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The legacy of Jean-Jacques Moreau in mechanics

Dynamics of a particle with friction and delay

Dynamique d'une particule avec frottement et retard

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ABSTRACT

We are interested in the motion of a simple mechanical system having a finite number of degrees of freedom subjected to a unilateral constraint with dry friction and delay effects (with maximal duration $\tau > 0$). At the contact point, we characterize the friction by a Coulomb law associated with a friction cone. Starting from a formulation of the problem that was given by Jean-Jacques Moreau in the form of a second-order differential inclusion in the sense of measures, we consider a sweeping process algorithm that converges towards a solution to the dynamical contact problem. The mathematical machinery as well as the general plan of the existence proof may seem much too heavy in order to treat just this simple case, but they have proved useful in more complex settings.

RÉSUMÉ

Nous nous intéressons au mouvement d'un système mécanique ayant un nombre fini de degrés de liberté soumis à une contrainte unilatérale avec frottement sec et des forces qui peuvent dépendre de l'histoire du mouvement avec facteur de retard τ. Au contact, nous caractérisons le frottement par une loi de Coulomb associée à un cône de frottement, en suivant la formulation du problème proposée par Jean-Jacques Moreau sous la forme d'une inclusion différentielle du second ordre au sens des mesures (la réaction et l'accéleration pouvant être des mesures). Un algorithme de type « sweeping process » permet de montrer l'existence d'une suite qui converge vers une solution du problème de contact dynamique. L'outillage mathématique ainsi que la démarche de la preuve semblent trop lourds pour traiter ce problème plutôt simple, mais les deux peuvent être utiles dans des cadres plus complexes.

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Introduction

Nous considérons un système mécanique avec un nombre fini de degrés de liberté soumis à une contrainte unilatérale avec frottement et à des forces pouvant manifester du retard. La configuration du système est associée à la position $q = (q_1, \ldots, q_d) \in E = \mathbb{R}^d$, où $q \in E$ désigne la représentation du système en coordonnées généralisées. Le mouvement du système est décrit par la fonction $q : I \to E$ avec $I = [0, T]$ $(T > 0)$, qui désigne un intervalle de temps. À chaque instant $t \in I$, on doit s'assurer que

$$q(t) \in L = \{ q \in E : g(q) \leq 0 \}$$

Nous faisons l'hypothèse que :

$$\bigtriangledown g(q) \neq 0 \quad \text{sur un voisinage de } \partial L = \{q \in E : \ g(q) = 0\}$$

La loi du mouvement lorsque les impacts sont considérés est décrite par

$$\ddot{q} = R + f(., q, p), \quad p = M(q)\dot{q} = \dot{q}$$

où p désigne la quantité de mouvement, $M(q) = I_{E \times E}$ la matrice d'inertie, que l'on suppose ici triviale (l'identité). La fonction f désigne les forces extérieures au système en l'absence de contact. La fonction R désigne la réaction en présence d'impact ou contact avec la frontière du domaine L, c'est-à-dire

$$g(q) = 0$$

On suppose qu'il y a du frottement sec et on prend la formulation donnée par Jean-Jacques Moreau dans [1]. En conséquence, nous adoptons la loi de frottement de Coulomb décrite dans [2,3] par :

$$R \in C = C(q) = \{v \in E \ : v.n(q) \geq |v| \cos(\alpha(q))\} = \{v \in E \ : -v. \bigtriangledown g(q) \geq c(q)|v|\}$$

avec $c(q) = |\bigtriangledown g(q)| \cos(\alpha(q))$; $\alpha(q) \in]0, \frac{\pi}{2}[$ est une fonction réelle positive (l'angle de frottement) et $n(q) = -\frac{\bigtriangledown g(q)}{|\bigtriangledown g(q)|}$ la normale intérieure.

En outre, nous supposons qu'il existe un facteur de retard $\tau \in]0, T]$, en ce sens que les forces agissant sur le système, autres que la réaction de contact, pourront dépendre à l'instant t de l'historique du mouvement pendant l'intervalle $[t - \tau, t]$. En exprimant cet historique par des fonctions q_t (ou q_t^τ, s'il y a risque de confusion) définies par

$$q_t^\tau(s) = q(t+s), \ s \in [-\tau, 0]$$

on peut décrire ces forces sous la forme $f = f(t, q_t, p) = f(t, q_t, \dot{q}) : I \times X \times E$, où $X = C([-\tau, 0], E)$ est un espace de fonctions continues.

Des systèmes avec retard sont souvent présents dans les applications ; c'est le cas, par exemple, des problèmes de contrôle où celui-ci est appliqué en fonction d'un signal acquis avec du retard. Aussi, au point de vue théorique, on espère que ce genre de problèmes modèles, moyennant un choix judicieux des termes manifestant du retard, ou bien l'adaptation du procédé, puisse contribuer à la compréhension d'autres formulations plus sophistiquées (voire plus réalistes) du frottement.

