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Static and dynamic behaviour of nonlocal elastic bar using integral strain-based and peridynamic models

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ABSTRACT

The static and dynamic behaviour of a nonlocal bar of finite length is studied in this paper. The nonlocal integral models considered in this paper are strain-based and relative displacement-based nonlocal models; the latter one is also labelled as a peridynamic model. For infinite media, and for sufficiently smooth displacement fields, both integral nonlocal models can be equivalent, assuming some kernel correspondence rules. For infinite media (or finite media with extended reflection rules), it is also shown that Eringen's differential model can be reformulated into a consistent strain-based integral nonlocal model with exponential kernel, or into a relative displacement-based integral nonlocal model with a modified exponential kernel. A finite bar in uniform tension is considered as a paradigmatic static case. The strain-based nonlocal behaviour of this bar in tension is analyzed for different kernels available in the literature. It is shown that the kernel has to fulfil some normalization and end compatibility conditions in order to preserve the uniform strain field associated with this homogeneous stress state. Such a kernel can be built by combining a local and a nonlocal strain measure with compatible boundary conditions, or by extending the domain outside its finite size while preserving some kinematic compatibility conditions. The same results are shown for the nonlocal peridynamic bar where a homogeneous strain field is also analytically obtained in the elastic bar for consistent compatible kinematic boundary conditions at the vicinity of the end conditions. The results are extended to the vibration of a fixed–fixed finite bar where the natural frequencies are calculated for both the strain-based and the peridynamic models.

1. Introduction

Nonlocal models are continuum models that are able to account for the change of scale in the analysis of a structure that contains some microstructure patterns. Among integral-based nonlocal models, strain-based and relative displacement-based models have been both developed in the literature. Strain-based nonlocal models relate the stress to the strain through an integral operator valid in the whole range of the solid, whereas relative-based displacement models expressed the balance equation through an integral operator of the displacement difference, which avoids the calculation of the strain through a gradient operator. Strain-based nonlocal models have emerged in the 1960's for bridging lattice mechanics with engineering

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continuum models, especially (but not only) to capture the wave dispersive properties of crystal materials (see for instance [1,2] or [3]). These models have been further cast in a consistent thermodynamics framework (see for instance [4] or more recently [5]). Altan [6] studied the uniqueness of the boundary value problem of strain-based nonlocal elasticity static problems. Strain-based integral nonlocal elasticity theories are widely reviewed in the seminal book of Eringen [7] – see the historical analysis of Maugin [8] on the topic. Relative displacement-based models have been introduced by Silling [9] through the terminology of peridynamic models, and then widely developed by the same author and his co-authors for several engineering applications ([10,11] for instance). These models can also be understood as the continualization of lattice models that account to short and long-range interactions, thus being also associated with the concept of physically-based nonlocal models [12]. A fractional peridynamic model has been recently reported by Lazopoulos [13].

Some mathematical properties have to be fulfilled by the kernel for both integral approaches, including the strain-based or relative displacement-based approaches. The kernel has to fulfil some normalization and end compatibility conditions in order to preserve the uniform strain field associated with this homogeneous stress state. For infinite media, analytical and numerical solutions have been found for various kernels, and the discussion of compatible boundary conditions is avoided. For finite-body problems, the kernel associated with the integral model has to be compatible with the boundary conditions of the problem. This incompatibility between the natural boundary conditions and the induced kernel-dependent boundary conditions prevent the use of some kernels including some elementary exponential-based kernels (as detailed by Fernández-Sáez et al. [14] or by Romano et al. [15] for nonlocal integral beam models). Many nonlocal strain measures developed in the literature violate the nonlocal invariance of the uniform strain field, which can be physically questionable. Consequently, there is a need to select appropriate kernels for engineering applications and to find relevant solutions in some benchmark cases for simple nonlocal structural mechanics applications.

The same remarks hold for relative displacement-based nonlocal models, whose available solutions have been mostly derived in statics and in dynamics for infinite one-dimensional media by Silling et al. [10], Mikata [16] or Bažant et al. [17]. An exception is the recent paper of Nishawala and Ostoja-Starzewski [18], who obtained an analytical solution for a finite bar in tension under various distributed loadings. Nishawala and Ostoja-Starzewski [18] also discussed some correction effects at the vicinity of the finite bar, by introducing some distributed load that affects the homogeneous stress state configuration of the problem. In a certain sense, the difficulties pointed out in the strain-based integral model are not avoided in the relative displacement-based models, and some clarifications are needed for both models.

