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The legacy of Jean-Jacques Moreau in mechanics

Moreau's hydrodynamic helicity and the life of vortex knots and links

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ABSTRACT

This contribution to an issue of *Comptes rendus Mécanique*, commemorating the scientific work of Jean-Jacques Moreau (1923–2014), is intended to give a brief overview of recent developments in the study of helicity dynamics in real fluids and an outlook on the growing legacy of Moreau's work. Moreau's discovery of the conservation of hydrodynamic helicity, presented in an article in the *Comptes rendus de l'Académie des sciences* in 1961, was not recognized until long after it was published. This seminal contribution is gaining a new life now that modern developments allow the study of helicity and topology in fields and is having a growing impact on diverse areas of physics.

Conservation laws, such as conservation of energy, underpin our most fundamental understanding of physical phenomena. The equations of fluid motion are a prime example: themselves an expression of conservation of momentum, mass and energy, they capture the myriad of fluid motions we see in the world. It is thus a rare privilege indeed to discover an additional conservation law. Jean-Jacques Moreau achieved just this in his seminal work of 1961 [1], by showing that within the Euler equations of fluid motion there lurked an additional conservation law: that of hydrodynamic helicity H:

$$H = \int u \cdot \omega \, \mathrm{d}V$$

where u is the flow velocity and $\omega = \nabla \times u$ the vorticity. Moreau's work followed a similar discovery in plasma physics by Woljer [2], and was followed by another seminal contribution by Keith Moffatt [3], who established the topological interpretation of both results (see Moffatt's contribution to this issue for a more precise account of events).

It is surprising that a new conservation law could be discovered so long after the Euler equations of motion. The origins of this long interval lie perhaps in the subtle geometric nature of helicity. Unlike momentum and energy, helicity does not originate in common space–time symmetries, such as translational symmetry or time-translation symmetry. Rather, as shown by Moffatt [3], helicity is intimately related to the topology of the flow.

This connection between helicity and topology can be intuitively understood by the following simple example: consider a thin vortex ring in which all the vorticity is parallel to the tube axis and lies within a tube D, having centerline c and cross section A, as depicted in Fig. 1. In this case the helicity simplifies to:

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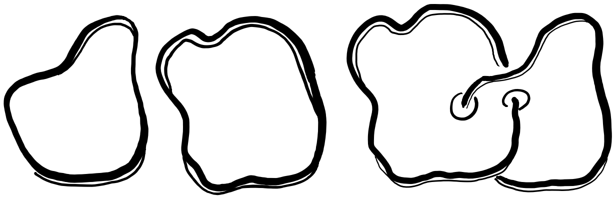

Fig. 1. Linked vortex lines carry helicity whereas unlinked vortex lines have zero helicity.

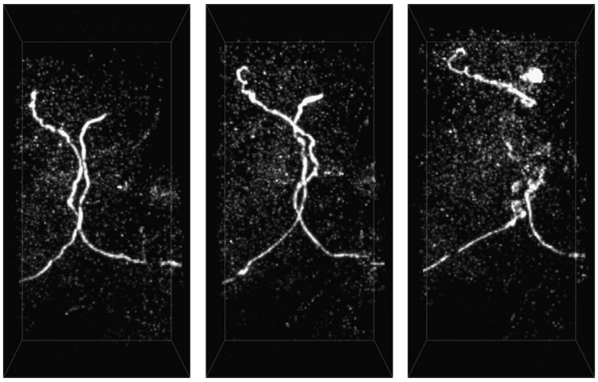

Fig. 2. A reconnection between vortex strands in water (from Kleckner and Irvine [8]). The image shows light scattered off air bubbles trapped in the core of the reconnecting vortex strands; the strands approach each other and re-connect.

$$H = \int_D u \cdot \omega \, dV = \Gamma \oint_c u \cdot dl$$

where $\Gamma = \int_A \omega \, dA$. By Stokes' theorem, the right-hand side will be equal to the amount of flux that threads the ring, which is zero for disconnected rings. If, however the rings are linked, then:

$$H = 2\Gamma^2$$

This intuitive idea generalizes to more intricate vortex geometries as well as continuous fields. The value $2\Gamma^2$ is in fact $2L_{1,2}\Gamma_1\Gamma_2$, where $L_{1,2}$ is the Gauss linking number of the two vortices, and it is multiplied by their respective circulations. For a general field, seen as a collection of N infinitesimal flux tubes, we have [4,5]:

$$H = \sum_{i,j}^{N\to\infty} \Gamma_i \Gamma_j L_{ij}$$

Helicity is thus the circulation-weighted sum of the topological linking number between all vortex line pairs.

