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Non-linear vibrations of sandwich viscoelastic shells

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ABSTRACT

This paper deals with the non-linear vibration of sandwich viscoelastic shell structures. Coupling a harmonic balance method with the Galerkin's procedure, one obtains an amplitude equation depending on two complex coefficients. The latter are determined by solving a classical eigenvalue problem and two linear ones. This permits to get the non-linear frequency and the non-linear loss factor as functions of the displacement amplitude. To validate our approach, these relationships are illustrated in the case of a circular sandwich ring.

1. Introduction

In the mechanical structures field, the viscoelastic material is widely used to reduce vibration and noise in many domains (e.g., aerospace industry). Indeed, it can induce an effective damping especially when it is sandwiched between two elastic hard layers. Generally, the damping properties are characterized by two modal parameters that are the frequency and the loss factor. Many investigations have been carried out on the linear dynamic analysis of viscoelastic structures. A major difficulty in their study is that the stiffness matrix is complex and depends non-linearly on the vibration frequency. The solution yield complex modes and complex eigenvalues whose real and imaginary parts are associated respectively with the frequencies and with the loss factors. Several procedures have been developed to determine these quantities. Analytical methods were devoted to simple structures [1–10], and numerical ones using finite element simulations were introduced to design structures with complex geometries and generic boundary conditions [11–22]. The simplest technique is the modal strain energy method used by Ma and He [12], which defines a rather good estimate of the loss factor from a sort of one-mode Galerkin approximation. One notes that from an engineering viewpoint, the most relevant quantity is the loss factor, which is associated with any mode.

In the case of non-linear viscoelastic structures, only a few investigations have been devoted to take into account the non-linear geometrical effects. For instance, these studies concern sandwich viscoelastic structures with simple geometry as beams or plates [23–26]. As it is well known, the non-linear geometrical effects induce some dependence between the frequencies and the loss factors with respect to the amplitude [25,27]. Recently, Boumediene et al. [28] developed a reduction method based on a high-order Newton algorithm and reductions techniques to determine the modal characteristics of

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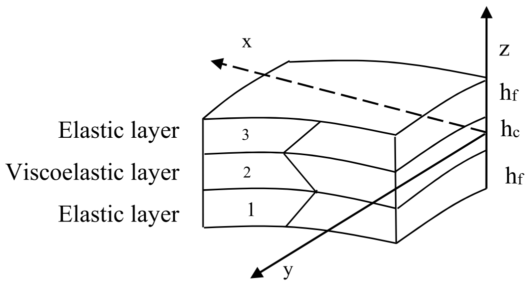

Fig. 1. Geometry of a 3D sandwich structure with two elastic layers and a central viscoelastic one.

viscoelastic sandwich structures. The forced harmonic response of viscoelastic sandwich structures with a reasonable computational cost was also studied, employing a reduction technique and the asymptotic numerical method [29]. Based on von Kármán's theory and taking into account geometric imperfections, the nonlinear vibrations of viscoelastic thin rectangular plates subjected to normal harmonic excitation are investigated by Amabili [30]. Lougou [31] proposed a double-scale asymptotic method for the vibration modeling of large repetitive sandwich structures with a viscoelastic core. In his work [32], Lampoh computes the sensitivity of eigensolutions using a homotopy-based asymptotic numerical method, then a first-order automatic differentiation to study the modeling of the linear free vibration of a sandwich structure including viscoelastic layers yields a complex nonlinear eigenvalue problem. The work of El Khaldi [33] presents a gradient method for viscoelastic behavior identification of damped sandwich structures devoted to the passive control of mechanical vibration.

The aim of this paper is to establish a much simple methodology for the non-linear vibration analysis of viscoelastic shell structures. The approach is based on a coupling of an approximated harmonic balance method with a Galerkin's procedure with one mode. The non-linear modal relationship giving the frequency (free and forced) and the loss factor, with respect to the displacement, are obtained by solving a classical eigenvalue problem and two linear ones [24,27]. To validate our approach, one gives an application to a sandwich viscoelastic ring.

2. Formulation

2.1. Kinematics and constitutive law of the model

Let us consider a thin symmetric sandwich shell having three layers, as shown in Fig. 1; the central layer is viscoelastic and the external ones are elastic. The shear deformation is neglected in the elastic layers, but, it is taken into account in the viscoelastic one; it is induced by the difference between the tangential displacements at the interfaces. For each layer, one denotes by u_i $(i = 1, 2, 3)$ the components of the displacement vector in the z direction and given by:

$$\begin{aligned} u_i(x,y,z,t) &= v_i(x,y,t) + (z - z_i)\beta_i(x,y,t) \quad i = 1,3 \\ u_2(x,y,z,t) &= v(x,y,t) + z\psi(x,y,t) \end{aligned} \tag{1}$$

where t is the time parameter, (x, y, z) is a coordinate system (z denotes the variation through the thickness). Because of the symmetry, one puts $z_1 = \frac{h_c+h_f}{2} = -z_3$, h_c and h_f being the thicknesses of the central and external layers, respectively. The subscript i indicates the layer variation, starting from the internal layer; 1 and 3 represent the elastic layers, while 2 is associated with the viscoelastic one. β_i and ψ denote the rotations of the cross-section, v_i $(i = 1, 3)$ and v denote tangential components of the displacement vector of the middle planes corresponding to the external and central layers, respectively.

The displacement continuity conditions at the interfaces between the central layer and the external ones permit to get:

$$\begin{aligned} v_1 &= v + \frac{h_c}{2}\psi + \frac{h_f}{2R_1}\beta_1 \\ v_3 &= v - \frac{h_c}{2}\psi - \frac{h_f}{2R_3}\beta_3 \end{aligned} \tag{2}$$

The Green–Lagrange strain in each layer can be decomposed into a linear part and a quadratic one:

$$\gamma_i = \gamma_i(u_i) + \gamma_{nl}(u_i, u_i) \tag{3}$$

For the elastic layers, the behavior is described by the classical Hook law, and it is given, for the viscoelastic one, by the classical convolution product $\otimes$ of the relaxation function $D(t)$ by the time derivative of the deformation:

$$\begin{aligned} S_i &= D(0)\dot{\gamma}_i \quad i = 1,3 \\ S_2 &= D \otimes \dot{\gamma}_2 \end{aligned} \tag{4}$$

where S_i is the second Piola–Kirchhoff stress tensor corresponding to the layer i and $D(0)$ is the delayed elasticity modulus.