In this paper, we explore some possible link between nonlocal elasticity, peridynamics theory also labelled as a nonlocal relative displacement-based theory and lattice mechanics. The statics and the vibration of a finite bar is investigated, and some exact analytical solutions are derived for possible benchmark testing.

2. General equations of integral-based nonlocal models for one-dimensional problems

A one-dimensional strain-based nonlocal model can be introduced from the following integral operator (see for instance [7]):

$$N(x) = EA\int_0^L G(x,y)\varepsilon(y)\,\mathrm{d}y \tag{1}$$

for a finite bar of length L, where N is the normal force, ε is the axial strain, E is the Young modulus, A is the area and $G(x,y)$ is the nonlocal kernel of the strain-based nonlocal model, which should verify the translational invariance principle for homogeneous isotropic media [7]:

$$G(x,y) = g(\xi) = g(-\xi) \quad \text{with } \xi = y - x \tag{2}$$

G has the dimension of the inverse of a length, i.e. L^{-1}. The normalization procedure, in order to leave the uniform strain unchanged, for finite structural elements (see also [7] or [19]; or [40] for the discussion of the criterion for finite solids) can be written as:

$$\int_0^L G(x,y)\,\mathrm{d}y = 1 \tag{3}$$

To avoid any difficulties at the limit of the finite domain, and following the methodology that is applied to lattice problems which may be also investigated within nonlocal mechanics (see also [7]), it is sometimes preferred to extend the finite domain outside its domain of definition following some symmetrical properties or periodic properties, leading to the nonlocal normal force strain-based definition:

$$N(x) = EA\int_{-\infty}^{+\infty} G(x,y)\varepsilon(y)\,\mathrm{d}y \tag{4}$$

In this case, the normalization procedure is simply reduced to

$$\int_{-\infty}^{+\infty} G(x, y)\,\mathrm{d}y = 1 \tag{5}$$

Two possible useful nonlocal kernels can be mentioned at this stage (which verifies the normalization procedure given by Eq. (5)), namely the symmetrical bilinear kernel function used by Eringen and Kim [20]:

$$\begin{cases} G(x, y) = \frac{1}{a^2}(a - |x-y|) & \text{if } |x-y| \le a \\ G(x, y) = 0 & \text{if } |x-y| \ge a \end{cases} \tag{6}$$

Another widely used nonlocal kernel is the exponential kernel:

$$G(x, y) = \frac{1}{2l_c} \mathrm{e}^{-\frac{|x-y|}{l_c}} \tag{7}$$

This kernel does not verify the normalization criterion for finite-length structures. Furthermore, as detailed by Benvenuti and Simone [21] for instance part (see also the recent discussions in [14] or in [15] for nonlocal beam problems), the choice of the kernel of the nonlocal model leads to an integro-differential equation that has to be compatible with the natural and essential boundary conditions of the mechanical problem. We will explore in the paper some other exponential-based kernels, which may satisfy the normalization criterion for finite-length problems.

Another class of nonlocal model is the stress gradient model of Eringen [22] based on the differential law:

$$N(x) - l_c^2 N''(x) = EA\varepsilon(x) \tag{8}$$

whose kernel depends on the boundary conditions, as will be discussed in the paper. It is easy to show that the kernel in Eq. (7) verifies the differential equation of the Eringen model [22]. However, as the normalization criterion applied to the kernel of Eq. (7) is only valid for an infinite domain, as characterized by Eq. (5), the correspondence between Eq. (7) and Eq. (8) is only valid for infinite domains (infinite domain, or finite domain extended at the infinite using periodicity boundary conditions).

Finally, the peridynamic model (also called relative displacement-based nonlocal model) can be formulated as [9,10,12]:

$$N'(x) = EA \int_0^L H(x, y)\left[u(y) - u(x)\right] \mathrm{d}y \tag{9}$$

where $H(x, y)$ is the nonlocal kernel of the peridynamics nonlocal model. H has the dimension of the inverse of a volume, i.e. L^{-3}.