L'étude des problèmes de contact dynamique a mis depuis longtemps en évidence les difficultés associées aux discontinuités qui se produisent aux instants de contact. Des travaux tels que [4,5], parmi beaucoup d'autres, ont contribué, soit à la genèse de plusieurs formulations de ce type de problème, soit à l'analyse de phénomènes tels que la non-unicité ou la non-régularité de la solution. Jean-Jacques Moreau a donné des formulations de ces problèmes de contact, inélastique, partiellement élastique ou avec frottement, sous la forme d'inclusions différentielles au sens des mesures, lesquelles donnent lieu à des algorithmes de type *sweeping process* en vue de l'obtention de solutions du problème de contact. Il a exploité ces algorithmes à grand profit, jusqu'à obtenir, par exemple, des avancées remarquables dans la compréhension des milieux granulaires.

L'étude mathématique de ces algorithmes, entamée dans [3], suivi de [6], a été developpée depuis, dans des directions de recherche que l'on peut dire génériquement influencées par Jean-Jacques Moreau. (On reviendra brièvement à ce sujet dans la version anglaise, la version française abrégée se terminant par la formulation du problème.) Il est absolument clair que les travaux de Jean-Jacques Moreau sont incontournables et fondateurs dans ce cadre, en fournissant de belles formulations, mécaniquement solides et mathématiquement inspiratrices.

Formulation du problème

La dynamique du mouvement est décrite par la relation

$$\mathrm{d}v = f(t, q_t, v)\,\mathrm{d}t + \mathrm{d}r$$

où v est la vitesse, dv l'accélération, exprimée par la mesure différentielle de v, dt la mesure de Lebesgue de l'intervalle et dr la mesure de réaction au contact.

Problème P. *Étant donnés la position initiale $q_{00} \in L$, la vitesse à droite initiale v_0 et l'historique du mouvement avant l'instant initial*

$$q_0 : [-\tau, 0] \longrightarrow L \quad \text{avec} \quad q_0(0) = q_{00}$$

on cherche une fonction continue à droite $v : I \longrightarrow E$ appartenant à $BV(I; E)$, l'espace des fonctions à variations bornées de I vers E et une fonction continue lipschitzienne $q : I \longrightarrow E$ définie par

$$q(t) = q_{00} + \int_0^t v(s)\mathrm{d}s \quad \forall t \in I$$

telles que, en prenant $q_t^\tau(s) = q_t(s) := q(t+s)$ $(s \in [-\tau, 0])$:

$$\begin{cases} q(0) &= q_{00} \in L \\ v(0) &= v_0 \\ q(t) &\in L \quad (\forall t \in I) \\ v(t) &\in V(q(t)) \quad (\forall t \in I) \\ dv &= f(t, q_t, v)\,\mathrm{d}t + \mathrm{d}r \end{cases}$$

au sens des mesures. Cela signifie que les fonctions de densité des mesures dr, dt *et* dv *par rapport à une mesure positive $\mu = \mathrm{d}\mu$, notées resp. r'_μ, t'_μ et v'_μ, satisfont*

$$v'_\mu(t) = f(t, q_t, v(t))t'_\mu(t) + r'_\mu(t), \ \mu\text{-p.p.}\ \textit{dans}\ I$$

En plus, la loi de contact avec frottement doit être satisfaite sous la forme présentée dans la version anglaise.

Les hypothèses considérées sont les suivantes.
(H1) La fonction g appartient à $C^1(E; \mathbb{R})$ et $\triangledown g$ ne s'annule pas dans un voisinage de l'hypersurface $\{q \in E : g(q) = 0\}$.
(H2) La fonction f est continue de $[0, T] \times X \times E$ (avec $T > 0$) à valeurs dans E et elle est lipschitzienne par rappport à ses deuxième et troisième variables.
(H3) L'application définissant le cône de frottement $C : q \mapsto C(q)$ est lipschitzienne (l'angle de frottement est une fonction lipschitzienne de la position).

Sous ces hypothèses, on considère un algorithme de discrétisation permettant d'approcher une solution du problème ; en effet, on démontre la convergence d'une sous-suite vers une solution.

1. Introduction

We consider a mechanical system with a finite number of degrees of freedom, which is subjected to a single unilateral constraint with friction and to forces that may contain a delay effect. The system configuration is associated with the position $q = (q_1, \ldots, q_d) \in E = \mathbb{R}^d$, where $q \in E$ denotes the generalized coordinates. The motion of the system is described by the function $q : I \to E$ where $I = [0, T]$ $(T > 0)$ is, say, a time interval. At each instant $t \in I$, one must have the relation

$$q(t) \in L = \{\, q \in E \ : \ g(q) \leq 0 \,\}$$

We assume that:

$$\triangledown g(q) \neq 0 \quad \text{in a neighborhood of } \partial L = \{q \in E : \ g(q) = 0\}$$

The set of admissible right-velocities at q is defined by

$$V(q) = \begin{cases} \{u \in E \ : \ \triangledown g(q).u \leq 0\}, & \text{if} \quad g(q) \geq 0 \\ E, & \text{otherwise} \end{cases}$$

The tangent hyperspace to L at q is

$$T(q)=\{u\in E\ :\ \nabla g(q).u=0\}$$

if $g(q)=0$ (or even if $g(q)>0$, for the purposes of discretization).

