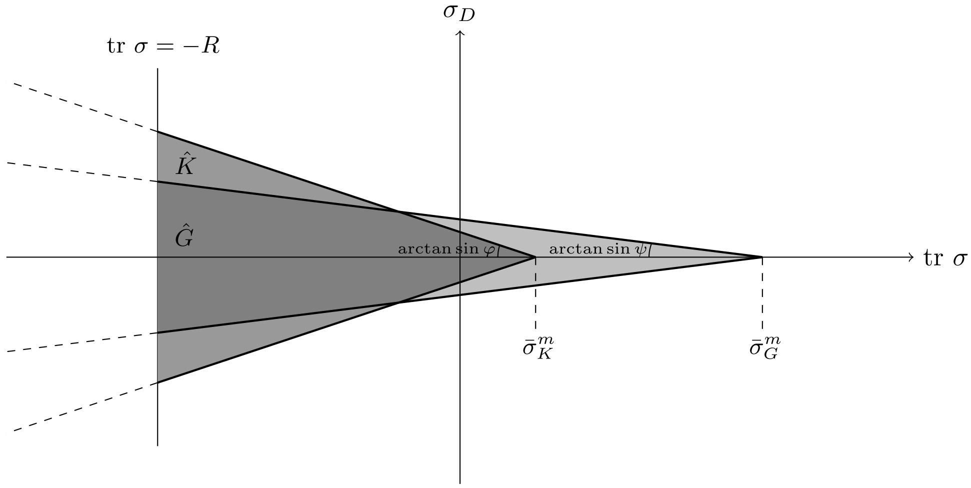

Fig. 2. The sets $\hat{K}$ and $\hat{G}$.

At this point, we appeal to a largely ignored result of P. Laborde [21, Proposition 4] (see also [22]), which is instrumental in the success of our derivation.

Lemma 3.1. *Consider a continuous function* $h : \mathbb{M}^{3\times 3}_{\text{sym}} \to \mathbb{R}$ *satisfying*

$$\begin{cases} h(\tau) = g(\tau) & \text{if} \quad f(\tau) = 0, \\ h(\tau) > g(\tau) & \text{if} \quad f(\tau) < 0, \\ h(\tau) < g(\tau) & \text{if} \quad f(\tau) > 0. \end{cases} \tag{3.6}$$

Define, for every $\sigma \in \mathbb{M}^{3\times 3}_{\text{sym}}$, *the closed and convex set*

$$K(\sigma) := \{\tau \in \mathbb{M}^{3\times 3}_{\text{sym}} : g(\tau) \le h(\sigma)\}.$$

Then, $\sigma \in K$ *if and only if* $\sigma \in K(\sigma)$, *and, in this case,* $\dot{p} \in \mathcal{A}\sigma$ *if and only if* $\dot{p} : (\tau - \sigma) \le 0$ *for any* $\tau \in K(\sigma)$. *Since* $K(\sigma)$ *is a closed and convex set in* $\mathbb{M}^{3\times 3}_{\text{sym}}$, *the latter property can be expressed as* $\dot{p} \in \partial I_{K(\sigma)}(\sigma)$.

Since $\hat{g}(0) < 0$, the sublevel set $\hat{G}_{\hat{g}(0)} := \{\tau \in \mathbb{M}^{3\times 3}_{\text{sym}} : \ \hat{g}(\tau) \le \hat{g}(0)\}$ is a cone with vertex at 0 and aperture $\arctan\psi$ cut by the plane $\sigma^m = -R - \hat{g}(0)$. In particular, for R sufficiently large, the set $\hat{G}_{\hat{g}(0)}$ is contained in the interior of $\hat{K}$. Hence, by continuity of g there exists $\delta > 0$ small enough so that, setting $\lambda := \hat{g}(0) + 2\delta$, we have

$$\hat{G}_\lambda \subset \operatorname{int} \hat{K}. \tag{3.7}$$

We now use Laborde's formulation, and we define

$$h(\sigma) := \min\left\{ \max\{\hat{g}(\sigma), \lambda\}, \, \mathcal{G} \right\} - \min\{\hat{f}(\sigma), \delta\}$$

for

$$\mathcal{G} := \max_{\hat{K}} \hat{g} + 1.$$

Since $\hat{K}$ is closed and bounded, the maximum of $\hat{g}$ on $\hat{K}$ exists and, using (3.7), h satisfies (3.6). Introducing

$$K(\sigma) := \{\tau \in \mathbb{M}^{3\times 3}_{\text{sym}} : g(\tau) \le h(\sigma)\}, \quad H(\sigma, p) := \sup_{\tau \in K(\sigma)} \tau \cdot p, \tag{3.8}$$

we end up with the following associated formulation of the Drucker–Prager (or Mohr–Coulomb) cap model:

$$\left.\begin{aligned} &\sigma(x,t) = Ae(x,t) \\ &\sigma(x,t) \in \partial H(\sigma(x,t), \dot{p}(x,t)). \end{aligned}\right\} \tag{3.9}$$

In (3.9), the set $K(\sigma)$ is defined as in (3.8).

In conclusion, here again, we have rendered the model associative at the expense of having an explicit dependence of the dissipation potential upon the stress σ. Of course, here again, this is just the start of a long story, because one should then prove the existence of an evolution. This is, here again, a rather complicated process, which also does require the kind of tweaking that was evoked in the previous section [12]... In other words, we have succeeded in imparting a variational structure to the Drucker–Prager and Mohr–Coulomb cap models.

4. Concluding remarks

As we have demonstrated on two examples, the variational template for non-associativity consists in defining in a correct way a state variable-dependent dissipation potential, or equivalently, a state-dependent yield domain. Of course, the correct definition may turn out to be quite involved, as demonstrated in the previous section.

We shied away from any mathematical considerations thus far. We should, however, note that this dependence creates a major mathematical hurdle in attempting to prove any kind of existence theorem, both in a static or in an evolutionary setting.

Up to now, all mathematical results have modified the model through a regularization of the dependence of the dissipation potential upon the state variable. That regularization is usually taken to be a convolution. In other words, the last item in, e.g., (2.8) or (3.9) is replaced by

$$\sigma(x,t) \in \partial H(\rho * v(x,t), \dot{p}(x,t))$$

where v stands for the relevant state variable and ρ is a regularizing convolution kernel.

There is, however, no major obstacle in implementing the associated reformulation without regularization in a numerical scheme, a worthy task that would require skills that are thoroughly beyond our abilities.

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