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The legacy of Jean-Jacques Moreau in mechanics

A duality recipe for non-convex variational problems

Un principe de dualité pour des problèmes variationnels non convexes

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A B S T R A C T

The aim of this paper is to present a general convexification recipe that can be useful for studying non-convex variational problems. In particular, this allows us to treat such problems by using a powerful primal–dual scheme. Possible further developments and open issues are given.

R É S U M É

Nous présentons un principe général de convexification permettant de traiter certains problèmes variationnels non convexes. Ce principe permet de mettre en œuvre les puissantes techniques de dualité en ramenant de tels problèmes à des formulation de type primal–dual. Quelques perspectives et problèmes ouverts sont évoqués.

1. Introduction

The role of duality techniques is nowadays very well established in applied mathematics, mechanics, and numerical analysis. In the context of infinite-dimensional vector spaces, convex analysis has been a powerful mathematical tool taking a major part in this success. The occasion is great here to honor the pioneering contributions of Jean-Jacques Moreau that go back to the 1960's [1] (lectures notes at the "Collège de France") almost concomitantly with the work of T. Rockafellar [2] focused on the finite dimensional case. Such a mathematical step in functional analysis was crucial in order to make a rigorous existence theory in elasticity theory (existence of equilibrium strain/stress tensors, quasi-static evolution), and it could extended to non-linear (but convex) situations, notably in plasticity theory [3,4]. Let us emphasize that the impact of duality and convexity encompasses a very broad area in optimization theory: in numerical analysis, many efficient and stable algorithms are based on min–max (or saddle points) schemes and still recent progress in this area are very influential (for instance around proximal projection algorithms); in optimal mass transport [5], the existence and the characterization of an optimal map often goes through the existence of a solution to a dual problem; in asymptotic analysis

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(dimension reduction, homogenization), a huge number of results have been obtained by combining duality arguments and Γ-convergence techniques (for the latter notion we refer to [6][7]).

Unfortunately, such a duality theory completely breaks down as soon as some non-convexity appears in the optimization problem under study. In particular, this drawback is often met in calculus of variations, where even very classical problems involve non-convex energy costs. As no systematical tool is available to characterize a global optimum, a dramatic consequence is that all currently available numerical methods loose their efficiency, because they are not able to rule out local minimizers and detect the global ones.

The aim of the present paper is to present some new perspectives for exploiting duality in a context of *non-convex* variational problems. We begin by presenting in Section 2 a general convexification recipe whose basic idea emerged several years ago from discussions with Antonin Chambolle (unpublished notes). It enlightens a new interpretation of the calibration field developed for the Munford–Shah segmentation problem [8][9], and suggests a new road for identifying the variational limit of non-convex functionals. In Section 3, we show how the recipe can be applied to integral functionals satisfying a generalized co-area formula. Then we present a survey of the primal–dual framework obtained in [10][11], and we sketch a new proof for the Γ-convergence of Cahn–Hilliard's models.

In the last Section 5, we discuss possible new developments and present a still unsolved conjecture.

2. General framework

We fix some preliminary notations. In this section, (X,τ) denotes a topological space. We assume that there exists a continuous embedding

$$\varphi : u \in X \mapsto \varphi_u \in Y$$

where Y is a topological locally convex vector space. The symbol $<\cdot,\cdot>$ will denote the duality mapping between Y and its dual Y^*. Our convexification procedure is based on the following assumption:

$$\begin{cases} \text{there exists a suitable compact metrizable convex subset } K \subset Y \\ \text{whose extremal set } \ddot{K} \text{ satisfies:} \quad \varphi(X) \subset \ddot{K} \end{cases} \tag{H1}$$

A typical situation is when Y is the dual of a separable Banach space Z equipped with the weak-star topology. Then Y^* can be identified with Z itself and every bounded and weakly-star closed convex subset of Y is compact metrizable. Recall that v is an extreme point of K if $v=\theta v_1+(1-\theta)v_2 \in K$ with $v_1, v_2 \in K$ and $\theta \in (0,1)$ cannot occur unless $v=v_1=v_2$.

We consider a sequence of proper functionals $F^\varepsilon : X \to [0,+\infty]$, which we assume to be uniformly coercive, that is:

$$\begin{cases} \text{for every } R, \text{ there exists a } \tau\text{-compact subset } C_R \text{ such that:} \\ \forall \varepsilon > 0\,,\ \{u \in X\,;\ F^\varepsilon(u) \le R\} \subset C_R \end{cases} \tag{H2}$$

Define $F_0^\varepsilon : Y \to [0,+\infty]$ by setting

$$F_0^\varepsilon(\varphi_u) = F^\varepsilon(u) \quad \text{for every } u \in X \quad , \quad F_0^\varepsilon(v) = +\infty \quad \text{if} \quad v \notin \varphi(X)$$

The Fenchel conjugate of F_0^ε is given on Y^* by:

$$(F_0^\varepsilon)^*(g) := \sup\{<v,g> - F_0^\varepsilon(v) \ :\ v \in Y\} = \sup\{<\varphi_u, g> - F^\varepsilon(u) \ :\ u \in X\}$$

Then (being K metrizable), the biconjugate $(F_0^\varepsilon)^{**}$ will coincide with the sequential convexification of F_0^ε, that is, for every $v \in Y$:

$$(F_0^\varepsilon)^{**}(v) = \inf\left\{ \liminf_h \sum_{i=1}^{n_h} t_i^h F^\varepsilon(u_i^h) \ :\ \sum_{i=1}^{n_h} t_i^h \varphi_{u_i^h} \to v \right\}$$

where $\{t_i^h : i=1,\cdots,n_h\}$ are real numbers in $[0,1]$ such that $\sum_i t_i^h = 1$.

The main result of this section states that the variational limit of F^ε at every $u \in X$ agrees with that of the convexification $(F_0^\varepsilon)^{**}$ at φ_u. To be more precise, let us consider (see [7]) the (sequential) Γ-limits of F_ε defined on (X,τ) by

$$\begin{aligned} \Gamma - \liminf F^\varepsilon(u) &= \inf\left\{ \liminf_{\varepsilon\to 0} F^\varepsilon(u_\varepsilon) \ :\ u_\varepsilon \to u \right\} \\ \Gamma - \limsup F^\varepsilon(u) &= \inf\left\{ \limsup_{\varepsilon\to 0} F^\varepsilon(u_\varepsilon) \ :\ u_\varepsilon \to u \right\} \end{aligned} \tag{1}$$

and in a similar way the Γ-limits of (F_0^ε) and $(F_0^\varepsilon)^{**}$ defined on Y. In order to simplify the notations, in the following, we will denote by:

– F', F'' the $\Gamma-\liminf$ and $\Gamma-\limsup$ of F^ε (defined on X),
– F_0', F_0'', the $\Gamma-\liminf$ and $\Gamma-\limsup$ of F_0^ε (defined on Y),
– G', G'', the $\Gamma-\liminf$ and $\Gamma-\limsup$ of $(F_0^\varepsilon)^{**}$.

Observe that due to (H2) by [12, Prop. 1.3.5], all sequential notions coincide with the topological ones. For instance, it holds:

$$F'(u)(u) = \sup_{V\in\mathcal{V}(u)} \liminf_{\varepsilon} \inf_V F^\varepsilon \quad , \quad F''(u)(u) = \sup_{V\in\mathcal{V}(u)} \limsup_{\varepsilon} \inf_V F^\varepsilon \tag{2}$$

On the other hand, in case of the constant sequence $F^\varepsilon = F$, the lower and upper Γ-limits F', F'' coincide and agree with the usual notion of lower semicontinuous envelope:

$$\overline{F}(u) = \sup\{\Phi(u) \ : \ \Phi \text{ lower semicontinuous}\ , \ \Phi \le F\ \}$$

Theorem 2.1. *Under (H1) and (H2), there holds, for every $u \in X$:*

$$\Gamma-\liminf F^\varepsilon(u) \le \Gamma-\liminf (F_0^\varepsilon)^{**}(\varphi_u) \le \Gamma-\limsup (F_0^\varepsilon)^{**}(\varphi_u) \le \Gamma-\limsup F^\varepsilon(u)$$

In particular, if $F^\varepsilon := F$ for every ε, then

$$\overline{F}(u) = (F_0)^{**}(\varphi_u)$$

The proof of Theorem 2.1 rests upon the following result. Let V an open subset of Y and set

$$\theta_V(w) := \begin{cases} \inf\Big\{\nu(V) : [\nu] = w\ ,\ \nu\in\mathcal{P}(K)\Big\} & \text{if } w\in K \\ +\infty & \text{otherwise} \end{cases} \tag{3}$$

where $\mathcal{P}(K)$ denotes the space of probability measures on Y supported on compact subset K and $[\nu]$ denotes the barycenter (i.e. $\int_K g\mathrm{d}\nu = g([\nu])$ for every continuous linear form $g \in Y^*$, see [13]).

Lemma 2.2. *The function θ_V is convex, l.s.c., and satisfies*

$$0 \le \theta_V \le 1 \text{ in } K\ ,\ \theta_V = 0 \text{ in } K\setminus V\ ,\ \theta_V = 1 \text{ in } \ddot{K}\cap V$$

Moreover, θ_V agrees with the convex l.s.c. envelope of the function $\mathbf{1}_V + \chi_K$ (where $\chi_K = 0$ on K and $\chi_K = +\infty$ on $Y \setminus K$). It vanishes identically on K whenever $V \cap \ddot{K} = \emptyset$.