For homogeneous isotropic media, the translational invariance principle is also fulfilled:

$$H(x, y) = h(\xi) = h(-\xi) \quad \text{with } \xi = y - x \tag{10}$$

The normalization procedure for the finite length peridynamic kernel can be defined from:

$$\int_0^L H(x, y)(y-x)^2\,\mathrm{d}y = 2 \tag{11}$$

For infinite domains, one has:

$$N'(x) = EA \int_{-\infty}^{+\infty} H(x, y)\left[u(y) - u(x)\right] \mathrm{d}y \tag{12}$$

where $H(x, y)$ is the nonlocal kernel of the peridynamics nonlocal model, which should verify the normalization procedure (see [10,16,17]) given by:

$$\int_{-\infty}^{+\infty} H(x, y)(y-x)^2\,\mathrm{d}y = 2 \tag{13}$$

The normal force can be calculated, by performing one integration (see for instance [12]), which can be written for an infinite bar as:

$$N(x) = EA \int_{y_1=-\infty}^{x} \int_{y_2=x}^{+\infty} H(y_1, y_2)\left[u(y_2) - u(y_1)\right] \mathrm{d}y_1\, \mathrm{d}y_2 \tag{14}$$

For a finite length problem, one cannot extend this double integral directly for finite domain:

$$N(x) \neq EA \int_{y_1=0}^{x} \int_{y_2=x}^{+L} H(y_1, y_2)\left[u(y_2) - u(y_1)\right] \mathrm{d}y_1\, \mathrm{d}y_2 \tag{15}$$

This is due to the fact that:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left[\int_{y_1=0}^{x} \int_{y_2=x}^{+L} H(y_1, y_2)\left[u(y_2) - u(y_1)\right] \mathrm{d}y_1\, \mathrm{d}y_2\right] &= \int_{0}^{L} H(x, y)\left[u(y) - u(x)\right] \mathrm{d}y \\ - \int_{x}^{L} H(0, y)\left[u(y) - u(0)\right] \mathrm{d}y &- \int_{0}^{x} H(L, y)\left[u(y) - u(L)\right] \mathrm{d}y \end{aligned} \tag{16}$$

In fact, there is a strong connection between peridynamic and strain-based integral models for infinite media: assuming the differentiability of the displacement field, peridynamic models can be converted into strain-based integral models by assuming a kernel correspondence between both nonlocal models:

$$h(\xi) = g''(\xi) = \frac{\mathrm{d}^2 g}{\mathrm{d}\xi^2} \tag{17}$$

The proof is given by Silling et al. [10] for general kernels and is also mentioned in [12] for exponential-based kernels. The proof is obtained by assuming that there exists a kernel operator g^* such as:

$$\frac{\mathrm{d}^2 g^*}{\mathrm{d}\xi^2} = h(\xi) \tag{18}$$

which asymptotically vanishes for infinite values, a property also valid for the first- and second-order derivative of this kernel. Rewriting the peridynamic governing equation (which couples the constitutive law with the balance equation) with the difference variable ξ gives:

$$N'(x) = EA \int_{-\infty}^{+\infty} \frac{\partial^2 g^*}{\partial \xi^2}\left[u(x+\xi) - u(x)\right] \mathrm{d}\xi \tag{19}$$

which can be integrated two times, thus leading to:

$$N'(x) = \left[EA \frac{\partial g^*(\xi)}{\partial \xi}\left[u(x+\xi) - u(x)\right]\right]_{-\infty}^{+\infty} - \left[EA g^*(\xi) \frac{\partial u(x+\xi)}{\partial \xi}\right]_{-\infty}^{+\infty} + EA \int_{-\infty}^{+\infty} g^*(\xi) \frac{\partial^2 u(x+\xi)}{\partial \xi^2} \mathrm{d}\xi \tag{20}$$

Assuming that the boundary terms are vanishing, this equation can be equivalently written as:

$$\frac{\mathrm{d}N}{\mathrm{d}x} = EA \frac{\partial}{\partial x} \int_{-\infty}^{+\infty} g^*(\xi) \frac{\partial u(x+\xi)}{\partial \xi} \mathrm{d}\xi \quad \Rightarrow \quad N(x) = EA \int_{-\infty}^{+\infty} g^*(\xi) \frac{\partial u(y)}{\partial y} \mathrm{d}y \tag{21}$$

with $\xi = y - x$, which shows that $g = g^*$ can be used as a possible kernel of the equivalent strain-based integral nonlocal model (for sufficiently smooth displacement fields).