The law of motion when impacts are considered is described by

$$\ddot{q}=R+f(.,q,p),\quad p=M(q)\dot{q}=\dot{q}$$

where p denotes the momentum and $M(q)$ is the inertia matrix, which we assume to be the identity matrix $I_{E\times E}$. The function f denotes the forces acting on the system (excluding contact). The function R denotes the reaction in the presence of impact or contact with the boundary of the domain L, i.e.

$$g(q)=0$$

To be precise, since at impact discontinuities of the velocity may be produced, we have to consider that the reaction forces are expressed by a measure, which we denote by dr. We suppose that there is dry friction and we take the formulation given by Jean-Jacques Moreau in [1], i.e. the Coulomb friction law as described in [3] by:

$$R\in C=C(q)=\{v\in E\ :v.n(q)\geq|v|\cos(\alpha(q))\}=\{v\in E\ :-v.\nabla g(q)\geq c(q)|v|\}$$

where $c(q)=|\nabla g(q)|\cos(\alpha(q)),\ \alpha(q)\in]0,\frac{\pi}{2}[$ is a positive real function and $n(q)=-\frac{\nabla g(q)}{|\nabla g(q)|}$.

Moreover, we assume that some delay effects of duration up to $\tau\in]0,T]$ may be present and that, more generally, the external forces at time t may also depend on the motion q between times $t-\tau$ and t. One defines φ_t or φ_t^τ by

$$\varphi_t(s):=\varphi(t+s)\ (s\in[-\tau,0])$$

for any (continuous) function φ defined in $[t-\tau,t]$. Let $X=C([-\tau,0],E)$. For continuous motions, all the q_t belong to X. In this setting, we may take the initial datum to be just $q:[-\tau,0]\longrightarrow L$, an admissible continuous motion taking place before the initial time $t_0=0$. This implies that the initial position $q(0)=:q_{00}\in L$ is known. If we assume that the left-velocity at $t=0$ exists, then the initial right-velocity should be equal to it or be determined from it by the contact law. Thus the essential initial data are implicitly contained in $q_0\in X=C([-\tau,0],E)$, as long as $q_0(s)=q(s)\in L$, for all $s\in[-\tau,0]$, and $v_0^-=\dot{q}^-(0)$ exists.

The force f may depend on the current time t, on the actual position $q(t)=q_t(0)$ (which is thus a function of q_t), on the previous motion during time-interval $[t-\tau,t]$, expressed by $q_t\in X$, and on the (right) velocity v. Thus, f may be described without loss of generality as a function of t, $q_t=\varphi\in X$ and v:

$$f=f(t,\varphi,v):[0,T]\times X\times E\longrightarrow E$$

1.1. Problem formulation

We are given a domain L, a friction cone C, a force f and $\varphi_0\in X$, a previous motion in L with left-velocity at $s=0$. The balance equation is

$$\mathrm{d}v=f(t,q_t,v)\mathrm{d}t+\mathrm{d}r$$

where v is the right-velocity, dv its Stieltjes or differential measure, dt the Lebesgue measure, and dr the reaction measure.

Problem P. *Given $\varphi_0\in X$, find a right-continuous function $v:I\longrightarrow E$, belonging to the space $BV(I;E)$ of functions with bounded variation from I to E, such that, together with the Lipschitz continuous function $q:I\longrightarrow E$ defined by*

$$q(t)=q_{00}+\int_0^t v(s)\mathrm{d}s\quad\forall t\in I$$

it will satisfy (if for $t\in I$, $q_t(s):=q(t+s)\ (s\in[-\tau,0])$):

$$\begin{cases} q_0 &= \varphi_0,\ \text{i.e. } q(s)=\varphi_0(s)\quad(s\in[-\tau,0]);\\ q(t) &\in L\quad(\forall t\in I)\\ v(t) &\in V(q(t))\quad(\forall t\in I)\\ \mathrm{d}v &= f(t,q_t,v)\,\mathrm{d}t+\mathrm{d}r\end{cases}$$

in the sense of measures. This means that there exist densities v'_μ, t'_μ and r'_μ of dv, dt, *and* dr *with respect to some positive measure μ and that these densities satisfy*

$$v'_\mu(t)=f(t,q_t,v(t))t'_\mu(t)+r'_\mu(t),\ \mu\text{-a.e. in } I$$

Moreover,

1) *the reaction density is zero, at μ-a.e. t, for which there is no contact ($g(q(t)) < 0$).*
Otherwise, the reaction density r'_μ is such that the friction law is satisfied μ-a.e. (and thus Lebesgue a.e.):
2) *if $g(q(t)) = 0$ and $v^-(t) \cdot \nabla g(q(t)) > 0$, then the velocity is discontinuous at t and*

$$v^+(t) = \mathrm{prox}\big(0, [v^-(t) + C(q(t))] \cap T(q(t))\big)$$

(the proximal point of 0 or the minimum norm point in the said set) or, equivalently,