Proof. We recall that K being compact and metrizable, the set of probability measures $\mathcal{K}$ is a weakly-star compact subset on which the affine map $\nu \mapsto [\nu]$ is continuous and takes values in K. It is then straightforward to check that the function $\theta_V : Y \to [0, +\infty]$ is convex l.s.c. on Y. If $w \in K$, by taking ν to be the Dirac mass at w in (3), we infer that $\theta_V(w) \in [0, 1]$, whereas $\theta_V(w) = 0$ if $w \notin V$. If $w \in V$ is an extreme point of K, the latter choice $\nu = \delta_w$ turns out to be the unique one compatible with the condition $[\nu] = w$, and, in this case, we get $\theta_V(w) = \delta_w(V) = 1$. In fact, it is a Choquet integral representation Theorem (see [13], Thm. 25, p. 283) that every $w \in K$ is the barycenter of a suitable probability measure supported on $\ddot{K}$, thus θ_V vanishes identically on K whenever $V \cap \ddot{K} = \emptyset$.

On the other hand, let us compute the Moreau–Fenchel conjugate of θ_V. For every $g \in Y^*$, we have

$$\begin{aligned}\theta_V^*(g) = \sup_{w\in K}\{<g,w> - \theta_V(w)\} &= \sup_{\nu\in\mathcal{P}(K)} \left\{ \int_K (<g,w> - \mathbf{1}_V(w))\,\nu(\mathrm{d}w) \right\} \\ &= \sup_{w\in K}\{<g,w> - \mathbf{1}_V(w)\} \ = \ (\mathbf{1}_V + \chi_K)^*(g)\end{aligned}$$

where, for the third equality, we used that the supremum over $\mathcal{P}(K)$ is reached by Dirac masses. As θ_V is convex l.s.c., we deduce that $\theta_V = (\theta_V)^{**} = (\mathbf{1}_V + \chi_K)^{**}$. Thus we have proved that θ_V coincides with the convex l.s.c. envelope of $\mathbf{1}_V + \chi_K$. $\square$

Proof of Theorem 2.1. By the assumption (H2), the sequential characterizations (1) for F', F'' can be used restricting ourselves to sequences $u_\varepsilon \to u$, where u_ε belongs to a fixed compact subset $C\,(= C_R) \subset X$. Since the embedding $\varphi : C \mapsto \varphi(C)$ is bicontinuous, the convergences $u_\varepsilon \to u$ or $\varphi_{u_\varepsilon} \to \varphi_u$ are equivalent. Thus, by the identity $F(u_\varepsilon) = F_0^\varepsilon(\varphi_{u_\varepsilon})$ (notice that $\Gamma-\liminf F_0^\varepsilon(v) = +\infty$ whenever $v \notin \varphi(X)$), we infer that

$$F_0' = F' \circ \varphi \quad , \quad F_0'' = F'' \circ \varphi$$

We are therefore reduced to showing that, for every $u \in X$:

i) $F'_0(\varphi_u) = \Gamma - \liminf F_0^\varepsilon(\varphi_u) \le \Gamma - \liminf (F_0^\varepsilon)^{**}(\varphi_u)$,
ii) $\Gamma - \limsup (F_0^\varepsilon)^{**}(\varphi_u) \le \Gamma - \limsup F_0^\varepsilon(\varphi_u) = F''_0(\varphi_u)$.

The inequality ii) is obvious since $(F_0^\varepsilon)^{**} \le F_0^\varepsilon$. Let us show i). Let v be in $\ddot{K}$ and choose a real $t < F'_0(v)$. Then, by using the topological characterization of F'_0 (see (2)), we may find a suitable open neighborhood V of v such that $t < \inf_V F_\varepsilon^0$ holds for ε small enough. For such ε, we have $F_0^\varepsilon \ge t\,\theta_V$. Then, by using Lemma 3 and by passing to the biconjugate, we obtain $(F_0^\varepsilon)^{**} \ge t\,(\theta_V)^{**} = t\,\theta_V$. We deduce

$$\Gamma - \liminf (F_0^\varepsilon)^{**}(v) \ \ge\ t\,\theta_V(v) \ =\ t$$

The claim i) follows, since by (H1), we have $\varphi_u \in \ddot{K}$ for every $u \in X$. $\square$

Homogeneous variant: In many cases the convex compact subset K appears to be the base of a closed convex cone. Namely, we make the additional assumption

There exists a continuous linear form $l_0 \in Y^*$ such that $l_0 = 1$ on K (H3)

This assumption allows us to simplify our duality scheme. For every $\varepsilon > 0$, we introduce the convex set of Y^*:

$$D^\varepsilon \ :=\ \{g \in Y^*\ :\ (F_0^\varepsilon)^*(g)\ <\ 0\}\ =\ \{g \in Y^*\ : < \varphi_u, g > <\ F^\varepsilon(u)\ \forall u \in X\}$$

and in a similar way, we define $D', D'' \subset Y^*$ as

$$D'\ :=\ \{g \in Y^*\ :\ (F'_0)^*(g)\ \le\ 0\} \quad , \quad D''\ :=\ \{g \in Y^*\ :\ (F''_0)^*(g)\ \le\ 0\} \tag{4}$$

Note that all results hereafter are unchanged if D^ε is defined alternatively with a large inequality. It turns out that functionals $(F_0^\varepsilon)^{**}, (F'_0)^{**}, (F''_0)^{**}$ agree on K with the support functions of D_ε, D', D'', respectively (they are one homogeneous convex, l.s.c. functionals on Y).

Lemma 2.3. *For every $v \in Y$ with $< v, l_0 >= 1$ (in particular for $v \in \varphi(X)$), one has*

$$(F_0^\varepsilon)^{**}(v)\ =\ \sup_{g \in D^\varepsilon} < v, g > \quad , \quad (F'_0)^{**}(v)\ =\ \sup_{g \in D'} < v, g > \quad , \quad (F''_0)^{**}(v)\ =\ \sup_{g \in D''} < v, g >$$

Proof. Clearly, we have for every $v \in Y$,

$$(F_0^\varepsilon)^{**}(v)\ =\ \sup_{g \in Y^*} < v, g > -(F_0^\varepsilon)^*(g) \ge \sup_{g \in D_\varepsilon} < v, g >$$

We need to prove that the converse inequality holds if v satisfies $< v, l_0 >= 1$. We notice that under (H3), we have, for every $g \in Y^*$ and $\lambda \in \mathbb{R}$,

$$(F_0^\varepsilon)^*(g - \lambda l_0)\ =\ \sup_{u \in X} \left\{ < \varphi_u, g - \lambda l_0 > -F^\varepsilon(u) \right\}\ =\ (F_0^\varepsilon)^*(g) - \lambda \tag{5}$$

In particular, by applying the identity above with $\lambda > (F_0^\varepsilon)^*(g)$, we obtain that $g_\lambda := g - \lambda\, l_0$ satisfies $(F_0^\varepsilon)^*(g_\lambda) < 0$ thus $g_\lambda \in D^\varepsilon$. Therefore, for every v such that $< v, l_0 >= 1$, one has

$$< v, g > -(F_0^\varepsilon)^*(g)\ = < v, g_\lambda > +\lambda - (F_0^\varepsilon)^*(g)\ \le\ \sup_{h \in D_\varepsilon} < v, h > +\lambda - (F_0^\varepsilon)^*(g)$$

hence the desired inequality letting $\lambda \searrow (F_0^\varepsilon)^*(g)$. The proof is the same for $(F'_0)^{**}$ and $(F''_0)^{**}$. $\square$

In the next lemma, we establish a comparison between the sets D', D'' defined above and the lower and upper Kuratowski limits of the sets D_ε in Y^*, where Y^* is equipped with the strong topology (that is, the topology of the uniform convergence on the compact subsets of Y). Let us denote by $Li(D_\varepsilon)$ and $Ls(D_\varepsilon)$ these lower and upper Kuratowski limits. There are closed subsets of Y^* whose indicator functions (see [7]) coincide, respectively, with $\Gamma - \liminf \chi_{D_\varepsilon}$ and $\Gamma - \limsup \chi_{D_\varepsilon}$. In other words:

i) $g \in Li(D_\varepsilon)$ iff there exist $g_\varepsilon \in D_\varepsilon$ such that $g_\varepsilon \to g$.
ii) $g \in Ls(D_\varepsilon)$ iff there exists a subsequence g'_ε, with $g'_\varepsilon \in D'_\varepsilon$, such that $g'_\varepsilon \to g$.

Lemma 2.4. *With the notations above and D', D'' defined by (4), we have the following inclusions:*

i) $(F'_0)^{**} \le G' \le G'' \le (F''_0)^{**}$.
ii) $D' \subseteq Li(D_\varepsilon) \subseteq Ls(D_\varepsilon) \subseteq D''$.

Proof. Since K is compact and by exploiting the definition of F_0' on a minimizing sequence, it is easy to check that, for every $g \in Y^*$, it holds

$$-\limsup_\varepsilon (F_0^\varepsilon)^*(g) = \liminf_\varepsilon \inf_K \{F_0^\varepsilon - <\cdot, g>\} \ \geq \inf_K \{F_0' - <\cdot, g>\} = -(F_0')^*(g) \tag{6}$$

Let us show (i). Since $(F_0^\varepsilon)^{**} \leq F_0^\varepsilon$, we have $G'' \leq F_0''$. The inequality $G'' \leq (F_0'')^{**}$ is then a consequence of the fact that G'' is l.s.c. and convex (as the $\Gamma-\limsup$-limit of sequence of convex functions). On the other hand, for every $g \in Y^*$ and every sequence $\{v_\varepsilon\}$ converging to v in Y, one has:

$$\liminf_\varepsilon (F_0^\varepsilon)^{**}(v_\varepsilon) \ \geq <v, g> - \limsup_\varepsilon (F_0^\varepsilon)^*(g) \geq <g, v> -(F_0')^*(g)$$

where we used Moreau–Fenchel inequality and (6). Thus $G'(v) \geq <g, v> -(F_0')^*(g)$. The inequality $(F_0')^{**} \leq G'$ follows by taking the supremum with respect to $g \in Y^*$.