2.2. Governing equations

Using the principal of virtual work, the equations describing the non-linear forced vibrations of a 3D sandwich viscoelastic structure can be written in the following general form:

$$L(U) + Q(U,U) + M(\ddot{U}) = f(t) \tag{5}$$

where $U = (u_1, S_1, u_2, S_2, u_3, S_3)$ is a mixed vector, its components are the generalized displacements and the stress corresponding to the three layers. $L(\cdot)$ is a linear operator, $Q(\cdot,\cdot)$ is a bilinear and symmetric one and $M(\cdot)$ is the inertial operator and $f(t)$ is the external applied load.

$$\langle L(U), \delta U \rangle = \int_{v_1} S_1 : \gamma_1(\delta u_1)\,\mathrm{d}v_1 + \int_{v_2} S_2 : \gamma_1(\delta u_2)\,\mathrm{d}v_2 + \int_{v_1} S_3 : \gamma_1(\delta u_3)\,\mathrm{d}v_3 \tag{6}$$

$$\begin{aligned}\langle Q(U,U), \delta U \rangle = &\int_{v_1} \left\{\delta S_1 : \gamma_{\mathrm{nl}}(u_1,u_1) + S_1 2\gamma_{\mathrm{nl}}(\delta u_1, u_1)\right\}\mathrm{d}v_1 + \int_{v_2} \left\{\delta S_2 : \gamma_{\mathrm{nl}}(u_2,u_2) + S_2 2\gamma_{\mathrm{nl}}(\delta u_2, u_2)\right\}\mathrm{d}v_2 \\ &+ \int_{v_3} \left\{\delta S_3 : \gamma_{\mathrm{nl}}(u_3,u_3) + S_3 2\gamma_{\mathrm{nl}}(\delta u_3, u_3)\right\}\mathrm{d}v_3\end{aligned} \tag{7}$$

$$\langle M(\ddot{U}), \delta U \rangle = \int_{v_1} \rho_1 \ddot{u}_1 \delta u_1\,\mathrm{d}v_1 + \int_{v_2} \rho_2 \ddot{u}_2 \delta u_2\,\mathrm{d}v_2 + \int_{v_3} \rho_3 \ddot{u}_3 \delta u_3\,\mathrm{d}v_3 \tag{8}$$

where ρ_i and v_i are respectively the mass densities and the reference configuration of the layer i.

3. Non-linear free vibration by an approximated harmonic balance method

The aim of this section is to get approximate solutions to the non-linear problem (5) and (4), assuming that $f(t) = 0$. As a first approximation, the solution is assumed to be harmonic in time and almost parallel to a single mode in space with arbitrary complex amplitude. This approximation assumes that the frequency is near the frequency of an associated linear elastic structure. As in non-linear elastodynamics, the harmonic response has to be corrected to balance the quadratic terms in (5) and (4). Thus, a non-linear complex frequency–amplitude relationship is obtained by using the one-mode Galerkin procedure.

3.1. First-order modal approximation

Let us consider a first approximated solution U_h to the problem (5) and (4), which is supposed harmonic and proportional to the linear mode:

$$U_\mathrm{h} = \frac{1}{2} U_n \left(a \mathrm{e}^{\mathrm{i}\omega t} + \mathrm{CC}\right) \tag{9}$$

where CC denotes the conjugate complex of the preceding term, a is an unknown complex amplitude, ω the frequency, U_n is the n-th linear vibration mode of the associated elastic system, defined by a classical real eigenvalue problem:

$$\begin{cases} L(U_n) - \omega_n^2 M(U_n) = 0 \\ S_n = D(0)\varepsilon(u_n) \end{cases} \tag{10}$$

One notes that this first approximation of the non-linear and complex problem is more valid when the damping is small, and it is used in the modal strain energy to determine the loss factor.

3.2. Computation of the correction term

Let us consider a second-order approximated solution to (5)–(4) by adding a corrective term U_c to the linear response (9):

$$U = U_\mathrm{h} + U_\mathrm{c} \tag{11}$$

The correction term is assumed to be small with respect to the main term. That is why the equations defining the correction are linearized with respect to U_c. This U_c balances the quadratic terms in (5)–(4):

$$L(U_\mathrm{c}) + M\ddot{U}_\mathrm{c} = -Q(U_\mathrm{h}, U_\mathrm{h}) \tag{12}$$

The correction term U_c combines a time-independent term and a harmonic term with a double frequency:

$$U_c = |a|^2 U_0 + \frac{1}{2}\left(a^2 U_2 e^{2i\omega t} + CC\right) \tag{13}$$

When restricted to the elastic case, the approximations (9)–(12) correspond to the two first terms of a Poincaré–Lindstedt expansion [34], which yields a parabolic approximation of the backbone curve. It holds for moderately large amplitude: the first harmonic term (9) is small ($O(a)$) and the correction term is smaller than the first one ($O(a^2)$). This way, the coupling term $Q(U_h, U_c)$ can be neglected in (12) ($O(a^3)$), as well as the quadratic term $Q(U_c, U_c)$ ($O(a^4)$).

The substitution of (13) into (12) leads to two linear time-independent problems satisfied by the amplitudes U_0 and U_2.

$$\begin{aligned} & L(U_0) = -\frac{1}{2} Q(U_n, U_n) \\ & S_{i0} + D(0)\left[\gamma_1(u_{i0}) + \frac{1}{2}\gamma_{nl}(u_{in}, u_{in})\right], \quad i = 1, 2, 3 \end{aligned} \tag{14}$$

$$\begin{aligned} & L(U_2) - 4\omega_n^2 M(U_2) = -\frac{1}{2} Q(U_n, U_n) \\ & S_{i2} = D(0)\left[\gamma_1(u_{i2}) + \frac{1}{2}\gamma_{nl}(u_{in}, u_{in})\right], \quad i = 1, 3 \\ & S_{22} = D(2i\omega_n)\left[\gamma_1(u_{22}) + \frac{1}{2}\gamma_{nl}(u_{2n}, u_{2n})\right] \end{aligned} \tag{15}$$

where $D(0)$ is the tensor of the delayed elasticity of the viscoelastic material and $D(2\omega)$ is the viscoelastic tensor at frequency 2ω. Thus, the general solution to (5) and (4) induces a principal harmonic and two secondary ones.

$$U = \frac{1}{2} U_n\left(a e^{i\omega t} + cc\right) + |a|^2 U_0 + \frac{1}{2}\left(a^2 U_2 e^{2i\omega t} + cc\right) \tag{16}$$

As previously said, the approximation (9) assumes that the structure oscillates with a frequency ω near the linear one ω_n. So, the tensor $D(2\omega)$ in (15) will be replaced by $D(2\omega_n)$.