Taking this elegant topological interpretation as a starting point, the conservation of helicity then follows simply from Helmholtz's laws of vortex motion that vorticity goes with the flow [6]: since vortex lines are transported by the velocity field, they cannot cross. This notion of topological robustness is exactly the concept that inspired Lord Kelvin's hypothesis that atoms are knotted vortices in the aether. It wasn't in fact till much later that this type of topological invariant was rationalized as a particle-relabeling symmetry and connected to the more commonly encountered circulation theorems [7].

This simple view of the conservation of helicity, is, however, shattered in the presence of viscosity, which mediates reconnection processes in which vortex lines are cut and spliced together in reconnections as shown in Fig. 2. Investigating the validity of Moreau's and Moffatt's law in real physical systems is difficult theoretically, as well as numerically, because of the singular nature of reconnections and the large scale separation often involved. Experimentally preparing vortex knots – the quintessential helicity-bearing excitations – had also remained an outstanding challenge since Lord Kelvin's conjecture, until recent advances.

In the past decade or so, a number of tools have changed the landscape. On the experimental side, additive manufacturing (3D printing) has opened the way of rapidly and cheaply creating objects with arbitrary shapes. Three-dimensional imaging and data-processing has become almost routine. Furthermore, many results of differential geometry and topology have percolated to the basic education of physicists. These advances made it possible to create the first vortex knots and links shown in Fig. 3. A physical embodiment of Kelvin's, Moreau's, and Moffatt's vision.

Once made, such vortex knots immediately distort and disconnect through reconnections (Fig. 4), as if drawn towards topological simplicity. For example a vortex link immediately decays in to two separate rings. This penchant for unraveling

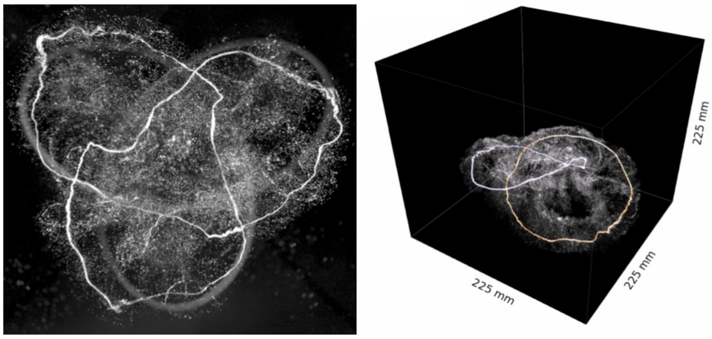

Fig. 3. A vortex knot (left) and a vortex link (right) in water (from Kleckner and Irvine [8] and Scheeler et al. [9]). These excitations of a fluid imagined by Lord Kelvin can now routinely be created in the laboratory.

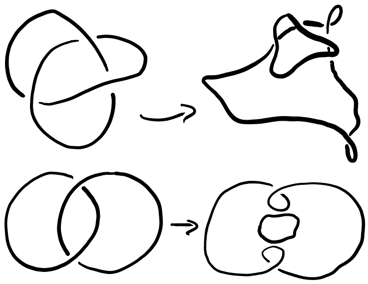

Fig. 4. The untying dynamics of a trefoil vortex knot and a pair of linked rings.