Let us show (ii): let $g \in D'$ and assume first that $(F_0')^*(g) < 0$. Then, by (6), one has $(F_0^\varepsilon)^*(g) \leq 0$ for ε small enough (hence $g \in D^\varepsilon$) so that g belongs to $Li(D^\varepsilon)$. This conclusion can be extended to an element $g \in D'$ such that $(F_0')^*(g) = 0$. Indeed, let $g_n := g - (1/n)\, l_0$. Then, by (5), $(F_0')^*(g_n) = -1/n < 0$. Therefore g_n belongs to the closed subset $Li(D^\varepsilon)$ while $g_n \to g$ as $n \to \infty$. Eventually, we have proved that $D' \subseteq Li(D_\varepsilon)$.

It remains to show that $Ls(D_\varepsilon) \subseteq D''$. Let $g \in Ls(D_\varepsilon)$ and $v \in Y^*$. By the (sequential) definitions of $Ls(D_\varepsilon)$ and of F_0'', there exists a sequence $(v_\varepsilon, g_\varepsilon) \in Y \times Y^*$ such that

$$g_\varepsilon \in D_\varepsilon \text{ and } g_\varepsilon \to g \text{ strongly in } Y^* \quad , \qquad v_\varepsilon \to v \text{ in } Y \text{ and } \limsup_\varepsilon F_0^\varepsilon(v_\varepsilon) \leq F_0''(v)$$

Then, by applying the Moreau–Fenchel inequality and (6), we are led to

$$<v, g> = \lim_\varepsilon <v_\varepsilon, g_\varepsilon> \leq \limsup_\varepsilon F_0^\varepsilon(v_\varepsilon) + \limsup_\varepsilon (F_0^\varepsilon)^*(g_\varepsilon) \ \leq \ F_0''(v) + 0$$

holding for every $v \in Y$. Thus, owing to the definition of D'' in (4), we get $g \in D''$. This proves that $Ls(D_\varepsilon) \subseteq D''$. $\square$

To summarize this section, we give the following practical result, which will be useful in the applications, where we need to identify the Γ-limit of a sequence $\{F^\varepsilon\}$. Notice that, by Kuratowski's compactness theorem [6] [7] and our assumptions (H1), (H2) (which allow us to treat X as a separable metric space) such a Γ-limit exists, at least for a subsequence of $\{F^\varepsilon\}$.

Theorem 2.5. *Assume that (H1), (H2), (H3) hold. Then the following three assertions are equivalent:*

i) *F^ε Γ-converges to a limit F in X (i.e. $F' = F'' = F$),*
ii) *$(F_0^\varepsilon)^{**}$ Γ-converges to a limit G in Y (i.e. $G' = G'' = G$),*
iii) *D_ε converges in the Kuratowski sense to a set D in the strong topology of Y^* (i.e. $D' = D'' = D$).*

In addition, if one of these assertions holds true, then F, G and D satisfy the relations

$$D \ = \ \{g \in Y^* \ : <\varphi_u, g> \leq \ F(u) \ \forall u \in X\} \tag{7}$$

$$G(v) \ = \ \sup_{g \in D} <v, g> \quad \textit{if } <v, l_0> = 1, \ +\infty \textit{ otherwise} \tag{8}$$

$$F(u) \ = \ G(\varphi_u) \ = \ \sup_{g \in D} <\varphi_u, g> \tag{9}$$

Proof. If $F^\varepsilon \xrightarrow{\Gamma} F$, then by definition $F = F' = F''$ so that $F_0' = F_0''$ and $D' = D''$. We conclude that ii) and iii) hold by invoking Lemma 2.4. We have $G = (F_0')^{**} = (F_0'')^{**}$ and $D = D' = D''$, showing (7), (8) and (9) as a consequence of Theorem 2.1.

Conversely, assume that ii) or iii) holds. By compactness, we consider F and a subsequence $\{F^{\varepsilon_k}\}$ that Γ-converges to F. Then, the reconstruction formula (9) shows that the limit F is uniquely determined. Hence the whole sequence $\{F^\varepsilon\}$ Γ-converges to F. $\square$

3. Application to non-convex variational problems

We now apply the framework developed in Section 2 to the following situation. Let Ω be a bounded Lipschitz domain of $\mathbb{R}^N$. We consider the embedding of $X = L^1(\Omega)$ into $Y = L^\infty(\Omega \times \mathbb{R})$ defined by:

$$\varphi : u \in X \mapsto \mathbf{1}_u \in Y \quad , \quad \mathbf{1}_u(x,t) := \begin{cases} 1 & \text{if } u(x) > t \\ 0 & \text{if } u(x) \leq t \end{cases}$$

Let us consider

$$K := \{v \in L^\infty(\Omega\times\mathbb{R}) \ : \ v(x,t) \in [0,1] \text{ a.e. } (x,t)\in\Omega\times\mathbb{R}\}$$

It is a compact subset of $L^\infty(\Omega\times\mathbb{R})$ equipped with its weak-star topology (we are in the case where $Y = Z^*$ if we set $Z = L^1(\Omega\times\mathbb{R})$). It is easy to check that $\mathbf{1}_u$ is an extreme point of K as it takes values in $\{0,1\}$. Moreover, the map $u \mapsto \mathbf{1}_u$ is continuous from $L^1(\Omega)$ to $L^\infty(\Omega\times\mathbb{R})$ (embedded with its weak-star topology).

Let $F : u \in L^1(\Omega) \to \mathbb{R}\cup+\infty$ be a possibly non-convex functional. We simply assume that F is l.s.c. (with respect to the strong convergence in $L^1(\Omega)$) and that the following coercivity assumption holds:

$$\begin{cases} F(u) \ \geq \ k\,\|u\| - \dfrac{1}{k}, & \text{for suitable constant } k>0 \\ \text{For every } R>0, \text{ the set } \{u : F(u)\leq R\} \text{ is a compact subset of } L^1(\Omega) \end{cases} \tag{10}$$

We consider the minimization problem

$$\inf\left\{F(u) \ : \ u\in L^1(\Omega)\right\} \tag{$\mathcal{P}$}$$

Under the assumption (10), this problem has at least one solution, and the set of solutions $\operatorname{argmin}\mathcal{P}$ is a non-void compact subset of $L^1(\Omega)$ (since F is not convex, we expect a priori multiple solutions).

Following the construction developed in Section 2, we define for every pair $(v,g)\in L^\infty(\Omega\times\mathbb{R})\times L^1(\Omega\times\mathbb{R})$

$$F_0^*(g) = \sup_{u\in L^1(\Omega)}\left\{\iint_{\Omega\times\mathbb{R}} g(x,t)\,\mathbf{1}_u\,\mathrm{d}x\,\mathrm{d}t - F(u)\right\}, \quad G(v) = \sup_{g\in L^\infty(\Omega\times\mathbb{R})}\left\{\iint_{\Omega\times\mathbb{R}} g\,v\,\mathrm{d}x\,\mathrm{d}t - F_0^*(g)\right\} \tag{11}$$

Since F coincides with its l.s.c. envelope, it holds, as a consequence of Theorem 2.1, that

$$G(\mathbf{1}_u) \ = \ F(u) \quad \text{for all } u\in L^1(\Omega) \tag{12}$$

Our convexification recipe leads to the following convex optimization problem

$$\inf\left\{G(v) \ : \ v\in L^\infty(\Omega\times\mathbb{R};[0,1])\right\} \tag{$\mathcal{Q}$}$$

whose set of solutions $\operatorname{argmin}\mathcal{Q}$ is a non-empty weakly star compact subset of $L^\infty(\Omega\times\mathbb{R};[0,1])$.

Lemma 3.1. *It holds* $\inf\mathcal{P} = \inf\mathcal{Q}$ *and the following equivalence holds:*

$$u\in\operatorname{argmin}\mathcal{P} \iff \mathbf{1}_u \in \operatorname{argmin}\mathcal{Q}$$

Proof. Applying (11) with $g=0$, we get

$$\inf\mathcal{Q} = -G^*(0) = -(F_0)^*(0) = -\sup\{-F(u) : u\in X\} = \inf\mathcal{P}$$

The equivalence statement follows by using the identity (12). □

The next step is twofold: first we have to identify the convexified energy in practice in order to settle a duality scheme for $\mathcal{Q}$; then, as some solutions v to $(\mathcal{Q})$ may take intermediate values in $(0,1)$ (i.e. v is not of the form $\mathbf{1}_u$), we have to specify how solutions to $(\mathcal{P})$ can be recovered.