3.3. Amplitude equation

To get the non-linear frequency–amplitude relationship, one applies the one-mode Galerkin procedure, which consists in projecting the equation (5) on $U_n e^{-i\omega t}$, the displacement being given by (16).

$$\int_0^{2\pi/\omega} \left\langle L(U) + Q(U, U) + M(\ddot{U}), U_n e^{-i\omega t}\right\rangle dt = 0 \tag{17}$$

The equation (17) leads to an equation for the complex amplitude in the following form:

$$a\left(k_l - \omega^2 m\right) + a|a|^2 k_{nl} = 0 \tag{18}$$

where k_l and k_{nl} are complex constants, which correspond, respectively, to the linear and non-linear modal stiffness; m is the modal mass.

$$k_l = \left\langle L(U_n), U_n\right\rangle, \qquad k_{nl} = \left\langle 2Q(U_n, U_0) + Q(U_n, U_2), U_n\right\rangle, \qquad m = \left\langle M(U_n), U_n\right\rangle \tag{19}$$

The amplitude equation can be considered as a generic bifurcation equation, which holds for any form of the non-linearity. It has been first derived in [24], but with a procedure that can only be applied in specific cases, as straight beams or flat plates. When it is restricted to an elastic material, the amplitude equation (15) coincides with the parabolic approximation of the backbone curve, which can be deduced, for instance, through the Poincaré–Lindstedt asymptotic procedure. The linearized form of (15) permits to recover the results of the modal strain energy method [3], which is a classical approach in the analysis of viscoelastic linear structures. The ratio $\frac{k_l}{m}$ permits to define the damped linear frequency Ω_l and the linear loss factor η_l.

$$\frac{k_l}{m} = \Omega_n^2(1 + i\eta_n), \qquad \Omega_n^2 = k_l^R/m, \qquad \eta_n = k_l^I/k_l^R \tag{20}$$

where (k_l^R, k_l^I) are, respectively, the real and imaginary parts of k_l. Equation (18) establishes that the non-linear complex frequency is a function of the amplitude $|a|$.

$$\omega^2 = \frac{k_l}{m} + |a|^2 \frac{k_{nl}}{m} \tag{21}$$

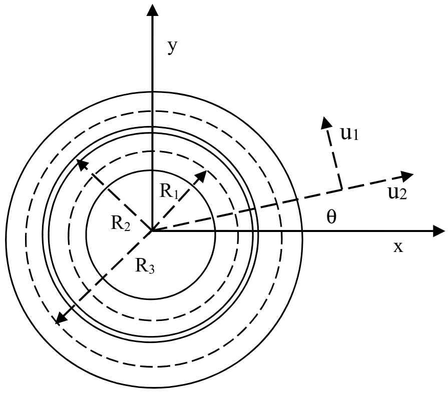

Fig. 2. Circular sandwich ring with two elastic external layers and a viscoelastic central one.

As in the modal strain energy, the non-linear modal frequency Ω_{nl}^2 and the non-linear modal loss factor η_{nl} are deduced from the complex frequency in the same way as in the linear case.

$$\Omega_{\text{nl}}^2 = \Omega_n^2\left(1 + C^{\text{R}}|a|^2\right), \qquad \eta_{\text{nl}} = \eta_n \frac{1 + C^{\text{I}}|a|^2}{1 + C^{\text{R}}|a|^2} \tag{22}$$

where $C^{\text{R}} = k_{\text{nl}}^{\text{R}}/k_{\text{l}}^{\text{R}}$ and $C^{\text{I}} = k_{\text{nl}}^{\text{I}}/k_{\text{l}}^{\text{I}}$.

4. Forced non-linear vibrations

The analysis is limited here to harmonic excitation $f(t) = f_0 \mathrm{e}^{\mathrm{i}\omega t}$, ω being a real number corresponding to the frequency of the excitation and f_0 its amplitude. Applying to Equation (5), the harmonic balance method and Galerkin method with one mode, one gets:

$$-\omega^2 Ma + K_{\text{l}}a + a|a|^2 K_{\text{nl}} = F \tag{23}$$

F represents the projection of f_0 on the mode ($F = \int_v f_0(\theta)U_n(\theta)\,\mathrm{d}\theta$).

The amplitude a is searched in the following form:

$$a = r\mathrm{e}^{\mathrm{i}\Theta} \tag{24}$$

where r is the real amplitude and Θ is the phase.

The solution to (23) permits to get the frequency and the phase versus the amplitude:

$$\omega^2 = \frac{\alpha \pm \sqrt{\alpha^2 - \beta}}{M}, \qquad \text{tg}(\Theta) = \eta_n \frac{1 + r^2 C^{\text{I}}}{-(\omega/\Omega_n)^2 + 1 + r^2 C^{\text{R}}} \tag{25}$$

where $\alpha = |K_{\text{l}}|\cos(\varphi) + r^2|K_{\text{nl}}|\cos(\psi)$, $\beta = |K_{\text{l}}|^2 + r^4|K_{\text{nl}}|^2 + 2r^2|K_{\text{l}}K_{\text{nl}}|\cos(\varphi - \psi) - \frac{F}{r^2}$, φ and ψ are the arguments of K_{l} and K_{nl}, respectively.

5. Application

In this section, the presented approach is applied to study the in-plane free non-linear vibrations of a sandwich viscoelastic circular ring shown in Fig. 2. In this analysis, the rotations are assumed to be moderate, the rotary inertia terms of the kinetic energy are neglected, and the shear deformation is taken into account for the viscoelastic layer and neglected for the elastic ones. The displacement field is given by:

$$u_{i1} = v_i + (z - z_i)\beta_i, \qquad u_{i2} = w, \quad i = 1, 3 \tag{26}$$

$$u_{21} = v + z\psi, \qquad u_{22} = w \tag{27}$$

The continuity condition of the displacements at the interfaces between the central layer and the external ones gets:

$$v_1 = \alpha_1\left(v + \frac{h_{\text{c}}}{2}\psi - \frac{h_{\text{f}}}{2R_1}w'\right), \qquad v_3 = \alpha_3\left(v - \frac{h_{\text{c}}}{2}\psi + \frac{h_{\text{f}}}{2R_3}w'\right), \qquad \alpha_1 = \frac{2R_1}{2R_1 - h_{\text{f}}}, \qquad \alpha_3 = \frac{2R_3}{2R_3 + h_{\text{f}}} \tag{28}$$

The Green–Lagrange deformations in each layer i are given by:

$$\gamma_i = \varepsilon_i + (z - z_i)k_i, \qquad \varepsilon_i = \frac{v_i' + w}{R_2} + \frac{1}{2}\beta_i^2, \qquad \beta_i = \frac{v_i - w'}{R_i}, \qquad k^i = \frac{\beta_i'}{R_i}, \quad i = 1, 3 \tag{29}$$

$$\gamma_2 = \varepsilon_2 + zk_2, \qquad \varepsilon_2 = \frac{v' + w}{R_2} + \frac{1}{2}\beta_2^2, \qquad \beta_2 = \frac{v - w'}{R_2}, \qquad k_2 = \frac{\psi'}{R_2}, \qquad 2\tau = \frac{w' - v}{r_2} + \psi \tag{30}$$

where v and w denote respectively the radial and tangential displacements of the central layer, β_i the rotation of the cross-section relative to the layer i and $()' = \frac{\mathrm{d}()}{\mathrm{d}\theta}$.