$$-v^+(t) \in \mathrm{proj}_{T(q(t))} N_{C(q(t))}\big(v^+(t) - v^-(t)\big)$$

(the r.h.s. is the orthogonal projection onto the tangent space of the outward normal cone to the friction cone at $v^+ - v^-$); and
3) *if $g(q(t)) = 0$ and $\nabla g(q(t)) \cdot v^-(t) = 0$, then*

$$-v^+(t) \in \mathrm{proj}_{T(q(t))} N_{C(q(t))}\big(r'_\mu(t)\big)$$

It may be shown that the last inclusion is equivalent to the previous one, even when the velocity is discontinuous at t, since then $r'_\mu(t) = v'_\mu(t) = \mu(\{t\})^{-1}(v^+(t) - v^-(t))$, while the sets involved are all cones. Also, 2) as well as 3) imply that μ-a.e.

$$r'_\mu(t) \in C(q(t))$$

1.2. Discretization algorithm

Let $h = h_N = \frac{T}{N}$, for $N \in \mathbb{N}$ such that $h_N < \tau$. We define a discretization of $[0, T]$ by taking:

$$t_{N,0} = 0; \quad t_{N,i} = t_{N,0} + ih, \text{ for } i \in \{0, ..., N\}$$

For each such N, we define an approximate motion q_N and its velocity v_N. One takes

$$q_N(s) := \varphi_0(s), \quad \text{for } s \in [-\tau, 0]$$

For the first subinterval $[0, h_N]$, the initial position is known: $q_N(0) = \varphi_0(0) \in L$; the left-velocity $v_{N,-1} = \dot{\varphi_0}^-(0)$ is also known, so this case may be treated as that of any other subinterval $[t_{N,i}, t_{N,i+1}]$. By recurrence, we assume that the initial position for that subinterval $q_N(t_{N,i}) = q_{N,i}$ is known, as well as a previous velocity $v_{N,i-1}$ (obtained from the past subinterval). We define

$$q_N(s) = q_{N,i} + (s - t_{N,i})\, v_{N,i} \quad (s \in [t_{N,i}, t_{N,i+1}]);\ q_{N,i+1} = q_{N,i} + h_N v_{N,i}$$

The updated velocity in the present interval, i.e. $v_{N,i}$, which is also the right-velocity at $t_{N,i}$ and the left-velocity at $t_{N,i+1}$, is obtained in possibly two steps. First, we consider the prospective velocity $w_{N,i}$, obtained from $v_{N,i-1}$ by taking into account only the non-frictional forces:

$$w_{N,i} = v_{N,i-1} + h_N f(t_{N,i}, (q_N)^\tau_{t_{N,i}}, v_{N,i-1})$$

If this is an admissible velocity at $q_{N,i}$, we take $v_{N,i} = w_{N,i}$; if not, we apply a discrete version of the friction contact law:

$$v_{N,i} = \mathrm{prox}\big(0, [w_{N,i} + C(q_{N,i})] \cap T(q_{N,i})\big)$$

Notice that this formula will encompass the other case, for q in the interior of L, by putting $C(q) = \{0\}$ and $T(q) = E$. Having now defined a function q_N on $[\tau, t_{N,i+1}]$, it is clear that all the $(q_N)_t \in X$ are also defined, for $t \in [0, t_{N,i+1}]$, and so we may continue this process.

In short, we defined a sequence of functions $(q_N)_N$ (approximate solutions) satisfying

$$\begin{cases} q_N(s) &= \varphi_0(s), \text{ in } [-\tau, 0] \\ q_{N,0} &= q_N(0) = \varphi_0(0) \in L \\ v_{N,-1} &= \dot{q_N}^-(0) = \dot{\varphi_0}^-(0) \end{cases}$$

and then, for $i = 0, ..., N-1$ and $s \in [t_{N,i},\ t_{N,i+1}]$:

$$\begin{cases} v_{N,i} &= proj\big(0, [v_{N,i-1} + h_N\, f(t_{N,i}, (q_N)^\tau_{t_{N,i}}, v_{N,i-1}) + C(q_{N,i})] \cap T(q_{N,i+})\big) \\ q_{N,i+1} &= q_{N,i} + h_N v_{N,i} \\ q_N(s) &= q_{N,i} + (s - t_{N,i})\, v_{N,i} \end{cases}$$

1.3. Convergence result

Assume that:

(H1) $g \in C^1(E;\mathbb{R})$ and $\nabla g \neq 0$ in a neighborhood of the hypersurface $\{q \in E : g(q) = 0\}$.
(H2) f is continuous from $[0,T] \times X \times E$ to E and Lipschitz continuous with respect to the second and third variables.
(H3) The friction cone $q \mapsto C(q)$ is Lipschitz continuous, in the sense that the angle of friction $\alpha = \alpha(q)$ is a Lipschitz continuous function of the state q.

We shall use the same general scheme of proof presented in [2,3].

We need preliminary estimates valid for the approximate velocities v_N and motions q_N ($N > \frac{T}{\tau}$). To simplify the outline of the proof, we assume that f is globally bounded, instead of locally (as per (H2)).