A complete answer to these two requirements will be obtained under an additional assumption on functional F. We will use the following slicing argument on the class

$$\mathcal{A} := \left\{v\in L^\infty(\Omega\times\mathbb{R}) \ : \ v(x,\cdot) \text{ non-increasing (a.e. } x\in\Omega), \ v(x,-\infty)=1, \ v(x,+\infty)=0\right\}$$

For every $v\in\mathcal{A}$ and $s\in[0,1]$, let us define

$$u_s(x) := \inf\left\{\tau\in\mathbb{R} \ : \ v(x,\tau)\leq s\right\} \tag{13}$$

Notice that, by construction, the subgraph of u_s agrees up to a Lebesgue negligible set with the level set $\{\tau\in\mathbb{R} : v(x,\tau)>s\}$, namely,

$$\mathbf{1}_{u_s}(x,t) = \mathbf{1}_{\{v>s\}}(x,t) \qquad \text{for a.e. } (x,t)\in\Omega\times\mathbb{R} \tag{14}$$

In what follows, we denote by v_0 the element of $\mathcal{A}$ defined by

$$v_0(x,t) := \mathbf{1}_{\{t>0\}} \quad (\text{that is } v_0 = \mathbf{1}_{u_0} \text{ with } u_0\equiv 0)$$

Definition 3.1. We say that a functional $J : L^\infty(\Omega\times\mathbb{R}) \to [0,+\infty]$ satisfies the *generalized co-area formula* if, for every $v \in L^\infty(\Omega\times\mathbb{R})$, the function $t \mapsto J(\mathbf{1}_{\{v>s\}})$ is Lebesgue-measurable on $\mathbb{R}$ and there holds

$$J(v) = \int_{-\infty}^{+\infty} J(\mathbf{1}_{\{v>s\}})\,\mathrm{d}s \qquad \forall v \in L^\infty(\Omega\times\mathbb{R}) \tag{15}$$

It is readily seen that a functional J satisfying the generalized co-area formula has to be positively 1-homogeneous (i.e. $J(\lambda v) = \lambda J(v)$ for all $\lambda \ge 0$) and that $J(v)$ vanishes for constant functions v.

Theorem 3.2. *Assume that F satisfies Eq. (10) and that there exists a convex and weakly-star l.s.c. functional $J : L^\infty(\Omega\times\mathbb{R}) \to [0,+\infty]$ satisfying the generalized co-area formula and such that*

$$J(\mathbf{1}_u) \;=\; F(u) \qquad \textit{for every } u \in L^1(\Omega) \tag{16}$$

Then, if $\{u_s;\, s\in[0,1]\}$ is the parametrized family associated with v through (13), it holds

$$G(v) = \begin{cases} \displaystyle\int_0^1 F(u_s)\,\mathrm{d}s & \textit{if } v \in \mathcal{A} \\ +\infty & \textit{otherwise} \end{cases} \tag{17}$$

Therefore, if $v \in \operatorname{argmin} G$, then $v \in \mathcal{A}$ and $u_s \in \operatorname{argmin} F$ for $\mathcal{L}^1$-a.e. $s \in (0,1)$. In particular, if the initial problem $\mathcal{P}$ admits a finite number of solutions $\{u^1,\dots,u^K\}$, then

$$\operatorname{argmin} G \;=\; \left\{ \sum_{k=1}^K t_k\, \mathbf{1}_{u^k}\,, \quad t_k \ge 0\,,\ \sum t_k = 1 \right\} \tag{18}$$

meaning that a solution v to problem $\mathcal{Q}$ must be a piecewise constant function.

It is remarkable consequence of Theorem 3.2 that a global minimizer for problem ($\mathcal{Q}$) suitably chosen (taking $t_k \in (0,1)$ in (18)) can encode all the possibly multiple solutions to problem ($\mathcal{P}$). We refer to [11] for the numerical illustration of this nice feature.

Before giving the proof, let us notice first that the co-area condition (16) is used merely to minorize G. An upper bound for G is provided in the general case owing to the following result.

Lemma 3.3. *Let $v \in L^\infty(\Omega\times\mathbb{R})$ such that $G(v) < +\infty$. Then $v \in \mathcal{A}$, and it holds*

$$G(v) \;\le\; \int_0^1 F(u_s)\,\mathrm{d}s \quad \textit{with } u_s \textit{ defined by } (13)$$

Remark. By a slight modification of the proof, it is possible to show that the conclusions of Lemma 3.3 still hold if the first condition in (10) is replaced by: $F(u) \ge \int_\Omega \beta(|u|)$ where $\beta : R^+ \to [0,+\infty]$ is non-decreasing with $\beta(+\infty) = \infty$.

Proof. By using Fubini formula, one checks easily that, for every $u \in L^1(\Omega)$, one has

$$\int_\Omega |u|\,\mathrm{d}x \;=\; \iint_{\Omega\times\mathbb{R}} |\mathbf{1}_u - v_0|\,\mathrm{d}x\,\mathrm{d}t$$

Therefore, by (10), for every v it holds:

$$F_0(v) \ge\; H(v) := \begin{cases} k \displaystyle\iint_{\Omega\times\mathbb{R}} |v - v_0|\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{k} & \text{if } v \in \mathcal{A} \\ +\infty & \text{otherwise} \end{cases}$$

It is easy to check that $H(v)$ is convex and weakly l.s.c. Indeed, if $\liminf J(v_n) < +\infty$ holds for a sequence v_n in $\mathcal{A}$ such that $v_n \to v$ is weakly star, then $v(x,\cdot)$ is still non-increasing and the inequality $\liminf \iint_{\Omega\times\mathbb{R}} |v_n - v_0| \ge \iint_{\Omega\times\mathbb{R}} |v - v_0|$ shows that $\int_{\mathbb{R}} |v - v_0|(x,t)\,\mathrm{d}t < +\infty$ for a.e. $x \in \Omega$. Thus $v(x,+\infty) = 0$ and $v(x,-\infty) = 1$. It follows that $v \in \mathcal{A}$ and $\liminf H(v_h) \ge H(v)$.

Now we may conclude by simply saying that $H = F_0^{**} \geq J$ so that $G(v) < +\infty$ implies that $H(v) < +\infty$ hence $v \in \mathcal{A}$ (in addition, we get $(v - v_0) \in L^1(\Omega \times \mathbb{R})$).

For the second assertion, we apply Jensen's inequality to the convex functional G and to the family of functions $\{v_s, s \in [0,1]\}$ where $v_s(x,t) = \mathbf{1}_{u_s}(x,t)$ (see (14)). One checks easily that $\int_0^1 v_s(x,t)\,\mathrm{d}s = v(x,t)$. Thus, recalling that $G(\mathbf{1}_{u_s}) = F(u_s)$ holds by Theorem 2.1, we conclude that

$$G(v) \leq \int_0^1 G(v_s)\,\mathrm{d}s = \int_0^1 F(u_s)\,\mathrm{d}s \qquad \square$$

Proof of Theorem 3.2. By the definition of F_0, it holds $J \leq F_0$. Thus, as J is convex l.s.c., by taking the biconjugates, we infer that $J \leq F_0^{**} = G$. Let $v \in \mathcal{A}$. By applying the assumption (16), we derive that $J(v) = \int_0^1 F(u_s)\,\mathrm{d}s$, being u_s defined by (13). Thus $G(v) \geq \int_0^1 F(u_s)\,\mathrm{d}s$. By invoking Lemma 3.3, we are led to the identity (17). Assume now that $v \in \operatorname{argmin} G$. By Lemma 3.1, F and G share the same infimum value, which is a finite real α. Therefore, as $0 = G(v) - \alpha = \int_0^1 (F(u_s) - \alpha)\,\mathrm{d}s$, we deduce that $u_s \in \operatorname{argmin} F$ for a.e. $s \in [0,1]$. Therefore, $(\mathcal{P})$ has infinitely many solutions unless v is piecewise constant. Conversely, if $(\mathcal{P})$ has a finite set of solutions $\{u^k : 1 \leq k \leq K\}$, then it is straightforward that the set of solutions to $(\mathcal{Q})$ coincides with the convex hull of $\{\mathbf{1}_{u^k} : 1 \leq k \leq K\}$. $\square$

4. Duality schemes and examples

A large class of functionals satisfying the assumptions required in Theorem 3.2 are of the kind

$$F_\lambda(u) = \begin{cases} \displaystyle\int_\Omega f(u, \nabla u)\,\mathrm{d}x - \lambda \int_\Omega p(x) u\,\mathrm{d}x\,, & \text{if } u \in W_0^{1,2}(\Omega) \\ +\infty & \text{otherwise} \end{cases}$$

where the integrand $f = f(t,z)$ is a function $f : \mathbb{R} \times \mathbb{R}^N \to (-\infty, +\infty]$ satisfying:

$$\forall t \in \mathbb{R}\,,\ f(t,\cdot) \text{ is convex}\ ,\ f \text{ is l.s.c. on } \mathbb{R}^N \times \mathbb{R}\,,\ \ f(t,z) \geq k|z|^2 - \frac{1}{k} \tag{19}$$

where $k > 0$, λ is a non-negative parameter and the source term $p(x)$ (load) belongs to $L^{r'}(\Omega)$ (r' conjugate exponent of r) where r is compatible with the embedding $W_0^{1,2}(\Omega) \subset L^r(\Omega)$ (that is $r \leq \frac{2N}{N+2}$ if $N \geq 3$, $r < +\infty$ if $N = 2$).

Notice that here the non-convexity of the energy density $f(u, \nabla u)$ involves only the dependence with respect to u. In fact, the convexity with respect to the gradient part is necessary to obtain lower semicontinuity for $F(u)$ and well-posedness for the primal problem. It turns out that the condition (16) is satisfied by considering the convex 1-homogeneous functional defined by:

$$J(v) := \int_{\Omega \times \mathbb{R}} h_f(t, Dv) \quad \text{where} \quad h_f(t, z^x, z^t) := \begin{cases} -z^t f\left(t, \dfrac{z^x}{-z^t}\right) & \text{if } z^t < 0 \\ +\infty & \text{if } z^t \geq 0 \end{cases} \tag{20}$$

We refer to the recent paper [11] for further details, namely for a proof of the co-area formula.