For infinite media, the normalization procedure of the peridynamic model can follow from the normalization criterion of the strain-based integral model:

$$\int_{-\infty}^{+\infty} h(\xi)\xi^2 \,\mathrm{d}\xi = \int_{-\infty}^{+\infty} g''(\xi)\xi^2 \,\mathrm{d}\xi = \left[g'(\xi)\xi^2\right]_{-\infty}^{+\infty} - \left[2\xi g(\xi)\right]_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} 2g(\xi)\,\mathrm{d}\xi = 2 \tag{22}$$

It is worth mentioning that there are generally no kernel correspondences between integral-based and peridynamic nonlocal models for finite-length structural elements. This correspondence can be eventually shown on some specific kernels such as the bilinear kernel considered by Eringen and Kim [20], and based on lattice interaction. Using Eq. (6), as shown by Eringen and Kim [20], it is possible to derive the continualized lattice-based equations:

$$N' = EA\frac{u(x+a) - 2u(x) + u(x-a)}{a^2} \tag{23}$$

Eq. (23) may be classified as a functional equation, which is the continualization of the associated difference equation (see [23]).

The proof of Eq. (23) is also based on the following integration by part:

$$\begin{aligned} \int_0^L G(x,y)\frac{\partial u}{\partial y}\,\mathrm{d}y &= \frac{1}{a}\int_{x-a}^{x+a}\left(1-\frac{|x-y|}{a}\right)\frac{\partial u}{\partial y}\,\mathrm{d}y \\ &= \left[\frac{1}{a}\left(1-\frac{|x-y|}{a}\right)u(y)\right]_{x-a}^{x+a} - \frac{1}{a^2}\int_{x-a}^{x+a}\mathrm{sgn}(x-y)u(y)\,\mathrm{d}y \\ &= -\frac{1}{a^2}\int_{x-a}^{x} u(y)\,\mathrm{d}y + \frac{1}{a^2}\int_{x}^{x+a} u(y)\,\mathrm{d}y \end{aligned} \tag{24}$$

The continualized lattice model is a particular case of the peridynamic model with the following kernel (see also [24]):

$$H(x,y) = \frac{\delta(x+a,y) - 2\delta(x,y) + \delta(x-a,y)}{a^2} \tag{25}$$

where the Dirac distribution has been used. Again, using Eq. (25), one is able to build Eq. (23), which is the continualized nonlocal lattice model with direct neighbouring interaction. It means that for some specific kernels, and for infinite media, strain-based nonlocal integral models are exactly coincident with relative-based displacement models. It is however worth mentioning the strong difference between the kernel of Eq. (25) and the one of Eq. (6), which shows that the kernel of peridynamic models cannot be chosen, in general, in the same form as the one of the strain-based nonlocal model. For this lattice-based nonlocal model, it is possible to show that the mathematical property of the kernel equivalence of Eq. (17) is fulfilled, at least if the derivative is understood in the distributed sense. We also mention some links between the discretized version of the peridynamic model with the finite difference formulation of local elasticity models (see [25]); the latter one can be also viewed as the difference equations of the lattice medium with direct neighbour interactions.

This correspondence (valid for infinite media) can be also achieved for the exponential kernel of the strain-based nonlocal model given by Eq. (7), whose peridynamic analogous would be formulated by (see also [12] for this specific model):

$$H(x,y) = \frac{1}{2l_c^3}\mathrm{e}^{-\frac{|x-y|}{l_c}} \tag{26}$$

Finally, we would like to emphasize that for infinite media (or finite media with extended reflection rules), Eringen's differential model given by Eq. (8) can be reformulated into a consistent strain-based integral nonlocal model with the exponential kernel of Eq. (7), or into a relative-based displacement integral nonlocal model with a modified exponential kernel given by Eq. (26). These nonlocal models are now applied to some simple structural cases, both in statics and in dynamics.