The behavior law is given by:

$$N_i = E_i A_i \varepsilon_i, \qquad M_i = E_i I_i k_i, \quad i = 1, 3 \tag{31}$$

$$N_2 = A_2 E_2^* \otimes \dot{\varepsilon}_2, \qquad M_2 = I_2 E_2^* \otimes \dot{k}_2, \qquad T_2 = A_2 G_2^* \otimes \dot{\tau} \tag{32}$$

where N_i, M_i, I_i, A_i are respectively the normal force, the bending moment, the inertia moment, and the cross-sectional area corresponding to the layer i ($i = 1, 2, 3$), T_2 is the shear transverse force relative to the layer 2. To simplify the analysis, one assumes that the complex Young and shear modulus are constants that do not depend on the frequency:

$$E_2(\alpha\omega) = E_{20}(1 + \mathrm{i}\eta_E), \qquad G_2(\alpha\omega) = G_{20}(1 + \mathrm{i}\eta_G), \quad \alpha = 0, 2 \tag{33}$$

where η_E and η_G are the material loss factor in extension and shear, in this analysis, one assumes that $\eta_E = \eta_G = \eta_v$, E_{20} and G_{20} are the delayed Young and shear delayed elasticity moduli, respectively.

The motion equations describing the non-linear free vibrations are given by:

$$-\alpha_1\left(N_1' + \frac{M_1'}{R_1}\right) - \alpha_3\left(N_3' + \frac{M_3'}{R_3}\right) - N_2' - T_2 + \alpha_1 N_1 \beta_1 + \alpha_3 N_3 \beta_3 + N_2 \beta_2 + m_{11}\ddot{v} + m_{12}\ddot{w}' + m_{13}\ddot{\psi} = f_1(t) \tag{34}$$

$$N_1 - \frac{M_1''}{R_1^2} - \frac{\alpha_1 h_\mathrm{f}}{2R_1}\left(N_1'' + \frac{M_1''}{R_1}\right) + N_3 - \frac{M_3''}{R_3} + \frac{\alpha_3 h_\mathrm{f}}{2R_3}\left(N_3'' + \frac{M_3''}{R_3}\right) + N_2 - T_2' + \left(1 + \frac{\alpha_1 h_\mathrm{f}}{2R_1}\right)(N_1\beta_1)' + \left(1 - \frac{\alpha_3 h_\mathrm{f}}{2R_3}\right)(N_3\beta_3)' + (N_2\beta_2)' + m_{21}\ddot{v}' + m_{22}\ddot{w}'' + m_{23}\ddot{w} + m_{24}\ddot{\psi}' = f_2(t) \tag{35}$$

$$-\frac{h_\mathrm{c}}{2}\left[\alpha_1\left(N_1' + \frac{M_1'}{R_1}\right) - \alpha_3\left(N_3' + \frac{M_3'}{R_3}\right)\right] + R_2 T_2 - M_2' + \frac{h_\mathrm{c}}{2}(\alpha_1 N_1 \beta_1 - \alpha_3 N_3 \beta_3) + \left(m_{31}\ddot{v} + m_{32}\ddot{w}' + m_{33}\ddot{\psi}\right) = f_3(t) \tag{36}$$

Neglecting the non-linear parts, assuming that $f(t) = 0$, and using the behavior law (22) with a real Young and shear moduli in (33), one gets a linear real eigenvalue problem, its solution gives the linear mode u_n and the corresponding eigenfrequency ω_n. The details are given in Appendix A.

$$u_n(\theta) = \begin{Bmatrix} v_n = V\cos(n\theta) \\ w_n = W\sin(n\theta) \\ \psi_n = \Psi\cos(n\theta) \end{Bmatrix} \tag{37}$$

where n is the circumferential wave number, V, W and Ψ are arbitrary constants determined by a normalization condition; here one assumes:

$$V^2 + W^2 + \Psi^2 = 1 \tag{38}$$

The obtained linear eigenvalues are in good agreement with those obtained by Patel et al. [25].

The correction term is obtained in the same way as in the general case. For the ring, the linear problems (14) and (15) give linear differential equations (see Appendix B), whose resolution gets $u_0 = (v_0, w_0, \psi_0)$ and $u_2 = (v_2, w_2, \psi_2)$ in the following form:

$$\begin{Bmatrix} v_0 \\ w_0 \\ \psi_0 \end{Bmatrix} = \begin{Bmatrix} v_{00} \\ w_{00} \\ \psi_{00} \end{Bmatrix} + \begin{Bmatrix} v_{01}\sin(2n\theta) \\ w_{01}\cos(2n\theta) \\ \psi_{01}\sin(2n\theta) \end{Bmatrix}, \qquad \begin{Bmatrix} v_2 \\ w_2 \\ \psi_2 \end{Bmatrix} = \begin{Bmatrix} v_{20} \\ w_{20} \\ \psi_{20} \end{Bmatrix} + \begin{Bmatrix} v_{21}\sin(2n\theta) \\ w_{21}\cos(2n\theta) \\ \psi_{21}\sin(2n\theta) \end{Bmatrix} \tag{39}$$

where $u_0 = (v_{0j}, w_{0j}, \psi_{0j})$ are real constants and $u_2 = (v_{2j}, w_{2j}, \psi_{2j})$ are complex ones.

Inserting (39) in the constitutive laws (14)–(15) and using (19), one gets the constants of the amplitude equation (18), for details, see Appendix C.

Table 1
Linear eigenfrequencies for circular ring with $R/h = 100$.

n	Present	Maple	Patel [25]	Belvins [29]	[30]
2	7.1622	7.1617	7.2000	7.2000	7.2001
3	57.2977	57.2967	57.6000	57.6000	57.6001
4	210.5630	210.6513	211.7651	211.7647	211.7656

Table 2
Vibration modal coefficients versus the circumferential wave number n.

n	Ω_n^2	η_n	C^R	C^I
2	1720.6795	$1.2493\ 10^{-2}$	−2.9408	−0.1507
4	49637.2070	$3.3344\ 10^{-3}$	−152.3605	$-1.9052\ 10^{-2}$
8	911401.0331	$8.4770\ 10^{-4}$	−3802.8828	0.3211

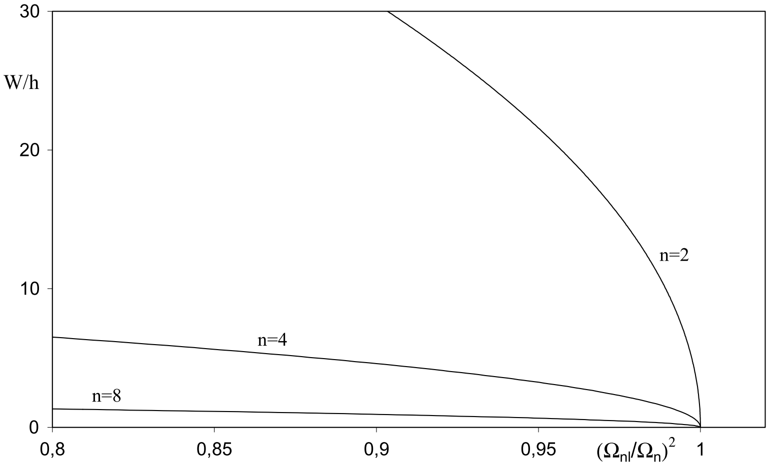

Fig. 3. Variation of the non-linear modal frequencies ratio (Backbone curves) versus the radial displacement near the linear frequencies Ω_n ($n = 2, 4, 8$). $\eta_v = 0.5$.