4.1. Dual problem in $\Omega \times \mathbb{R}$

Let us describe the dual problem in the simpler case where f is of the form $f(t,z) = g(t) + \varphi(z)$ being $\varphi : \mathbb{R}^N \to \mathbb{R}^+$ convex continuous with $\varphi(0) = 0$ and $g : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ a lower semicontinuous function with possibly countably many discontinuities. The primal problem reads

$$\inf \left\{ \int_\Omega (\varphi(\nabla u) + g(u))\,\mathrm{d}x - \lambda \int_\Omega p(x) u\,\mathrm{d}x \ :\ u \in H_0^1(\Omega) \right\} \tag{$\mathcal{P}_\lambda$}$$

and its convexified version

$$\inf \left\{ \iint_{\Omega \times \mathbb{R}} h_f(t, Dv) - \lambda \iint_{\Omega \times \mathbb{R}} p(x) v\,\mathrm{d}x\,\mathrm{d}t \ :\ v \in \mathcal{A}\,,\ v - v_0 \in BV_0(\Omega \times \mathbb{R}) \right\} \tag{$\mathcal{Q}_\lambda$}$$

where $BV_0(\Omega \times \mathbb{R})$ denotes the set of integrable functions with bounded variations on $\Omega \times \mathbb{R}$ and whose trace vanishes on the lateral boundary $\partial\Omega \times \mathbb{R}$ (see [14]).

The dual problem to our non-convex problem $(\mathcal{P}_\lambda)$ is then recovered be applying classical duality to problem $(\mathcal{Q}_\lambda)$. The competitors of this dual problem $(\mathcal{P}^*_\lambda)$ are vector fields $\sigma = (\sigma^x, \sigma^t) : \Omega \times \mathbb{R} \to \mathbb{R}^N \times \mathbb{R}$ that we take in the class

$$X_1(\Omega \times \mathbb{R}) = \left\{ \sigma \in L^\infty(\Omega \times \mathbb{R}; \mathbb{R}^{N+1}) \ : \ \operatorname{div} \sigma \in L^1_{\mathrm{loc}}(\overline{\Omega} \times \mathbb{R}) \right\}$$

and $(\mathcal{P}^*_\lambda)$ consists of the following maximal flux problem:

$$\sup \left\{ -\int_\Omega \sigma^t(x,0)\,\mathrm{d}x \ : \ \sigma \in \mathcal{K}\,, \ -\operatorname{div}\sigma = \lambda\, p \quad \text{in } \Omega \times \mathbb{R} \right\} \tag{$\mathcal{P}^*_\lambda$}$$

where $\sigma \in \mathcal{K}$ means that the vector field $\sigma \in X_1(\Omega \times \mathbb{R})$ satisfies the pointwise (convex) constraints:

$$\begin{cases} \sigma^t(x,t) \geq \varphi^*(\sigma^x(x,t)) - g(t) \ \text{ for } \mathcal{L}^{N+1}\text{-a.e. } (x,t) \in \Omega \times \mathbb{R} \\ \sigma^t(x,t) \geq -g(t) \quad \forall t \in S_g \text{ and for } \mathcal{L}^N\text{-a.e. } x \in \Omega \end{cases} \tag{21}$$

where S_g is the set of discontinuities of g.

Notice here that the regularity condition $\sigma \in X_1(\Omega \times \mathbb{R})$ is required in order to be able to define the normal trace of σ on a N-dimensional rectifiable subset of $\mathbb{R}^{N+1}$ (in particular for every t, $\sigma^t(\cdot,t)$ is well defined for a.e. $x \in \Omega$). This allows us also to compute the flux of σ through the graph of any competitor u for $(\mathcal{P}_\lambda)$. By applying a generalized Green formula for duality pairings (v,σ), it is easy to check that, for every admissible pair (u,σ), one has

$$\int_\Omega f(u,\nabla u)\,\mathrm{d}x - \lambda \int_\Omega p(x) u\,\mathrm{d}x \quad \geq \quad -\int_\Omega \sigma^t(x,0)\,\mathrm{d}x$$

thus it holds $\inf(\mathcal{P}_\lambda) \geq \sup(\mathcal{P}^*_\lambda)$. The core of our duality theory is the following no-gap result (see [11] for a complete proof in the case $p = 0$).

Theorem 4.1.

$$\inf(\mathcal{P}_\lambda) \quad = \quad \sup(\mathcal{P}^*_\lambda)$$

Among consequences of Theorem 4.1, we can derive (see [11]) necessary and sufficient conditions for a global optimum of $(\mathcal{P}_\lambda)$, thus allowing us to rule out local minimizers that are non-global ones. A second consequence is a saddle point characterization that fits to the implementation of efficient primal–dual algorithms ([15]). This is described in the next subsection.

Remark 4.2. The result above can be extended to mixed Dirichlet–Neumann conditions. In particular, if $u = u_0$ is prescribed on a subset $\Gamma_0 \subset \partial\Omega$ for some $u_0 \in W^{1,2}(\Omega)$, then the competitors σ for $(\mathcal{P}^*_\lambda)$ have to satisfy a vanishing normal trace condition on $(\partial\Omega \setminus \Gamma_0) \times \mathbb{R}$, while the linear term to be maximized becomes $\int_{G_{u_0}} \sigma \cdot \nu_{u_0}$, being G_{u_0} the graph of u_0 with unit normal ν_{u_0} pointing downwards. For the more delicate case of Robin-type conditions, we refer the reader to [11].

Remark 4.3. When the boundary datum on Γ_0 is a bounded function u_0, there exist in general a priori lower and upper bounds for the minimizers of the primal problem $(\mathcal{P}_\lambda)$. In this case, the infimum is unchanged if we impose u to take values in a suitable closed interval $\overline{I} := [m, M]$ of the real line. We are thus led to consider the variant of the primal problem $(\mathcal{P}_\lambda)$, where the class of admissible functions is restricted to $\left\{ u \in W^{1,2}(\Omega; I) \ : \ u = u_0 \text{ on } \Gamma_0 \right\}$. The duality result continues to hold (with a simpler proof), provided the admissible fields in the dual problem $(\mathcal{P}^*_\lambda)$ are taken in the class $\mathcal{K}(\Gamma_0, I)$ of elements $\sigma \in X_1(\Omega \times (m, M))$ satisfying the pointwise constraints (21) on $\Omega \times \overline{I}$ and the equilibrium conditions

$$-\operatorname{div}\sigma = \lambda\, p(x) \ \text{ in } \Omega \times I \quad , \quad \sigma^x \cdot \nu_\Omega = 0 \ \text{ on } \ (\partial\Omega \setminus \Gamma_0) \times I$$

Accordingly, the convexified problem $(\mathcal{Q}_\lambda)$ becomes

$$\inf \left\{ \iint_{\Omega \times I} h_f(t, Dv) - \lambda \iint_{\Omega \times I} p(x) v\,\mathrm{d}x\,\mathrm{d}t \ : \ v \in \mathcal{A}(u_0, \Gamma_0, I) \right\} \tag{$\mathcal{Q}_\lambda$}$$

where the set of admissible functions v is given by

$$\mathcal{A}(u_0, \Gamma_0, I) := \left\{ v \in BV(\Omega \times I; [0,1]) \ : \ v = 1 \text{ on } \Omega \times \{m\}\,, \ v = 0 \text{ on } \Omega \times \{M\}\,, \ v = \mathbf{1}_{u_0} \text{ on } \Gamma_0 \times I \right\}$$

(the condition $\iint_{\Omega \times I} h_f(t, Dv) < +\infty$ implies implicitly that $v(x,\cdot)$ is monotone non-increasing).

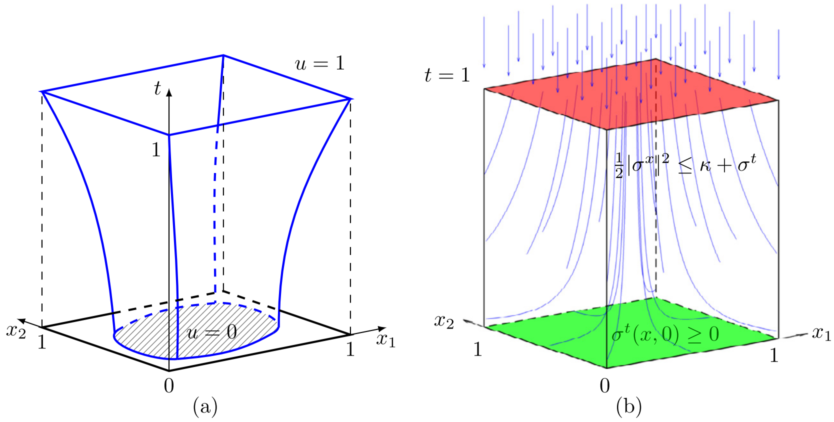

Fig. 1. (a) The free boundary problem (22). (b) The optimal flow problem (23).

Remark 4.4. The growth condition (19) with exponent $r=2$ can be considered with a different exponent $r \in (1,+\infty)$ and all the statements can be reformulated accordingly. The case $r=1$ works pretty well for the dual problem – see [10] – but a lot of attention has to be devoted to the existence and compactness issue in the primal problem. Indeed, the functional F is no more l.s.c. in $L^1(\Omega)$ and has to be relaxed in the space $BV(\Omega)$, whereas, so that $\inf(\mathcal{P}_\lambda) > -\infty$, we need to chose λ in a finite interval $[0,\lambda^*)$ (this is in relation with the limit load problem is plasticity [16]).

To have in mind a prototype situation, let us mention, for instance, the free boundary problem studied in the seminal paper [17]:

$$\inf\left\{\int_\Omega \frac{1}{2}|\nabla u|^2\,\mathrm{d}x + \kappa\,\big|\{u>0\}\big| \;:\; u \in W^{1,2}(\Omega)\,,\; u=1 \text{ on } \partial\Omega\right\} \tag{22}$$

the free boundary being the frontier of the positivity set $\{u>0\}$ (see Fig. 1, in which $\Omega=(0,1)^2 \subset \mathbb{R}^2$).