3. Static case – finite domain

3.1. Loading configuration

We are studying a uniform bar of length L under pure tension, which may be expressed from the principle of virtual work:

$$\int_0^L \sigma A\delta u'\,\mathrm{d}x - \sigma_0 A\delta u(L) = 0 \tag{27}$$

where σ_0 is the uniform tension stress applied at the end vicinity, and A is the cross-sectional area. The bar is fixed at $x=0$, which means, after an integration by part, that

$$\int_0^L -(\sigma A)'\delta u\,\mathrm{d}x + [\sigma A\delta u]_0^L - \sigma_0 A\delta u(L) = 0 \tag{28}$$

The boundary conditions are obtained:

$$u(0) = 0 \quad \text{and} \quad \sigma(L) = \sigma_0 \tag{29}$$

and the stress is homogeneous for a uniform bar $A'(x) = 0$:

$$N'(x) = 0 \quad \Rightarrow \quad \sigma(x) = \sigma_0 \tag{30}$$

3.2. Eringen's differential model

For Eringen's differential model [22], the nonlocal elasticity model degenerates into the local model:

$$N'(x) = 0 \quad \Rightarrow \quad N - l_c^2 N'' = N = EAu' \tag{31}$$

and we then obtain the local elasticity solution:

$$u(x) = \frac{\sigma_0 x}{E} \tag{32}$$

3.3. Exponential kernel with infinite domain

The displacement field has to be extended outside the domain to be compatible with natural and essential boundary conditions of the bar problem. Recently, Sumelka [26] used a similar concept based on the extension of the domain outside its initial definition. One possibility is to take the continuation rule:

$$u^-(x) = u^+(x) \quad \text{for } x \le 0 \quad \text{and} \quad u^-(x) = u^+(x) \quad \text{for } x \ge L \tag{33}$$

In this case, it is easy to check that the uniform solution:

$$u(x) = \frac{\sigma_0}{E} x \quad \Rightarrow \quad u'(x) = \frac{\sigma_0}{E} \quad \text{for } x \in [-\infty; +\infty] \tag{34}$$

is solution to:

$$N(x) = EA \int_{-\infty}^{+\infty} \frac{1}{2l_c} \mathrm{e}^{-\frac{|x-y|}{l_c}} \varepsilon(y)\,\mathrm{d}y = \sigma_0 A \tag{35}$$

There is an equivalence between the integral model for infinite domain with periodic boundary conditions and the Eringen's stress gradient model also extended to an infinite domain. The correspondence (valid for infinite media) can be also achieved for the peridynamic model with the exponential kernel given by Eq. (26).

3.4. Exponential kernel with finite domain – direct nonlocal measure

If now one considers the integral model with a truncated exponential kernel, one would obtain:

$$N(x) = EA \int_{0}^{L} \frac{1}{2l_c} \mathrm{e}^{-\frac{|x-y|}{l_c}} \varepsilon(y)\,\mathrm{d}y = \sigma_0 A \tag{36}$$

As shown by Benvenuti and Simone [21], such integral equation implicitly contains some constraints on the normal force at the boundary such as:

$$N'(0) - \frac{1}{l_c} N(0) = 0 \quad \text{and} \quad N'(L) + \frac{1}{l_c} N(L) = 0 \tag{37}$$

Clearly, these boundary conditions violate the natural boundary conditions of the problem, and the kernel of Eq. (36) is not admissible for the loading case considered in this part (see also the recent discussions in [14] or in [15] for nonlocal beam problems – see also the discussion in Appendix A). Furthermore, the kernel chosen in Eq. (36) does not verify the normalization criterion given by Eq. (3).

3.5. Exponential kernel with finite domain – combination of nonlocal and local measures

Following the methodology of Eringen [27], it is possible to consider a combination of local and nonlocal measures given by:

$$N(x) = EA[\xi_1 \varepsilon + \xi_2 \overline{\varepsilon}] \quad \text{with } \xi_1 + \xi_2 = 1,\ \xi_1 \in [0;1] \text{ and } \xi_2 \in [0;1] \tag{38}$$

The requirement $\xi_1 \in [0;1]$ is necessary to preserve the positivity of the nonlocal kernel operator. For $\xi_1 > 1$, $\xi_2 = 1 - \xi_1$ becomes negative and the associated nonlocal strain-based energy functional loses its definite positiveness, thus preventing to obtain the uniqueness property of the integral nonlocal boundary value problem [6,5]. It is eventually possible to express this model in term of nonlocal strain and strain gradient energy functional (see [28,29] or [30]), which may eventually restore the positive definiteness of the nonlocal strain and strain gradient energy functional. However, the paper is mainly focused on strain-based nonlocal models and we will impose that both ξ_1 and ξ_2 belong to [0; 1]. As shown by Challamel and Wang [28] for nonlocal bending problems or by Challamel et al. [29] for nonlocal axial rods, the nonlocal kernel can be introduced from the following nonlocal measure (which is used by Peerlings et al. [31] for nonlocal damage applications):

$$\overline{\varepsilon} - l_c^2 \overline{\varepsilon}'' = \varepsilon \tag{39}$$