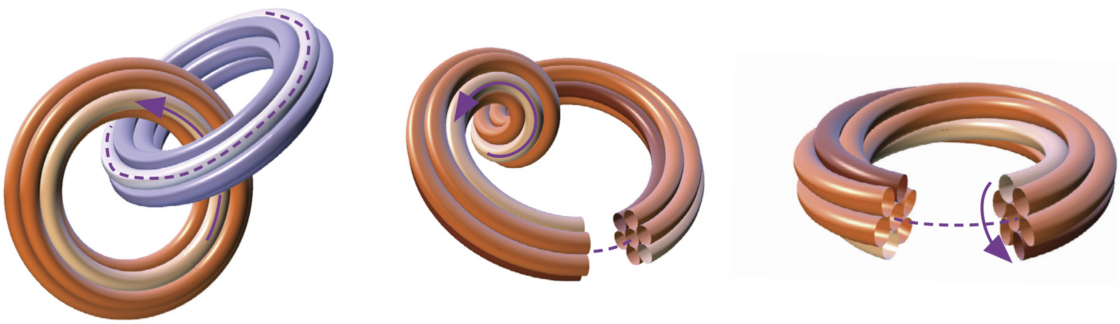

Fig. 5. A representation of link (left), writhe (center) and twist (right) contributions to helicity for vortex bundles.

appears to threaten Moreau's edifice of conservation. A careful analysis of the helicity content of the disconnected rings however reveals that this is not necessarily so.

Beyond enabling reconnections, the presence of viscosity causes vorticity to diffuse away from the centerline of the vortex, endowing even the initially infinitesimally thin vortex with a non-vanishing thickness. This makes the structure of vortex loops in a viscous fluid more like a loop of yarn than a line. Accounting for the geometry and topology of the vorticity field becomes thus a more nuanced endeavor, however for thin vortex tubes (or magnetic flux tubes) with a smooth core, the separation of scales enables helicity to be conveniently expressed as [4,5,10,11] (see Fig. 5 for a graphical representation):

$$H = Lk + Wr + Tw$$

where Lk is the Gauss linking number between the *centerlines* of the vortex *tubes* and exactly accounts for linking between all the strands (or sub-tubes) in *separate* tubes. Wr and Tw are, respectively, the writhe of the vortex tube's centerline and the twist of the bundle *strands* about the centerline. They account for the circulation-weighted linking (or asymptotic linking) between all the strands within a tube. They can be naturally broken down as follows: Twist measures how much the strands rotate *locally* about the centerline where rotation is measured with respect to the parallel transport framing. This is the only contribution that is locally detectable. Writhe measures how much the bundle fibers wind around each other in the absence of any local twist, as a result of parallel transport along the centerline. This contribution is not locally

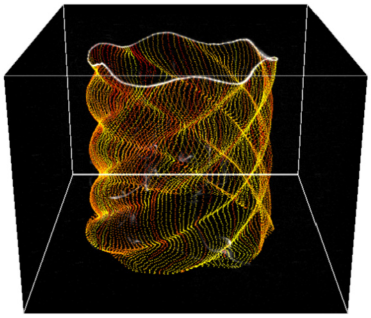

Fig. 6. A spiral vortex tube with dyed blobs in its core (white). Overlaid are the trajectories of the dyed blobs. From Scheeler et al. [17].

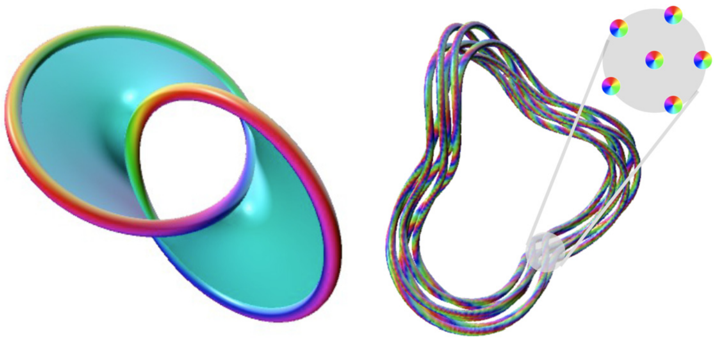

Fig. 7. (Left) A pair of linked superfluid vortex rings (from Kleckner, Kauffman and Irvine [16]). The surface joining the two loops corresponds to a phase iso-surface of the complex scalar field that fills the surrounding space. (Right) A superfluid bundle (from Kedia et al. [18]).

detectable, as in the case of linking between tubes. Thus $Lk + Tw$ can be measured from knowledge of the centerline alone, whereas Tw requires measuring the flows within the vortex core. See Fig. 5 for a visual breakdown of the three.