Clearly, problem (22) falls into this general framework, by taking $\Gamma_0=\partial\Omega$, $u_0 \equiv 1$, $p\equiv 0$, and $\varphi(z)=\frac{1}{2}|z|^2$ and $g(t)=\kappa\,\mathbf{1}_{(0,+\infty)}(t)$ which jumps at $t=0$. It is easy to check that solutions u exist and satisfy $0\le u \le 1$ a.e. so that Remark 4.3 applies and the dual problem can be restricted to vector fields defined in $\Omega\times(0,1)$. Let $\mathcal{K}$ be the set of $\sigma \in X_1(\Omega\times(0,1))$ such that

$$\sigma^t(x,t)+\kappa \geq \frac{1}{2}|\sigma^x(x,t)|^2 \quad \text{a.e. on } \Omega\times\mathbb{R}\,, \qquad \sigma^t(x,0)\geq 0 \text{ a.e. on } \Omega$$

The dual problem reads:

$$\sup\left\{-\int_\Omega \sigma^t(x,1)\,\mathrm{d}x \;:\; \sigma\in\mathcal{K}\,,\; \operatorname{div}\sigma=0 \quad \text{in } \Omega\times(0,1)\right\} \tag{23}$$

Notice that the integral on Ω represents the flux of σ across the graph of the boundary datum $u_0\equiv 1$. Thus problem (23) has a nice fluid mechanic interpretation: it consists in maximizing the downflow through the top face $\Omega\times\{1\}$ of an incompressible fluid constrained into the cylinder $\Omega\times\mathbb{R}$, whose speed σ satisfies the conditions above, preventing in particular the fluid from passing across the bottom face (see Fig. 1(b)).

4.2. Saddle point characterization

We consider the variant described in Remark 4.3 where competitors for $(\mathcal{Q}_\lambda)$ are in the class $\widehat{\mathcal{A}} := \mathcal{A}(u_0,\Gamma_0,I)$ (here $I=(m,M)$) and competitors σ for $(\mathcal{P}^*_\lambda)$ are in the class $\widehat{\mathcal{K}} := \mathcal{K}(u_0,\Gamma_0,I)$. Let us introduce, for every pair (v,σ), with $v\in BV(\Omega\times I;[0,1])$ and $\sigma\in X_1(\Omega\times(m,M))$, the following Lagrangian

$$L(v,\sigma) := \iint_{\Omega\times I} (\sigma\cdot \mathrm{D}v) - \lambda \iint_{\Omega\times I} p(x)\,v\,\mathrm{d}x\,\mathrm{d}t \tag{24}$$

Theorem 4.5. *There holds*

$$\inf(\mathcal{P}_\lambda) \;=\; \inf_{v\in\widehat{\mathcal{A}}}\,\sup_{\sigma\in\widehat{\mathcal{K}}} L(v,\sigma) \;=\; \sup_{\sigma\in\widehat{\mathcal{K}}}\,\inf_{v\in\widehat{\mathcal{A}}} L(v,\sigma) \;=\; \sup(\mathcal{P}^*_\lambda)$$

Moreover, a pair $(\overline{v}, \overline{\sigma})$ is optimal for $(\mathcal{Q}_\lambda)$ and for the dual problem $(\mathcal{P}_\lambda^*)$ if and only if it is a saddle point for L, namely

$$L(\overline{v}, \sigma) \le L(\overline{v}, \overline{\sigma}) \le L(v, \overline{\sigma}) \qquad \forall (v, \sigma) \in \widehat{\mathcal{A}} \times \widehat{\mathcal{K}}$$

The proof is straightforward. Different numerical schemes (explicit and implicit) in order to solve the saddle-point problem above are presented in [11] and [15]. In particular, in the case of the 2d-example (22), some threshold value κ^* can be computed, for which a numerical solution $v(x_1, x_2, t)$ is piecewise constant, taking three values $0, \theta, 1$ ($\theta \in (0, 1)$). The upper level sets of v thus determine two global minimizers u_1, u_2 for the original free boundary problem ($u_1 \equiv 1$ remains a solution for $\kappa < \kappa^*$, and it is the unique one).

4.3. An example of Γ-convergence

We revisit here the celebrated asymptotic analysis of the Modica–Mortola functional, which arises in the sharp interface model for Cahn–Hilliard fluids, showing how the duality approach developed in Section 2 can be used efficiently. In fact, we can treat a slightly more general model where we consider a family of functionals $(F^\varepsilon)_{\varepsilon>0}$, indexed with a (small) scale parameter $\varepsilon > 0$, of the following form (see [18])

$$F^\varepsilon(u) := \frac{1}{\varepsilon} \int_\Omega f(u(x), \varepsilon \nabla u(x))\, dx$$

where Ω is a bounded domain of $\mathbb{R}^N$ with Lipschitz boundary, and we make the following assumptions on $f : \mathbb{R} \times \mathbb{R}^N \to [0, +\infty)$:

i) f is continuous in the first variable and convex in the second;
ii) there exist two real numbers $0 < \alpha < \beta$ such that $f(t, 0) > 0$ if $t \neq \alpha, \beta$, $f(\alpha, 0) = f(\beta, 0) = 0$, and for every $z \neq 0$ and every t, $f(t, z) > f(t, 0)$;
iii) there exists $M > \beta$ such that $f(t, \cdot)$ is locally bounded, uniformly in $t \in [0, M]$;
iv) there exists a function ψ with superlinear growth at ∞, such that $f(t, p) \ge \psi(p)$ for every $t \in \mathbb{R}$ and $p \in \mathbb{R}^N$.

Under such assumptions, it is not difficult to show that the family $\{F^\varepsilon, \varepsilon > 0\}$ is equicoercive in $X = L^1(\Omega; [0, M])$ (that is, it satisfies the condition (H2) in Section 2). Our aim is to compute the Γ-limit of F_ε as $\varepsilon \to 0$. For every $t \in \mathbb{R}$ and $z \in \mathbb{R}^N$, we define

$$f_\varepsilon(t, z) = \frac{f(t, \varepsilon z)}{\varepsilon} \quad , \quad f_c(t, z) = \inf_{\varepsilon>0} f_\varepsilon(t, z) \quad , \quad h(z) = \int_\alpha^\beta f_c^{**}(t, z)\, dt$$

By construction, the conical envelope of f_c is one-homogeneous in z. It follows that h is a convex and one homogeneous function of z. An easy computation involving Moreau–Fenchel conjugates, for fixed t and with respect to the variable z, shows that

$$f_\varepsilon^*(t, z^*) = \frac{1}{\varepsilon} f^*(t, z^*) \quad , \quad f_c^{**}(z) = \sup_{z^* \in \mathbb{R}^N} \{z \cdot z^* \ : \ f^*(t, z^*) \le 0\} \tag{25}$$

Under these assumptions, we can show the following result.

Theorem 4.6. *As ε goes to zero, F^ε Γ-converges in $L^1(\Omega; [0, M])$ to the functional F given by*

$$F(u) = \begin{cases} \displaystyle\int_{S_u \cap \Omega} h(\nu_u)\, dH^{N-1} & \text{if } u \in BV(\Omega; \{\alpha, \beta\}) \\ +\infty & \text{otherwise} \end{cases}$$

Here S_u denotes the discontinuity set of u given in the form $u = \alpha \mathbf{1}_A + \beta \mathbf{1}_{\Omega \setminus A}$, $\nu_u = \frac{Du}{|Du|}$ represents the inwards pointing normal to the interface $\partial A \cap \Omega$ and the integral on S_u is taken with respect to the $N-1$ dimensional measure.

The Modica–Mortola functional corresponds to taking $f(t, z) = \frac{1}{2}|z|^2 + W(t)$, where the double-well potential $W : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function such that

$$\lim_{t \to \infty} \frac{W(t)}{t} = +\infty \quad , \quad W(t) = 0 \iff t \in \{\alpha, \beta\}$$

In that case, we recover an isotropic interface energy $h(z) = c\,|z|$, where the surface tension coefficient is determined by $c = \int_\alpha^\beta \sqrt{2W(s)}\, ds$ (see [19]).

As an alternative to [18], we propose here a proof by duality exploiting Theorem 2.5 in Section 2. Let us sketch the different steps. We consider $X = L^1(\Omega; [0, M])$, $Y := L^\infty(\Omega \times (-1, M))$ (endowed with the weak-star topology), and $\varphi : u \in X \mapsto \mathbf{1}_u(x, y)$. The assumptions (H1), (H2) are fulfilled as well as (H3) if we consider the (weakly star) continuous linear form $l_0(v) := |\Omega|^{-1} \int_{\Omega\times(-1,0)} v \,\mathrm{d}x\,\mathrm{d}t$ (as $u \in X$ is non-negative, it holds $\mathbf{1}_u = 1$ on $\Omega \times (-1, 0)$ so that $l_0(\mathbf{1}_u) = 1$).