$$k_l = \int_0^{2\pi} \{l_v v_n + l_w w_n + l_\psi \psi_n\} \, d\theta, \qquad k_{nl} = \int_0^{2\pi} \{q_v v_n + q_w w_n + q_\psi \psi_n\} \, d\theta$$

$$m = \int_0^{2\pi} \{m_v v_n + m_w w_n + m_\psi \psi_n\} \, d\theta \tag{40}$$

In this application, the geometrical data are: radii $R_1 = 0.9997$, $R_2 = 1$ and $R_3 = 1.003$, thicknesses $h_c = h_f/2 = 0.002$ and a width $b = 0.012$. The structure is described by one degree of freedom and by the angle θ ($0 \le \theta \le 2\pi$). In Table 1, one gives the first linear frequencies. In Table 2, one presents the constants C^R and C^I for various vibration modes, the same results are given by Maple and a Fortran program. One notes that C^R is a negative number and that $|C^R|$ is greater than $|C^I|$. In Figs. 3 and 4, one presents the backbone curves corresponding to non-linear modal frequencies and the modal loss factors with respect to the adimensionalized radial displacement when the ring vibrates near the linear frequencies associated with $n = 2, 4$ and 8. It is clearly seen that the frequencies decrease (non-linearity of soft type) while the loss factor increases with the displacement. The increase and decrease in frequencies and loss factor, respectively, are more important for higher vibration modes. In Fig. 5 and Fig. 6, one gives the forced non-linear response for various excitation amplitudes and material loss factors. The forced response parts, tangent to the non-linear free response ($F = 0$) and situated below it, are instable, so the structure can jump between several equilibrium positions. In Fig. 7, the variation of the non-linear phase versus the excitation frequency is presented for various material losses factor.

6. Conclusion

In this study, an amplitude equation has been presented for the nonlinear vibrations analysis of viscoelastic shells structures. This amplitude equation is obtained by coupling an approximated harmonic balance method with the one-mode Galerkin procedure. It involves two modal parameters C^R and C^I, which account for the non-linear effects. These constants

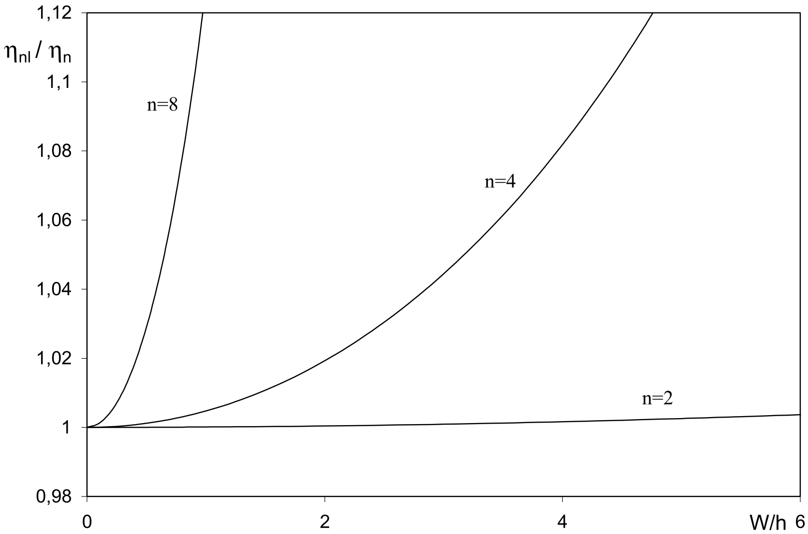

Fig. 4. Variation of the non-linear modal loss factor ratio versus the radial displacement near the linear frequencies Ω_n $(n = 2, 4, 8)$ for various wave circumferential numbers n. $\eta_v = 0.5$.

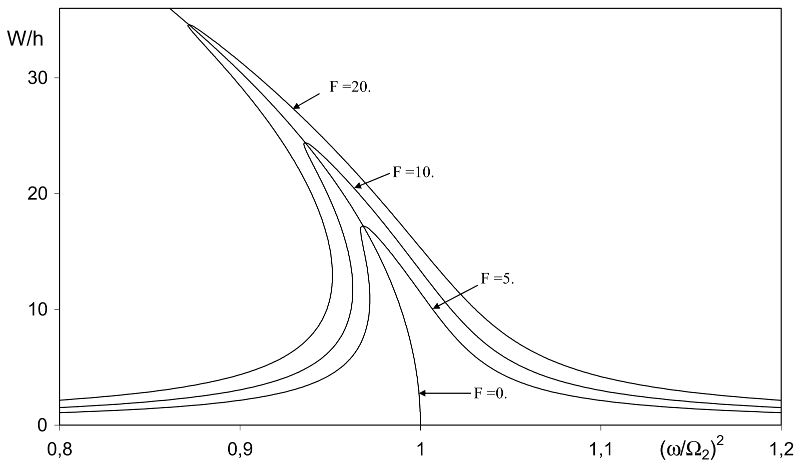

Fig. 5. Variation of the non-linear response with the load amplitude ($n = 2$, $\eta_v = 0.5$).

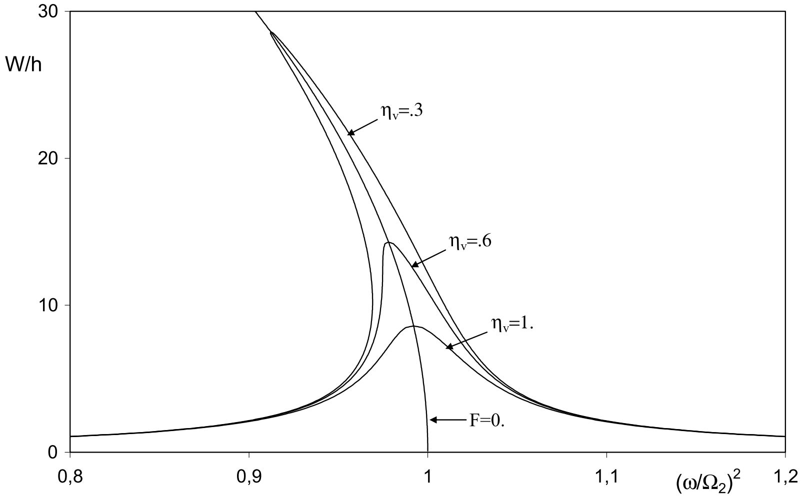

Fig. 6. Variation of the non-linear response with the loss factor ($n = 2$, $F = 5$).