Lemma 1.1. *Let us consider the approximate velocities $v_N = \dot{q}_N$, which are given by*

$$v_N(t) = v_{N,i} \quad \textit{if} \quad t \in [t_{N,i}, t_{N,i+1}[$$

and $v_N(T) = v_{N,N}$. Then, the sequence of functions $(v_N)_N$ is uniformly bounded in the norm.

Proof. The discrete form of the friction law, i.e. the maps

$$w \mapsto v = \text{prox}\big(0, [w + C(q)] \cap T(q)\big)$$

are dissipative in the sense that $|v| \leq |w|$. Indeed, either $v = w$, or, since the friction cone contains the inward normal direction, $\text{prox}(w, T(q)) \in w + C(q)$ and so $|v - 0| \leq |\text{prox}(w, T(q)) - 0| \leq |w|$. Therefore, $|v_{N,i}| \leq |w_{N,i}| \leq |v_{N,i-1}| + h_N ||f||_\infty$ and so, in $[0,T]$:

$$||v_N||_\infty \leq |v_0| + T||f||_\infty =: R \qquad \square$$

Lemma 1.2. *The finite sequences $(q_{N,i})_{i\geq 0}$ (for all $N > T/\tau$) are uniformly bounded. Then the sequence of function $(q_N)_N$ is uniformly bounded and equi-Lipschitz-continuous.*

Proof. This is a straightforward consequence of the previous lemma. We have $|q_{N,i}| \leq |q_{N,i-1}| + h|v_{N,i}| \leq |q_{N,i-1}| + hR$. It follows that $|q_{N,i}| \leq |q_{N,0}| + RT$ and since q_N is piecewise affine

$$|q_N(s)| \leq A := |q_{00}| + RT, \forall s \in [0,T]$$

Due to the definition of the sequences of functions $(v_N, q_N)_N$ and the previous estimates, we have

$$|q_N(s) - q_N(s')| \leq R|s - s'| \qquad \square$$

In view of these two lemmas, by Ascoli–Arzelà's theorem, we may extract subsequences of $(q_N)_N$ that converge uniformly. In the next subsection, we outline the proof of the following:

Theorem 1.3 *(Solution to the problem). Let a function $q \in C([0,T],E)$ be obtained as the uniform limit of a subsequence $(q_N)_{N\in\mathbb{N}^*}$ of the approximate solutions defined by using the above numerical scheme; i.e. $\mathbb{N}^*$ is an infinite subset of $\mathbb{N}$ and*

$$\max_{t\in[0,T]}|q_N(t) - q(t)| \longrightarrow 0 \quad (N \in \mathbb{N}^*, N \to \infty)$$

Then q is a solution to problem P.

The proof is based on the method used in [3] (or [2] or in e.g. [7,8], in the partially elastic setting).

1.4. Properties of a limit function

We need to consider the acceleration of q, so that we have to show that q possesses more regularity properties than stated in the theorem. We shall prove that the approximate velocities (v_N) have bounded total variation on I and indeed that these are uniformly bounded.

Lemma 1.4. *There exists a constant $B > 0$ such that for all N, $var(v_N; I) \leq B$.*

Proof. To simplify the notations, subsequences of the original sequences of motions and velocities, etc. are denoted as if they were the initial sequences. We also denote $f_i = f(t_{N,i}, (q_N)^{\tau}_{t_{N,i}}, v_{N,i})$, $w_i = v_{N,i-1} + hf_i$ and C_i, T_i the friction and tangent cones at $q_{N,i}$, etc. The proof in [3] (pp. 96–99) was somewhat involved. Here, we simply give hints on what is going on. The total variation of v_N is

$$\mathrm{var}(v_N, I) = \sum_i |v_{N,i} - v_{N,i-1}| = \sum_i |v_i - v_{i-1}|$$

in shorter notation. If $v_i = w_i = v_{i-1} + hf_i$ (the "free" updated velocity), then

$$|v_i - v_{i-1}| = |w_i - v_{i-1}| \le h||f||_\infty$$

When the discretized friction law is applied, we use its special form. Having restricted friction angles away from $\pi/2$, we may assume that $\cos\alpha(q_i) \ge c > 0$ (locally at least), and we obtain

$$|v_i - w_i| \le \frac{1}{c}|\mathrm{proj}(w_i, T_i) - w_i| = \frac{1}{c}\mathrm{dist}(w_i, T_i)$$

Since $\mathrm{dist}(w_i, T_i) \le |w_i - v_{i-1}| + \mathrm{dist}(v_{i-1}, T_i)$, by the triangle inequality:

$$|v_i - v_{i-1}| \le h(1 + \frac{1}{c})||f||_\infty + \frac{1}{c}\mathrm{dist}(v_{i-1}, T_i)$$