Remarkably, a measurement of the geometry of the vortex before and after reconnection reveals that far from destroyed, the vortex geometry through the reconnection process appears to obey a simple rule:

$$Lk + Wr = \text{constant}$$

The link helicity is thus transferred almost seamlessly to the writhe helicity. This surgical interchange can be traced to the vanishing contribution of anti-parallel strands to the crossings in any projection of a knot or link [3,9,12] rescuing helicity from the apparent doom presented by reconnections.

Since the mechanism for conservation of $Lk + Wr$ through reconnections neglects twist contributions to helicity and thus requires knowledge of only the geometry of the vortex centerline, it is natural to ask whether there exists a simpler model system that mimics both the centerline dynamics of vortices and allows reconnections to occur. Compressible superfluids, such as Bose gases provide a natural candidate (Fig. 6).

Taking a superfluid state to be represented by a complex scalar function $\psi = \sqrt{\rho}e^{i\phi}$ – related to the superfluid velocity and density by: $v = \nabla\phi$ and $\rho = |\psi|^2$ – a vortex corresponds to a phase singularity that winds around a line. Given a desired vortex line geometry, the corresponding space-filling phase field $\phi(x)$ [9,13,14] takes the value of the solid angle subtended by the vortex line at a point x. An example of a phase iso-surface for a pair of linked rings is shown in Fig. 7. The corresponding density field, drops to zero smoothly in a vicinity η of the vortex line, where η is the healing length of the condensate.

Numerical integration of the Gross–Pitaevskii equation reveals that a superfluid trefoil vortex knot evolves in a way that mirrors closely the evolution of a trefoil knot in experiment [15]: it stretches and disconnects through quasi-antiparallel reconnections into disjoint writhing vortex loops. A comprehensive survey [16] reveals that these dynamics are general in that they occur in knots that start in a quasi-ideal shape. Tracking their centerline it is then possible to verify that $Lk + Wr$ of the centerline is approximately conserved, with the approximation improving as the separation of scales – between the overall vortex size and the core size – becomes large. Although these findings are restricted to the centerline evolution, and are thus only indirect witnesses of the evolution of total helicity in either viscous or superfluids, they show that reconnections need not be a fundamental threat to helicity conservation.

To go beyond centerline dynamics, and study the dynamics of distributed vorticity, requires a precise measurement of the structure of the vortex core – a more challenging endeavor.

Recent experiments [17] have made a step in this direction: using inhomogeneous distributions of dye placed within the core of a vortex, a measurement of the flow along the vortex core was made possible. For thin vortices, this in turn enables the measurement of the total helicity and thus – by simultaneously measuring the writhe – resolution of the individual dynamics of Wr and Tw.

In experiments in water in which helical vortex loops were stretched or compressed by a second vortex, almost seamless transfer of writhe to twist was observed, as would occur in an inviscid fluid.

An experiment that tracked the evolution of an individual vortex tube over time, enabled the observation of the effects of viscous dissipation on the total helicity. Remarkably, total helicity was observed to rise from an initial value close to zero, to the value of the writhe of the helical vortex loop. This seemingly paradoxical dynamic in which the presence of viscosity leads to the generation of helicity can be understood by noting that as the only *local* contribution, twist is the only contribution that is dissipated, leaving $Lk + Wr$ as the controllers of the fate of helicity. Thus even when viscosity dominates, geometry and topology provide insights into the evolution of the flow.

An analogous picture can be developed for superfluid vortex bundles, either directly constructed numerically by winding superfluid strands about a centerline [18], or spontaneously formed in a complex flow [19]. In both cases the observation is that the centerline helicity behaves in a similar manner to the classical helicity.

The work described above, a tale of the resilience of Moreau's conservation law in the face of dissipation, is but an example of the extraordinary legacy of Moreau's article of 1961. Beyond simple fluids and superfluids, Moreau's work continues to inspire research in areas ranging from plasma physics [20,21], to turbulence [22,23] (see Moffatt's account in this issue). Even more broadly, the idea that topology has a role to play in the physics of fields is having an impact across physics, in theory and experiment. From topological field theories [24–27], to superconductors [25,28], to electromagnetism [29,30], from liquid crystals [31–34], to optical fields [35,36], to Bose condensates [15,37], the time is ripe for discovery and Jean-Jacques Moreau's extraordinary legacy is expanding at an ever increasing rate.

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