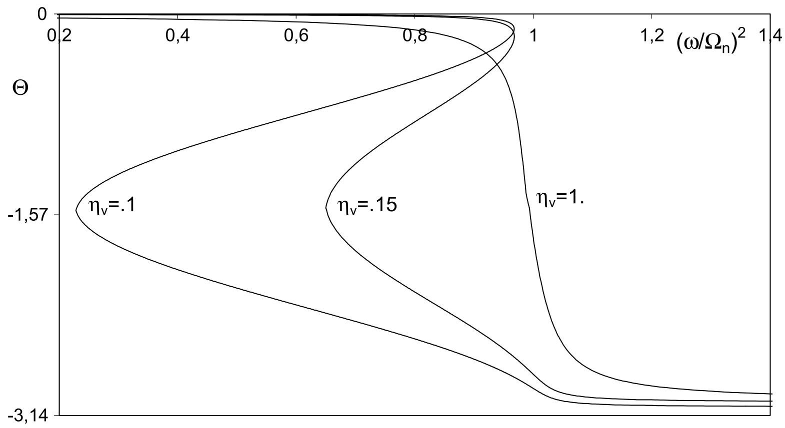

Fig. 7. Variation of the phase versus the excitation frequency for various loss factors η_v $(F = 5)$.

are determined by solving three classical problems. The first one is a real eigenvalue problem that allows one to define the linear frequency and the linear loss factor. The two others are linear problems. In the case of free vibrations, the backbone corresponding to the non-linear modal frequency and the modal non-linear loss factor, with respect to the displacement, are obtained. The non-linear forced response and the corresponding non-linear phase are also determined. This approach will be extended to harmonic forcing vibration problems and to others viscoelastic sandwich or composite shell structures such as the cylinder.

Appendix A. Computation of linear vibration modes

The linear part of (37) gets:

$$\begin{aligned} L_{11}^{i} v_n + L_{12}^{i} w_n + L_{13}^{i} \psi_n &= M_{11}\ddot{v}_n + M_{12}\ddot{w}_n + M_{13}\ddot{\psi}_n \\ L_{21}^{i} v_n + L_{22}^{i} w_n + L_{23}^{i} \psi_n &= M_{21}\ddot{v}_n + M_{22}\ddot{w}_n + M_{24}\ddot{\psi}_n \\ L_{31}^{i} v_n + L_{32}^{i} w_n + L_{33}^{i} \psi_n &= M_{31}\ddot{v}_n + M_{32}\ddot{w}_n + M_{33}\ddot{\psi}_n \end{aligned} \tag{41}$$

with the following operators:

$$\begin{aligned} &L_{11}^{i} = A_{11}^{i}\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + A_{12}^{i}, \quad L_{12}^{i} = A_{13}^{i}\frac{\mathrm{d}^3}{\mathrm{d}\theta^3} + A_{14}^{i}\frac{\mathrm{d}}{\mathrm{d}\theta}, \quad L_{13}^{i} = A_{15}^{i}\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + A_{16}^{i}, \quad L_{21}^{i} = A_{21}^{i}\frac{\mathrm{d}^3}{\mathrm{d}\theta^3} + A_{22}^{i}\frac{\mathrm{d}}{\mathrm{d}\theta} \\ &L_{22}^{i} = A_{23}^{i}\frac{\mathrm{d}^4}{\mathrm{d}\theta^4} + A_{24}^{i}\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + A_{25}^{i}, \quad L_{23}^{i} = A_{26}^{i}\frac{\mathrm{d}^3}{\mathrm{d}\theta^3} + A_{27}^{i}\frac{\mathrm{d}}{\mathrm{d}\theta}, \quad L_{31}^{i} = A_{31}^{i}\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + A_{32}^{i} \\ &L_{32}^{i} = A_{33}^{i}\frac{\mathrm{d}^3}{\mathrm{d}\theta^3} + A_{34}^{i}\frac{\mathrm{d}}{\mathrm{d}\theta}, \quad L_{33}^{i} = A_{35}^{i}\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + A_{36}^{i}, \quad M_{11} = m_{11}, \quad M_{12} = m_{12}\frac{\mathrm{d}}{\mathrm{d}\theta}, \quad M_{13} = m_{13} \\ &M_{21} = m_{21}\frac{\mathrm{d}}{\mathrm{d}\theta}, \quad M_{22} = m_{22}\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + m_{23}, \quad M_{23} = m_{24}\frac{\mathrm{d}}{\mathrm{d}\theta}, \quad M_{31} = m_{31}, \quad M_{32} = m_{32}\frac{\mathrm{d}}{\mathrm{d}\theta}, \quad M_{33} = m_{33} \end{aligned}$$

The constants A_{jk}^{i} $(i = 0, 2,\ j = 1, 2, 3, \text{ and } k = 1, \ldots, 7)$ are given by:

$$\begin{aligned} &A_{11}^{i} = \frac{\alpha_1^2 E_1}{R_1}\left(A_1 + \frac{I_1}{R_1^2}\right) + \frac{\alpha_3^2 E_3}{R_3}\left(A_3 + \frac{I_1}{R_3^2}\right) + \frac{E_2(ij\omega)A_2}{R_1}, \quad A_{12}^{i} = -\frac{k' G(ij\omega)A_2}{R_2} \\ &A_{13}^{i} = -\frac{\alpha_1 E_1}{R_1^2}\left[\frac{\alpha_1 A_1 h_{\mathrm{f}}}{2} + \frac{I_1}{R_1}\left(1 + \frac{\alpha_1 h_{\mathrm{f}}}{2R_1}\right)\right] + \frac{\alpha_3 E_3}{R_3^2}\left[\frac{\alpha_3 A_3 h_{\mathrm{f}}}{2} - \frac{I_3}{R_3}\left(1 - \frac{\alpha_3 h_{\mathrm{f}}}{2R_3}\right)\right] \\ &A_{14}^{i} = \frac{\alpha_1 E_1 A_1}{R_1} + \frac{\alpha_3 E_3 A_3}{R_3} + \frac{E(ij\omega)A_2}{R_2} + \frac{k' G_2(ij\omega)A_2}{R_2}, \quad A_{15}^{i} = \frac{h_{\mathrm{c}}}{2}\left[\frac{\alpha_1^2 E_1}{R_1}\left(A_1 + \frac{I_1}{R_1^2}\right) - \frac{\alpha_3^2 E_3}{R_3}\left(A_3 + \frac{I_1}{R_3^2}\right)\right] \\ &A_{16}^{i} = k' A_2 G_2(ij\omega), \quad A_{21}^{i} = \frac{\alpha_1 E_1}{R_1^2}\left[\frac{\alpha_1 A_1 h_{\mathrm{f}}}{2} + \frac{I_1}{R_1}\left(1 + \frac{\alpha_1 h_{\mathrm{f}}}{2R_1}\right)\right] - \frac{\alpha_3 E_3}{R_3^2}\left[\frac{\alpha_3 A_3 h_{\mathrm{f}}}{2} - \frac{I_3}{R_3}\left(1 - \frac{\alpha_3 h_{\mathrm{f}}}{2R_3}\right)\right] \\ &A_{22}^{i} = -A_{14}^{i}, \quad A_{23}^{i} = -\frac{E_1}{R_1^3}\left[\frac{\alpha_1^2 A_1 h_{\mathrm{f}}^2}{4} + \frac{I_1}{R_1}\left(1 + \frac{\alpha_1 h_{\mathrm{f}}}{2R_1}\right)^2\right] - \frac{E_3}{R_3^3}\left[\frac{\alpha_3^2 A_3 h_{\mathrm{f}}^2}{4} + \frac{I_3}{R_3}\left(1 - \frac{\alpha_3 h_{\mathrm{f}}}{2R_3}\right)^2\right] \end{aligned}$$