The first type of terms add up to $T(1 + \frac{1}{c})||f||_\infty$, while to control the distances a finer analysis is required. We may replace the tangent hyperspace by the cone of admissible right-velocities $V(q)$, in the non-trivial cases. Also one uses the fact that $v_{i-1} \in V(q_{i-1})$ and the dependence of this set on q, and that initially $v_0 \in V(q_{00})$. If all these and further details, concerning the distance to tangent hyperspaces, are taken into consideration, then it is not surprising that an estimate of the form

$$\mathrm{var}(v_N, I) \le RT$$

is obtained, for some $R \ge (1 + \frac{1}{c})||f||_\infty$. Special versions of this inequality are needed to study the several cases in the friction law. □

The sequence of functions $(v_N)_{N\ge 0}$ is uniformly bounded and bounded in variation in $[0, T]$. By Helly's theorem, possibly by extracting a subsequence still denoted $(q_N, v_N)_N$, there exists $u \in BV(I, E)$ such that

$$v_N(t) \longrightarrow u(t) \quad \forall t \in [0, T]$$

We define $v(t) = u^+(t)$; v is a right-continuous function with bounded variation. By using the same method of proof as in [3], we have the following lemma.

Lemma 1.5. *Every $q(t)$ with $t \in I$ belongs to L, that is,*

$$g(q(t)) \le 0 \quad \forall t \in I$$

Indeed, although $q_N(t)$ may violate the constraint, i.e. $g(q_N(t)) > 0$, it can nevertheless be shown that, for some constant K, $g(q_N(t)) \le Kh_N$, and we may take limits.

It follows from the Lemma 1.5 that $v(t) \in V(q(t))$.

We introduce a base measure $\mathrm{d}\mu = |\mathrm{d}v| + \mathrm{d}t$, with respect to which there exist densities of the differential measure or Stieltjes (vector) measure $\mathrm{d}v$, of its measure of total variation $|\mathrm{d}v|$ and of the Lebesgue measure.

We are going to consider several cases.

1) In the interior of the domain L

Let $q(t)$ be an interior point of the domain L, i.e. $g(q(t)) < 0$.

Let $I(\epsilon) = [t - \epsilon, t + \epsilon]$ be a neighborhood of t such that all the $q(s)$ with $s \in I(\epsilon)$ are interior points; moreover, thanks to uniform convergence, $q_N(s) \in \mathrm{int}(L)$, for large N. Let $[t_{N,j-1}, t_{N,k}]$ be the smallest such interval that contains $I(\epsilon)$. We have eventually:

$$v_N(t+\epsilon) - v_N(t-\epsilon) = \sum_{i=j}^{k}(v_{N,i} - v_{N,i-1}) = \sum_{i=j}^{k} h_N\, f(t_{N,i}, (q_N)^{\tau}_{t_{N,i}}, v_{N,i-1})$$

which is a Riemann sum for the integral of $s \mapsto f(s, (q_N)^{\tau}_s, v_N^-(s))$ on the above interval, which "converges" to $I(\epsilon)$, as $N \to \infty$. In the limit, as $N \to \infty$ and so $h_N \to 0$, we have

$$\mathrm{d}v(I(\epsilon) = v(t+\epsilon) - v(t-\epsilon) = \int_{t-\epsilon}^{t+\epsilon} f(s, q_s, v(s))\mathrm{d}s$$

since $q_N \to q$ uniformly, so that $(q_N)_s^\tau \to q_s^\tau = q_s$, and v is continuous at a.e. s. This yields classically that $\dot{v}(t) = f(t, q_t, v(t))$ (divide by 2ϵ and take limits). In the present setting, we may divide by $\mathrm{d}\mu(I(\epsilon))$ and take the limit when $\epsilon \to 0^+$. We obtain for a.e. t, where Lebesgue results on the differentiation and densities of measures apply:

$$v'_\mu(t) = f(t, q_t, v(t))\, t'_\mu(t), \text{ so } r'_\mu(t) = 0$$

2) Impacts

Now, we consider contact, that is $g(q(t)) = 0$ $(t \geq 0)$. Since the motion takes place in L, for $s < t$ close to t, one has $g(q(s)) \leq g(q(t)) = 0$, implying that the left-derivative is non-negative $v^-(t) \cdot \bigtriangledown g(q(t)) \geq 0$.

Let $g(q(t)) = 0$ and $v^-(t) \cdot \bigtriangledown g(q(t)) > 0$.

On the other hand, $v^+(t) \cdot \bigtriangledown g(q(t)) \leq 0$, so that the right-velocity must be different from the left-velocity: an impact occurs. The time t is an atom for the measure $\mathrm{d}v$ and for the measure of total variation $|\mathrm{d}v|$, with $\mathrm{d}v(\{t\}) = v^+(t) - v^-(t)$ and $|\mathrm{d}v|(\{t\}) = |v^+(t) - v^-(t)|$; t must also be an atom of μ: $\mu(\{t\}) > 0$. Since $\mathrm{d}t(\{t\}) = 0$, we have

$$r'_\mu(t) = v'_\mu(t) = \frac{\mathrm{d}v}{\mathrm{d}\mu}(t) = \frac{1}{\mu(\{t\})}(v^+(t) - v^-(t))$$

According to the statement of the problem, in subsection 1.1, our goal is to prove that

$$v^+(t) = \mathrm{prox}\big(0, [v^-(t) + C(q(t))] \cap T(q(t))\big)$$

Notice that this implies that $v^+(t) - v^-(t) \in C(q(t))$, hence $r'_\mu(t) \in C(q(t))$.