Let us set $Q := \Omega \times (-1, M)$, $Q^- := \Omega \times (-1, 0)$, $Q^+ := \Omega \times (0, M)$. First we compute:

$$D^\varepsilon := \left\{ p \in L^1(Q) : \iint_Q \mathbf{1}_u \, p \,\mathrm{d}x\,\mathrm{d}t < F^\varepsilon(u) \quad \forall u \in X \right\}$$

We observe that $p \in D^\varepsilon$ iff it holds $\inf\{F^\varepsilon(u) - \iint_{Q^+} p\,\mathbf{1}_u\} > \iint_{Q^-} p$. Thus, by applying the duality result of Theorem 4.1 (in the variants described in Remark 4.2 and Remark 4.3), we get

$$D^\varepsilon = \Big\{ p \in L^1(Q) \ : \ \exists \sigma \in \mathcal{K}_\varepsilon \,,\ -\operatorname{div}\sigma = p \quad \text{in } Q^+ \,,\ \sigma^x \cdot \nu_\Omega = 0 \quad \text{on } \partial\Omega \times (0, M)$$
$$\int_\Omega \sigma^t(x, 0)\,\mathrm{d}x + \iint_{Q^-} p < 0 \Big\}$$

where, in view of (25) and of the continuity of $f(\cdot, z)$,

$$\mathcal{K}_\varepsilon = \{\sigma \ \in L^\infty(Q^+; \mathbb{R}^{N+1}) \ : \ f^*(t, \sigma^x) \le \varepsilon\,\sigma^t \quad \text{a.e. in } Q^+\}$$

Next we define D to be the closure in $L^1(Q)$ of the set D_0 given by

$$D_0 = \Big\{ p \in L^1(Q) \ : \ \exists \sigma \in \mathcal{K}_0 \ : ,\ -\operatorname{div}\sigma = p \quad \text{in } Q^+ \,,\ \sigma^x \cdot \nu_\Omega = 0 \quad \text{on } \partial\Omega \times (0, M)$$
$$\int_\Omega \sigma^t(x, 0)\,\mathrm{d}x + \iint_{Q^-} p < 0 \Big\}$$

where

$$\mathcal{K}_0 = \{\sigma \ \in C^1(\overline{Q^+}) \ : \ \sigma(x,t) \in \Gamma_0(t) \quad \forall (x,t) \in Q^+\}$$
$$C_0(t) = \Big\{ q = (z^*, \tau) \in \mathbb{R}^N \times \mathbb{R} : f^*(t, z^*) < 0 \,,\ \tau > 0 \text{ if } t \in \{\alpha, \beta\} \Big\}$$

We introduce the following functional G defined on $L^1(Q)$:

$$G(v) \ := \ \sup_{p \in D} \iint_Q v \cdot p \,\mathrm{d}x\,\mathrm{d}t \ = \ \sup_{p \in D_0} \iint_Q v \cdot p \,\mathrm{d}x\,\mathrm{d}t$$

By a straightforward computation, we observe that the support function of the convex constraint associated with subset $\mathcal{K}_0$ i.e. $C_0(t) = \big\{ q = (z^*, \tau) \in \mathbb{R}^N \times \mathbb{R} : f^*(t, z^*) < 0 \,,\ \tau > 0 \text{ if } t \in \{\alpha, \beta\} \big\}$ is given by:

$$h(t, z^x, z^t) = f_c^{**}(t, z^x) \quad \text{if } z^t = 0 \quad \text{or } \big[z^t \le 0 \text{ and } t \in \{\alpha, \beta\} \big] \quad , \quad h(t, z^x, z^t) = +\infty \quad \text{otherwise}$$

As a consequence of a commutation argument between the symbols sup and $\iint$ (see [20]), we infer that $G(v) = \iint_Q h(t, \mathrm{D}v)$. In particular, $G(v)$ is finite only for those functions $v(x, t)$ that are piecewise constant with respect to t and such that $v = 1$ for $t < \alpha$, $v = 0$ for $t > \beta$ and $v = \theta(x)$ for $t \in (\alpha, \beta)$ being θ an element of $BV(\Omega; [0, 1])$. Then $G(v) = \int_\alpha^\beta \int_\Omega f_c^{**}(t, \mathrm{D}\theta) = \int_\Omega h(\mathrm{D}\theta)$. If $v = \mathbf{1}_u$, then we have $\theta = \mathbf{1}_A$ for a suitable subset $A \subset \Omega$ with finite perimeter so that $u = \alpha \mathbf{1}_A + \beta \mathbf{1}_{\Omega\setminus A}$ and $G(\mathbf{1}u) = F(u)$. Therefore, $G(v)$ (which satisfies the co-area formula) is nothing else but the convexified functional associated with the limit F given in Theorem 4.6. In other words, we have showed that the set D_0 above satisfies

$$\overline{D_0} \ = \ D \ = \ \left\{ p \in L^1(Q) \ : \ \iint_Q p\,\mathbf{1}_u \,\mathrm{d}x\,\mathrm{d}t \le F(u) \quad \forall u \in L^1(Q) \right\} \tag{26}$$

Owing to Theorem 2.5, we deduce Theorem 4.6 by invoking the following result.

Lemma 4.7. *With the notations above, D_ε converges in the Kuratowski sense to D in $L^1(Q)$.*

Proof. First we prove that $D_0 \subset Li(D_\varepsilon)$ (thus $D \subset Li(D_\varepsilon)$). Let $p \in D_0$ and $\sigma \in \mathcal{K}_0$ associated. By continuity, σ^t is positive in a neighborhood of $t \in \{\alpha, \beta\}$, while in the complementary $f^*(t, \sigma^x)$ is majored by a negative constant. Therefore, $\sigma \in \mathcal{K}_\varepsilon$ and $p \in D^\varepsilon$ for ε small enough.

Let us show now that $Ls(D_\varepsilon) \subset D$. Let $p \in Ls(D_\varepsilon)$. Then there exists a sequence such that $p_\varepsilon \to p$ in $L^1(Q)$ and $p_\varepsilon = -\operatorname{div} \sigma_\varepsilon$ in Q^+, where σ_ε has a vanishing normal trace on $\partial\Omega \times (0, M)$ and satisfies

$$f^*(t, \sigma_\varepsilon^x) \le \varepsilon\, \sigma_\varepsilon^t \quad \text{in } Q^+ \quad , \quad \int_\Omega \sigma_\varepsilon^t(x, 0)\, \mathrm{d}x + \iint_{Q^-} p_\varepsilon < 0 \tag{27}$$

By the Gauss–Green formula applied on $\Omega \times [0, s]$ and by exploiting the vanishing normal trace condition on $\partial\Omega \times (0, M)$, we obtain that for every $s \in (0, M)$, there holds:

$$\int_\Omega \sigma_\varepsilon^t(x, s)\, \mathrm{d}x - \int_\Omega \sigma_\varepsilon^t(x, 0)\, \mathrm{d}x = \iint_{\Omega \times [0,s]} p_\varepsilon\, \mathrm{d}x\, \mathrm{d}t \le \|p_\varepsilon\|_{L^1(Q)}$$

Thus, by integrating in s over $(0, M)$ and taking into account (27), we deduce that

$$\iint_{Q^+} f^*(t, \sigma_\varepsilon^x) \le \varepsilon \iint_{Q^+} \sigma_\varepsilon^t(x, t)\, \mathrm{d}x\, \mathrm{d}t \ \le\ M\, \varepsilon\, \|p_\varepsilon\|_{L^1(Q)}$$

As $f^*(t, \sigma_\varepsilon^x) \ge -f(t, 0)$ is minorized, we infer that $\iint_{Q^+} f^*(t, \sigma_\varepsilon^x) \to 0$ and that σ_ε^t is bounded in $L^1(Q^+)$. By the assumption iii) on f, this implies that $\{\sigma_\varepsilon^x\}$ is bounded and equi-integrable in $L^1(Q^+)$, hence up to a subsequence, we may assume that $\sigma_\varepsilon^x \rightharpoonup \sigma^x$ in $L^1(Q^+; \mathbb{R}^N)$ for a suitable σ^x. By the convexity of $f^*(t, \cdot)$, it is easy to show that the weak limit σ^x satisfies $f^*(t, \sigma^x) \le 0$ a.e. in Q^+. Let $u \in L^1(\Omega)$ such that $F(u) < +\infty$. Then we have $u = \alpha \mathbf{1}_A + \beta \mathbf{1}_{\Omega \setminus A}$, where A is a subset of finite perimeter. We represent by ν_A the outward pointing normal to A, which is well defined a.e. on the essential boundary of A denoted ∂A (its essential boundary).

Considering Eq. (26), in order to show that $p \in D$, we are reduced to check that

$$\iint_Q p\, \mathbf{1}_u\, \mathrm{d}x\, \mathrm{d}t \le F(u) = \int_{\partial A \cap \Omega} h(\nu_A)\, \mathrm{d}H^1$$

Recalling that $p_\varepsilon \to p$ in $L^1(Q)$ while $p_\varepsilon = -\operatorname{div} \sigma_\varepsilon$ in Q^+ and $\mathbf{1}_u = 1$ in Q^-, we have

$$\begin{aligned}
\iint_Q p\, \mathbf{1}_u\, \mathrm{d}x\, \mathrm{d}t \ &= \ \lim_{\varepsilon \to 0} \left(\iint_{Q^-} p_\varepsilon\, \mathrm{d}x\, \mathrm{d}t - \iint_{Q^+} \mathbf{1}_u \operatorname{div} \sigma_\varepsilon \right) \\
&\le \limsup_{\varepsilon \to 0} \left(- \int_A \sigma_\varepsilon^t(x, \beta)\, \mathrm{d}x - \int_{\Omega \setminus A} \sigma_\varepsilon^t(x, \alpha)\, \mathrm{d}x + \iint_{\partial A \times (\alpha, \beta)} \sigma_\varepsilon^x(x, t) \cdot \nu_A\, H^1(\mathrm{d}x) \otimes \mathrm{d}t \right) \\
&\le \iint_{\partial A \times (\alpha, \beta)} \sigma^x(x, t) \cdot \nu_A\, H^1(\mathrm{d}x) \otimes \mathrm{d}t
\end{aligned}$$

where:

- in the second line we applied the generalized Gauss–Green formula on subset Q^+ taking into account the right-hand side inequality in (27) and the fact that $\sigma_\varepsilon^x \cdot \nu_\Omega$ vanishes on $\partial\Omega \times (0, M)$,
- in the third line, we used the fact that $\sigma_\varepsilon^t(\cdot, t)$ is nonnegative for $t \in \{\alpha, \beta\}$ together with the weak convergence of the normal trace of σ_ε on $\partial A \times (\alpha, \beta)$.