$$A^i_{24} = h_f\left(\frac{\alpha_1 E_1 A_1}{R_1^2} - \frac{\alpha_3 E_3 A_3}{R_3^2}\right) + \frac{k' A_2 G_2(ij\omega)}{R_2}, \qquad A^i_{25} = -\left(\frac{E_1 A_1}{R_1} + \frac{E_2(ij\omega)A_2}{R_2} + \frac{E_3 A_3}{R_3}\right)$$

$$A^i_{26} = \frac{h_c}{2}\left\{\frac{\alpha_1 E_1}{R_1^2}\left[\frac{\alpha_1 A_1 h_f}{2} + \frac{I_1}{R_1}\left(1 + \frac{\alpha_1 h_f}{2R_1}\right)\right] + \frac{\alpha_3 E_3}{R_3^2}\left[\frac{\alpha_3 A_3 h_f}{2} - \frac{I_3}{R_3}\left(1 - \frac{\alpha_3 h_f}{2R_3}\right)\right]\right\}$$

$$A^i_{27} = -\frac{h_c}{2}\left(\frac{\alpha_1 E_1 A_1}{R_1} - \frac{\alpha_3 E_3 A_3}{R_3}\right) + \frac{k' G_2(ij\omega) A_2}{R_2}, \qquad A^i_{31} = A^i_{15}, \qquad A^i_{32} = A^i_{16}, \qquad A^i_{33} = -A^i_{26}$$

$$A^i_{34} = -A^i_{27}, \qquad A^i_{35} = \frac{h_c^2}{4}\left[\frac{\alpha_1^2 E_1}{R_1}\left(A_1 + \frac{I_1}{R_1^2}\right) + \frac{\alpha_3^2 E_3}{R_3}\left(A_3 + \frac{I_1}{R_3^2}\right)\right] + \frac{E_2(ij\omega) I_2}{R_2}, \qquad A^i_{36} = -R_2 A^i_{32}$$

$$m_{11} = \rho_1 A_1 R_1 \alpha_1^2 + \rho_2 A_2 R_2 + \rho_3 A_3 R_3 \alpha_3^2, \qquad m_{12} = -\frac{h_f}{2}(\rho_1 A_1 \alpha_1^2 - \rho_3 A_3 \alpha_3^2) = -m_{21}$$

$$m_{13} = \frac{h_c}{2}(\rho_1 A_1 R_1 \alpha_1^2 - \rho_3 A_3 R_3 \alpha_3^2) = m_{31}, \qquad m_{22} = -\frac{h_f^2}{4}\left(\frac{\rho_1 A_1 \alpha_1^2}{R_1} + \frac{\rho_3 A_3 \alpha_3^2}{R_3}\right)$$

$$m_{23} = \rho_1 A_1 R_1 + \rho_2 A_2 R_2 + \rho_3 A_3 R_3, \qquad m_{24} = \frac{h_c h_f}{4}(\rho_1 A_1 \alpha_1^2 + \rho_3 A_3 \alpha_3^2) = -m_{32}$$

$$m_{11} = \frac{h_c^2}{4}(\rho_1 A_1 R_1 \alpha_1^2 + \rho_3 A_3 R_3 \alpha_3^2)$$

The general solution is given by:

$$u_n(\theta) = \left\{\begin{array}{l} v_n = V\cos(n\theta) \\ w_n = W\sin(n\theta) \\ \psi_n = \Psi\cos(n\theta) \end{array}\right\} e^{i\omega_n t} \tag{42}$$

where ω_n is a real number corresponding to the real frequency.

Injecting (42) in (41), one gets a real linear eigenvalue problem allowing one to have the real linear mode and the associated linear frequencies.

$$\begin{bmatrix} n^2 A^0_{11} - A^0_{12} & n^3 A^0_{13} - n A^0_{14} & n^2 A^0_{15} - A^0_{16} \\ -n^3 A^0_{21} + n A^0_{22} & -n^4 A^0_{23} + n^2 A^0_{24} - A^0_{25} & -n^3 A^0_{26} + n A^0_{27} \\ n^2 A^0_{31} - A^0_{32} & n^3 A^0_{33} - n A^0_{34} & n^2 A^0_{35} - A^0_{36} \end{bmatrix} \begin{Bmatrix} V \\ W \\ \Psi \end{Bmatrix}$$
$$= \omega_n^2 \begin{bmatrix} m_{11} & n m_{12} & m_{13} \\ -n m_{21} & -n^2 m_{22} + m_{23} & -n m_{24} \\ m_{31} & n m_{32} & m_{33} \end{bmatrix} \begin{Bmatrix} V \\ W \\ \Psi \end{Bmatrix} \tag{43}$$

Appendix B. Computation of U_2 and U_0

In the case of the ring, equation (15) gives:

$$\begin{aligned} &\alpha_1\left(N'_{12} + \frac{M'_{12}}{R_1}\right) + \alpha_3\left(N'_{32} + \frac{M'_{32}}{R_3}\right) + N'_{22} + T_{22} + 4\omega_0^2(m_{11} v_2 + m_{12} w_2 + m_{13}\psi_2) \\ &= \frac{1}{2}[\alpha_1(N_{1m}\beta_{1m}) + \alpha_3(N_{3m}\beta_{3m}) + N_{2m}\beta_{2m}] \\ &\quad - N_{12} + \frac{M''_{12}}{R_1^2} + \frac{\alpha_1 h_f}{2R_1}\left(N''_{12} + \frac{M''_{12}}{R_1}\right) - N_{32} + \frac{M''_{32}}{R_3} - \frac{\alpha_3 h_f}{2R_3}\left(N''_{32} + \frac{M''_{32}}{R_3}\right) - N_{22} + T'_{22} \\ &\quad + 4\omega_0^2(m_{21} v'_2 + m_{22} w'_2 + m_{23} w_2 + m_{24}\psi'_2) \\ &= \frac{1}{2}\left[\left(1 + \frac{\alpha_1 h_f}{2R_1}\right)(N_{1m}\beta_{1m})' + \left(1 - \frac{\alpha_3 h_f}{2R_3}\right)(N_{3m}\beta_{3m})' + (N_{2m}\beta_{2m})'\right] \\ &\quad \frac{h_c}{2}\left[\alpha_1\left(N'_{12} + \frac{M'_{12}}{R_1}\right) - \alpha_3\left(N'_{32} + \frac{M'_{32}}{R_3}\right)\right] - R_2 T_{22} + M'_{22} + 4\omega_0^2(m_{31} v_2 + m_{32} w'_2 + m_{33}\psi_2) \\ &= \frac{h_c}{4}(\alpha_1 N_{1m}\beta_{1m} - \alpha_3 N_{3m}\beta_{3m}) \end{aligned} \tag{44}$$