We argue that the approximate motions q_N and velocities v_N must exhibit an application of the friction law at least at some discretization node $t_{N,i}$ near t; otherwise, they would have small variation on an interval $I(\epsilon)$ (as it was the case for interior motions), which would contradict the variation of the limit velocity v being greater than $|v^+(t) - v^-(t)|$.

Then, we use the continuity properties of the projection on the tangent hyperplanes $T(q)$ (for $q \notin \mathrm{int}(L)$) and, more precisely, of the ensuing continuity for the "friction operator"

$$P : (w, q) \in (E \setminus V(q)) \times (E \setminus \mathrm{int}(L)) \mapsto \mathrm{prox}\big(0, [w + C(q)] \cap T(q)\big)$$

Now, for some $t_{N,i}$ close to t the friction law is applied with $v_{N,i-1}$ close to $v^-(t)$. Since $v_{N,i} = P(v_{N,i-1}) + hf_i, q_{n,i})$ and $|hf_i| \leq h||f||_\infty$, then $v_{N,i}$ will be close to $P(v^-(t), q(t))$. Since we have some control on the variation after such a "friction node", this will finally lead to the desired result, namely that

$$v^+(t) = P(v^-(t), q(t))$$

3) Contact with continuous velocity

Let $g(q(t)) = 0$ and $v^-(t). \bigtriangledown g(q(t)) = 0$, which is the remaining case.

We may still use the same method of proof as that developed in [3]. First, the finer study of the variation of the v_N allows us to conclude that if the left-velocity $v^-(t)$ is admissible $(v^-(t) \in V(q(t)))$, then the velocity is continuous at t (cf. Lemma 3.10(b), ch. 3, in [3]). Then, we use a kind of variational characterization of the inclusion that we need to prove; cf. case 3 in the statement of the Problem P). The proof of

$$-v(t) \in \mathrm{proj}_{T(q(t))} N_{C(q(t))}\big(r'_\mu(t)\big)$$

is therefore reduced to that of

i) $v(t) \in T(q(t))$, $r'_\mu(t) \in C(q(t))$;
ii) $\forall x \in \mathrm{int}\, C(q(t)):\quad \big(r'_\mu(t) \cdot \bigtriangledown g(q(t))\big)\big(v(t) \cdot x\big) \leq \big(r'_\mu(t) \cdot v(t)\big)\big(\bigtriangledown g(q(t)) \cdot x\big)$

(see Lemma 3.13, *idem*). This is too long to be presented here.

2. Final comments

2.1. Delay terms

If we take $f(t, \phi, v) = F(t, \phi(-\tau), \phi(0), v)$, for some function $F : I \times E \times E \times E \to E$, then the force f takes the form

$$f(t, q_t, v(t)) = F(t, q(t-\tau), q(t), \dot{q}(t))$$

allowing a classical delay term.

More generally, one is allowed to consider a dependence of the forces on the motion during $[t, t-\tau]$ and one may speak of a functional differential inclusion.

For instance, f might contain terms depending on scalar quantities such as

$$\beta = \beta\big(\max\{\,|g(q_t(s))|\,:\, s \in [-\delta, 0]\}\big) = \beta\big(\max\{\,|g(q(s))|\,:\, s \in [t-\delta, t]\}\big)$$

where $\delta \in]0, \tau]$ is small enough and β is a smooth function. Then β would depend smoothly on $(q_t)^\tau$ and it might give a measure of how near the immediate past trajectory was to the boundary (how "tangential", in some sense). Similar considerations might apply to the question of distinguishing between a statical and a dynamical angle of friction α. For that purpose, one would like to have α depend on some measure of the deviation from rest before the present time, by considering, say,

$$\alpha = \alpha\big(\max\{\,|q(s) - q(t)|\,:\, s \in [t-\delta, t]\,\}\big)$$

2.2. A very, very short reference list

In connection with the subject of this communication, one might easily expand the reference list and still forget many works, but it is hoped that fairly all the contributors to the field will be mentioned in the whole of the present volume. Here, we mention mainly some contributions that we are aware of and that are closer to ours, inviting the reader to learn more from other works from those and other authors. We do not follow a strict chronological order.

The first author had the privilege of being supervised by Jean-Jacques Moreau in the framework of his Ph.D. thesis (Lisbon) [2] on this and connected topics, after having defended his "Thèse de 3^{e} cycle" (Montpellier) on different matters [9]. Some twenty years later, he co-supervised with L. Paoli the thesis of the second author [7], prepared in Lisbon and Saint-Étienne.

In the last few years, L. Paoli pursued this line of research, analyzing the friction problems, including the so-called "frictional catastrophes" (such as the notorious "Painlevé paradox") – cf. [10–12]. We can say that – although at a slower pace, befitting the nature of the "sweeping process" –, Moreau's approach is fulfilling its promise also with respect to mathematical proofs.