Next we observe that, thanks to $f^*(t, \sigma^x) \le 0$, we have $f_c^*(t, \sigma^x) = 0$ and, by the Moreau–Fenchel inequality, there holds: $\sigma_\varepsilon^x(x, t) \cdot \nu_A \le f_c^{**}(t, \nu_A)$. We can therefore conclude that

$$\iint_Q p\, \mathbf{1}_u\, \mathrm{d}x\, \mathrm{d}t \ \le \iint_{\partial A \times (\alpha, \beta)} f_c^{**}(t, \nu_A)\, H^1(\mathrm{d}x) \otimes \mathrm{d}t \ = \int_{\partial A} h(\nu_A)\, \mathrm{d}H^1 \ = \ F(u) \qquad \square$$

5. Perspectives and open problems

5.1. Functionals involving vector-valued functions

The arguments in Section 2 have been developed merely in the case of scalar functions. Namely, the space X for which we construct an embedding in the extreme points of some convex compact subset has always been $L^1(\Omega)$. An extension of the method working for vector-valued functions requires to construct another embedding. A very simple choice would be to associate with a vector field $u \in L^1(\Omega, \mathbb{R}^N)$ the Dirac mass at $u(x)$, so that $\varphi(u)$ becomes the Young measure on $\Omega \times \mathbb{R}^N$, defined by:

$$< \varphi(u), \psi >= \int_\Omega \psi(x, u(x))\, \mathrm{d}x\,, \;\; \psi \in C^0(\Omega \times \mathbb{R}^N)$$

If $|\Omega| = 1$ and $u(x)$ is assigned to stay in a given convex compact subset $K \subset \mathbb{R}^N$, then $\varphi(u)$ is an extreme point in the set of probabilities measures on $\Omega \times K$ whose first marginal agrees with the Lebesgue measure on Ω. However, the explicit computation of the convexified functional seems difficult in this framework. A possible issue would be to consider more involved tools of geometric measure theory as Cartesian currents or varifolds.

5.2. Functionals involving second-order gradients

Going back to the scalar case, many problems involve functionals of the kind

$$F(u) \;=\; \int_\Omega f(\nabla u, \nabla^2 u)\, \mathrm{d}x$$

being $f(z, M) : \mathbb{R}^N \times \mathbb{R}^{N\times N}_{\mathrm{sym}}$ a function convex in M, but not in z. Applying the convexification procedure like in Section 3 seems to be a nice perspective in this context; it will involve the curvature tensor at each point $(x, u(x))$ of the graph of u.

5.3. Munford–Shah functional

The free discontinuity problems have been a late motivation for the convexification recipe presented in this paper. The goal was to prove the optimality of some specific configurations for the image segmentation problem described hereafter. Let Ω be a bounded and Lipschitz domain of $\mathbb{R}^2$ and $g : \Omega \to [0, 1]$ (grey-level data).

$$\inf_{u\in SBV(\Omega)} \left\{ \int_{\Omega\setminus S_u} \frac{1}{2}|\nabla u|^2\, \mathrm{d}x + H^1(S_u) + \frac{1}{2}\int_\Omega |u - g(x)|^2\, \mathrm{d}x \right\} \qquad (\mathcal{P})$$

This setting turns out to be well posed (existence of at least one minimizer) in the space $SBV(\Omega)$ of functions $u \in L^1(\Omega)$ whose distributional gradient Du consists of a regular part ∇u (coinciding with the a.e. defined gradient) and a singular part concentrated on the jump set S_u, which is a rectifiable one-dimensional (unknown) subset of Ω, whose total length is denoted by $H^1(S_u)$. A more mechanical formulation of $(\mathcal{P})$ (popular in fracture mechanics) reads

$$\inf \left\{ \int_{\Omega\setminus K} \frac{1}{2}|\nabla u|^2\, \mathrm{d}x + H^1(K) + \frac{1}{2}\int_\Omega |u - g(x)|^2\, \mathrm{d}x, \quad K \text{ closed subset} \subset \Omega\,,\; u \in C^1(\Omega \setminus K) \right\}$$

As the source term g satisfies $0 \le g \le 1$, by using a trivial truncation argument, one checks easily that the infimum of $(\mathcal{P})$ is unchanged if restricted to competitors u taking values in $[0, 1]$. Accordingly, we consider the metric space $X = L^1(\Omega; [0, 1])$, on which we define the functional

$$F(u) := \begin{cases} \displaystyle\int_{\Omega\setminus S_u} \frac{1}{2}|\nabla u|^2\, \mathrm{d}x + H^1(S_u) + \frac{1}{2}\int_\Omega |u - g(x)|^2\, \mathrm{d}x & \text{if } u \in SBV(\Omega; [0, 1]) \\ +\infty & \text{otherwise} \end{cases}$$

Then F turns out to be coercive and l.s.c. Recalling the construction in Section 3 (see also Remark 4.3), we can define a convex functional G on $L^\infty(\Omega \times [0, 1])$ by setting:

$$G(v) = \sup_{g \in L^\infty(\Omega\times[0,1])} \left\{ \iint\limits_{\Omega\times[0,1]} g\, v\, \mathrm{d}x\, \mathrm{d}t - F_0^*(g) \right\} , \quad F_0^*(g) = \sup_{u\in X} \left\{ \iint\limits_{\Omega\times[0,1]} g(x,t)\, \mathbf{1}_u\, \mathrm{d}x\, \mathrm{d}t - F(u) \right\}$$

so that $G(\mathbf{1}_u) = F(u)$ for every $u \in X$ and $\inf(\mathcal{P}) = \inf\{G(v) : v \in L^\infty(\Omega \times [0,1]; [0,1])\}$.

Unfortunately, this functional G cannot be recovered by using the co-area formula (17) and, to our knowledge, no explicit formula for G is available. Alternatively, in [8], another convex l.s.c. functional J was used such that $J \le G$, but satisfying $J(\mathbf{1}_u) = G(\mathbf{1}_u) = F(u)$ for every $u \in X$. Although it is not known whether or not J shares the same infimum as G, a duality scheme applied to J has been unexpectedly useful for checking the optimality of some competitors for problem $(\mathcal{P})$ (see many examples in [8]). In this framework, the dual problem reads as follows:

$$\sup \left\{ -\int\limits_{\Omega} \sigma^t(x,0)\, \mathrm{d}x\ ,\ \sigma \in \mathcal{K}\ ,\ \mathrm{div}\sigma = 0 \ \text{ on } \Omega\times[0,1]\ ,\ \sigma^x \cdot \nu_\Omega = 0 \ \text{ on } \partial\Omega\times[0,1] \right\} \tag{$\mathcal{Q}$}$$

where the convex constraint $\sigma \in \mathcal{K}$ splits into the two conditions:

i) $\frac{1}{2}|\sigma^x|^2 \le \sigma^t + \frac{1}{2}|t - g(x)|^2$ a.e. in $\Omega \times [0,1]$,

ii) $\left| \int_{t_1}^{t_2} \sigma^x(x,s)\, \mathrm{d}s \right| \le 1$ a.e. $x \in \Omega$ and for every $(t_1, t_2) \in [0,1]^2$.

The second condition takes into account the jump energy in $F(u)$ and *is non-local*. The functional J defined in $BV(\Omega \times \mathbb{R}; [0,1])$ can be recovered by duality:

$$J(v)\ =\ \sup \left\{ \iint\limits_{\Omega\times\mathbb{R}} (\mathrm{D}v \cdot \sigma)\ :\ \sigma \in \mathcal{K}\ ,\ \sigma \in C^1(\Omega\times\mathbb{R}) \right\}$$

As claimed before, this convex functional satisfies $J(\mathbf{1}_u) = F(u)$, whereas $J(v) < +\infty$ implies that $v(x,\cdot)$ is non-increasing. It is then possible to prove:

Proposition 5.1. *Let $g \in L^\infty(\Omega; [0,1])$. Then it holds $\inf(\mathcal{P}) \ge \sup(\mathcal{Q})$ with equality if, for an admissible pair (u,σ), one has*

$$\begin{aligned} \sigma(x, u(x)) \ &=\ (\nabla u(x), \tfrac{1}{2}(|\nabla u|^2 - |u-g|^2)) \quad \text{a.e. } x \in \Omega \\ \int_{u^-(x)}^{u^+(x)} \sigma^x(x,t)\cdot \nu_u \ &=\ 1 \quad H^1 \text{ a.e. } x \in S_u \end{aligned} \tag{28}$$

where $u^\pm$ denote the upper and lower approximate limits of u and $S_u = \{u^+ > u^-\}$.

Let us notice that the latter result is useful merely when it is possible to guess particular pairs (u,σ) satisfying conditions (28). When a competitor u is candidate to be a global minimizer, finding a σ provides a sufficient condition of optimality. This calibrating vector field σ, if it exists, is determined on the graph of u by relations (28). The difficulty is to extend it outside the graph of u while preserving the constraints i) and ii) and the divergence-free condition. We refer to [8] for explicit constructions in case of particular Dirichlet boundary data. Unfortunately, a calibration field for proving the optimality of a function of crack tip type could not yet be found. A very challenging issue that will be worth for further investigations is the following.

Conjecture. *The following equality holds:* $\inf(\mathcal{P}) = \sup(\mathcal{Q})$.

Postulating a priori the validity of such a conjecture, numerical schemes based on a primal–dual algorithm are actually used to solve problem $(\mathcal{P})$ (see [21]). To our knowledge, no numerical gap disproving the conjecture has ever been evidenced.

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