The corresponding behavior law are given by:

$$
\begin{aligned}
&N_{12}=E_1S_1\left\{\frac{1}{R_1}\left[\alpha_1\left(v_2'+\frac{h_c}{2}\psi_2'-\frac{h_f}{2R_1}w_2''\right)+w_2\right]+\frac{\beta_{1m}^2}{4}\right\}\\
&M_{12}=\frac{E_1I_1}{R_1^2}\left[\alpha_1\left(v_2'+\frac{h_c}{2}\psi_2'\right)-\left(1+\frac{\alpha_1h_f}{2R_1}\right)w_2''\right],\qquad \beta_{32}=\frac{1}{R_3}\left[\alpha_3\left(v_2-\frac{h_c}{2}\psi_2\right)-\left(1-\frac{\alpha_3h_f}{2R_3}\right)w_2'\right]\\
&N_{32}=E_3A_3\left\{\frac{1}{R_3}\left[\alpha_3\left(v_2'-\frac{h_c}{2}\psi_2'+\frac{h_f}{2R_3}w_2''\right)+w_2\right]+\frac{\beta_{3m}^2}{4}\right\}\\
&M_{32}=\frac{E_3I_3}{R_3^2}\left[\alpha_3\left(v_2'-\frac{h_c}{2}\psi_2'\right)-\left(1-\frac{\alpha_3h_f}{2R_3}\right)w_2''\right],\qquad \beta_{22}=\frac{v_2-w_2'}{R_2}\\
&N_{22}=E_2(2\mathrm{i}\omega)A_2\left(\frac{v_2'+w_2}{R_2}+\frac{\beta_{2m}^2}{4}\right)\\
&M_{22}=\frac{E_2(2\mathrm{i}\omega)I_2}{R_2}\psi_2',\qquad T_{22}=k'G_2(2\mathrm{i}\omega)A_2\left(\frac{w_2'-v_2}{R_2}+\psi_2\right)
\end{aligned}
\tag{45}
$$

Injecting the last behavior law in the equation, one gets the following linear complex system:

$$
\begin{bmatrix}
-4n^2A_{11}^2+A_{12}^2+4\omega_n^2m_{11} & 8n^3A_{13}^2-2nA_{14}^2-8n\omega_n^2m_{12} & -4n^2A_{15}^2+A_{16}^2+4\omega_n^2m_{13}\\
-8n^3A_{21}^2+2nA_{22}^2+8n\omega_n^2m_{21} & \left\{\begin{matrix}16n^4A_{23}^2-4n^2A_{24}^2+A_{25}^2+\\ 4\omega_n^2(-4n^2m_{22}+m_{23})\end{matrix}\right\} & -8n^3A_{26}^2+2nA_{27}^2+8n\omega_n^2m_{24}\\
-4n^2A_{31}^2+A_{32}^2+4\omega_n^2m_{31} & 8n^3A_{33}^2-2nA_{34}^2-8n\omega_n^2m_{32} & -4n^2A_{35}^2+A_{36}^2+4\omega_n^2m_{33}
\end{bmatrix}
\times\begin{Bmatrix}v_{21}\\ w_{21}\\ \psi_{21}\end{Bmatrix}=\begin{Bmatrix}c_{11}^2\\ c_{22}^2\\ c_{31}^2\end{Bmatrix}
\tag{46}
$$

U_0 is obtained as a particular case from U_2 by putting $\omega_n=0$ and using real Young and shear moduli in the behavior corresponding to layer 2.

Appendix C

$$
\begin{aligned}
&l_v=-\alpha_1\left(N_{1n}'+\frac{M_{1n}'}{R_1}\right)-\alpha_3\left(N_{3n}'+\frac{M_{3n}'}{R_3}\right)-N_{2n}'+T_{2n}\\
&l_w=N_{1n}-\frac{M_{1n}''}{R_1^2}-\frac{\alpha_1h_f}{2R_1}\left(N_{1n}''+\frac{M_{1n}''}{R_1}\right)+N_{3n}-\frac{M_{3n}''}{R_3}+\frac{\alpha_3h_f}{2R_3}\left(N_{3n}''+\frac{M_{3n}''}{R_3}\right)-N_{2n}-T_{2n}'\\
&l_\psi=-\frac{h_c}{2}\left[\alpha_1\left(N_{1n}'+\frac{M_{1n}'}{R_1}\right)-\alpha_3\left(N_{3n}'+\frac{M_{3n}'}{R_3}\right)\right]+R_2T_{2n}-M_{2n}'\\
&m_v=m_{11}v_n+m_{12}w_n'+m_{13}\psi_n\\
&m_w=m_{21}v_n'+m_{22}w_n''+m_{23}w_n+m_{24}\psi_n'\\
&m_\psi=m_{31}v_n+m_{32}w_n'+m_{33}\psi_n\\
&q_v=\alpha_1\left[N_{1n}\left(\beta_{10}+\frac{1}{2}\beta_{12}\right)+\beta_{1n}\left(N_{10}+\frac{1}{2}N_{12}\right)\right]+\alpha_3\left[N_{3n}\left(\beta_{30}+\frac{1}{2}\beta_{32}\right)+\beta_{3n}\left(N_{30}+\frac{1}{2}N_{32}\right)\right]\\
&\qquad+N_{2n}\left(\beta_{20}+\frac{1}{2}\beta_{22}\right)+\beta_{2n}\left(N_{20}+\frac{1}{2}N_{22}\right)\\
&q_w=\left(1+\frac{\alpha_1h_f}{2R_1}\right)\left[N_{1n}'\left(\beta_{10}'+\frac{1}{2}\beta_{12}'\right)+\beta_{1n}'\left(N_{10}'+\frac{1}{2}N_{12}'\right)\right]\\
&\qquad+\left(1-\frac{\alpha_3h_f}{2R_3}\right)\left[N_{3n}'\left(\beta_{30}'+\frac{1}{2}\beta_{32}'\right)+\beta_{3n}'\left(N_{30}'+\frac{1}{2}N_{32}'\right)\right]\\
&\qquad+N_{2n}'\left(\beta_{20}'+\frac{1}{2}\beta_{22}'\right)+\beta_{2n}'\left(N_{20}'+\frac{1}{2}N_{22}'\right)\\
&q_\psi=\frac{h_c}{2}\left\{\alpha_1\left[N_{1n}\left(\beta_{10}+\frac{1}{2}\beta_{12}\right)+\beta_{1n}\left(N_{10}+\frac{1}{2}N_{12}\right)\right]-\left[N_{3n}\left(\beta_{30}+\frac{1}{2}\beta_{32}\right)+\beta_{3n}\left(N_{30}+\frac{1}{2}N_{32}\right)\right]\right\}
\end{aligned}
\tag{47}
$$

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