In [6], M. Mabrouk had in the meantime extended the case of inelastic contact in [3] to partially elastic contact, which was also addressed later in [7,8].

M. Schatzman, among the first to consider differential inclusions in the sense of measures [5], dedicated continued attention to the subject, as in, e.g., [13] and up to [14] on the "Painlevé paradox".

D.E. Stewart, who was working on the subject [16], put together the complementarity approach with measure differential inclusions in [17] (the first analysis of the "Painlevé paradox" in this mathematical setting) and has pursued his work on the subject, including a book [18].

The works of P. Ballard (and students), from [19] to [20], not to mention the recent incursion into deformable bodies cases, constitute what is a very coherent and solid treatment of these subjects. In particular, it is shown that, under analyticity assumptions, uniqueness results may be obtained.

B. Brogliato and co-workers (such as V. Acary, F. Génot, D. Goeleven, R. Leine, H. Nijmeier, to name a few) have applied Moreau's approach in a larger sense to many applied problems, in, e.g., robotics and automatic control, even extending the notion of differential inclusions, and they also studied the "Painlevé paradox". The third edition of the book [15], with a list of more than 1300 references, is ample proof of the impact of these studies.

The reference list below is strikingly short with respect to Jean-Jacques Moreau's contribution. We refer to the other papers in this volume for a general overview of his seminal works in the field of non-smooth mechanics.

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References

[1] J.J. Moreau, Une formulation du contact à frottement sec; application au calcul numérique, C. R. Acad. Sci. Paris, Ser. II 302 (1986) 799–801.
[2] M.D.P. Monteiro Marques, Inclusões diferenciais e choques inelásticos, Ph.D. thesis, Universidade de Lisboa, Lisbon, 1988.
[3] M.D.P. Monteiro Marques, Differential Inclusions in Non-Smooth Mechanical Problems: Shocks and Dry Friction, Birkhauser, 1993.
[4] A. Bressan, Questioni di regolarità e di unicità del moto in presenza di vincoli olonomi unilaterali, Rend. Semin. Mat. Univ. Padova 29 (1959) 271–315.
[5] M. Schatzman, A class of non linear differential equations of second order in time, Nonlinear Anal. 2 (1978) 355–378.
[6] M. Mabrouk, A unified variational model for the dynamics of perfect unilateral constraints, Eur. J. Mech. A, Solids 17 (1998) 819–842.
[7] R. Dzonou, Convergence des schémas numériques pour des problèmes d'impact, thèse de doctorat, Université Jean-Monnet, Saint-Étienne, France, 2007.
[8] R. Dzonou, M. Marques, L. Paoli, A convergence result for a vibro-impact problem with a general inertial operator, J. Nonlinear Dyn. 58 (2009) 361–384.
[9] M.D.P. Monteiro Marques, Quelques questions d'analyse et géométrie posées par la mécanique des milieux continus, thèse de troisième cycle, USTL, Montpellier, France, 1983.
[10] L. Paoli, Vibro-impact problems with dry friction – Part I: existence result, SIAM J. Math. Anal. 47 (2015) 3285–3313.

[11] L. Paoli, Vibro-impact problems with dry friction – Part II: tangential contacts and frictional catastrophes, SIAM J. Math. Anal. 48 (2016) 1272–1296.
[12] L. Paoli, Multibody dynamics with unilateral constraints and dry friction: how the contact dynamics approach may handle Coulomb's law indeterminacies?, J. Convex Anal. 23 (2016) 849–876.
[13] L. Paoli, M. Schatzman, A numerical scheme for impact problem II. The multidimensional case, SIAM J. Numer. Anal. 40 (2002) 734–768.
[14] M. Schatzman, Penalty approximation of Painlevé problem, in: P. Alart, O. Maisonneuve, R.T. Rockafellar (Eds.), Nonsmooth Mechanics and Analysis, Springer, 2006, pp. 129–143.
[15] B. Brogliato, Nonsmooth Mechanics. Models, Dynamics and Control, 3rd edition, Comm. Control Engrg. Ser., Springer, 2016.
[16] D.E. Stewart, J.C. Trinkle, An implicit time-stepping scheme for rigid body dynamics with inelastic collisions and Coulomb friction, Int. J. Numer. Methods Eng. 39 (1996) 2673–2691.
[17] D.E. Stewart, Convergence of a time-stepping scheme for rigid body dynamics and resolution of Painlevé's paradoxes, Arch. Ration. Mech. Anal. 145 (1998) 215–260.
[18] D.E. Stewart, Dynamics with Inequalities. Impacts and Hard Constraints, Society for Industrial and Applied Mathematics (SIAM), 2011.
[19] P. Ballard, The dynamics of discrete mechanical systems with perfect unilateral constraints, Arch. Ration. Mech. Anal. 154 (2000) 199–274.
[20] A. Charles, P. Ballard, Existence and uniqueness of solutions to dynamical unilateral contact problems with Coulomb friction: the case of a collection of points, Modél. Math. Anal. Numér. 48 (2014) 1–25.