Furthermore, the following boundary conditions have been obtained by Challamel and Wang [28] or Challamel et al. [29] from the application of a variational principle:

$$\overline{\varepsilon}'(0) = \overline{\varepsilon}'(L) = 0 \quad \Rightarrow \quad \overline{\varepsilon} = \int_0^L F(x,y)\varepsilon(y)\,\mathrm{d}y \quad \text{with}$$
$$\begin{cases} F(x,y) = \frac{1}{l_c}\frac{\cosh\left(\frac{L-y}{l_c}\right)}{\sinh\left(\frac{L}{l_c}\right)}\cosh\left(\frac{x}{l_c}\right) & \text{if } x \le y \\ F(x,y) = \frac{1}{l_c}\frac{\cosh\left(\frac{L-x}{l_c}\right)}{\sinh\left(\frac{L}{l_c}\right)}\cosh\left(\frac{y}{l_c}\right) & \text{if } x \ge y \end{cases} \tag{40}$$

Such a kernel can be found in [32] as the Green function associated with the differential equation associated with the considered boundary conditions (see also [33] for nonlocal damage mechanics applications).

These higher-order boundary conditions can be re-expressed in term of mixed normal force–strain variable as:

$$\frac{1}{\xi_2}\left[\frac{N'(0)}{EA} - \xi_1 \varepsilon'(0)\right] = \frac{1}{\xi_2}\left[\frac{N'(L)}{EA} - \xi_1 \varepsilon'(L)\right] = 0 \tag{41}$$

We note that the uniform local solution for both the stress and the strain is compatible with these boundary conditions:

$$N(x) = \sigma_0 A = AE\varepsilon(x) \tag{42}$$

We finally have:

$$G(x,y) = \xi_1 \delta(x,y) + \xi_2 F(x,y) \quad \text{with } \xi_1 + \xi_2 = 1 \tag{43}$$

Note that this model is also equivalent to a nonlocal coupled strain gradient elasticity model:

$$N(x) - l_c^2 N''(x) = EA\left[\varepsilon(x) - a^2 \varepsilon''(x)\right] \quad \text{with } \xi_1 = \left(\frac{a}{l_c}\right)^2 \in [0;1] \tag{44}$$

Applying this model to the uniform bar under tension gives the local solution and the uniform strain model. Note that this model can be cast in the so-called micromorphic approach [29,34]. Exact solutions for nonlocal beam problems (including bending, buckling and vibration) have been elaborated by Zhang et al. [35]. Lim et al. [30] also showed the thermodynamic background of this model, which can be classified also as a nonlocal strain gradient model.

3.6. Alternative of the exponential kernel with finite domain – nonlocal solution affected by the small-length-scale terms

In this model, used by Benvenuti and Simone (2013), Eqs. (38) and (39) are used (Eq. (44) is also valid) but the boundary conditions for defining the nonlocal strain differ and are defined by:

$$\overline{\varepsilon}'(0) - \frac{1}{l_c}\overline{\varepsilon}(0) = 0 \quad \text{and} \quad \overline{\varepsilon}'(L) + \frac{1}{l_c}\overline{\varepsilon}(L) = 0 \tag{45}$$

which can be re-expressed in term of mixed normal force–strain boundary conditions as:

$$\frac{1}{\xi_2}\left[\frac{N'(0)}{EA}-\xi_1\varepsilon'(0)\right]-\frac{1}{\xi_2 l_c}\left[\frac{N(0)}{EA}-\xi_1\varepsilon(0)\right]=\frac{1}{\xi_2}\left[\frac{N'(L)}{EA}-\xi_1\varepsilon'(L)\right]+\frac{1}{\xi_2 l_c}\left[\frac{N(L)}{EA}-\xi_1\varepsilon(L)\right]=0 \tag{46}$$

We note that the uniform local solution given by Eq. (42) for both the stress and the strain is not compatible with these boundary conditions, which would imply that:

$$N'=0 \quad \text{and} \quad \varepsilon'=0 \quad \Rightarrow \quad N=EA\xi_1\varepsilon \tag{47}$$

Equation (44) which is also valid for this model, can be equivalently presented in the following differential form:

$$N(x)-l_c^2N''(x)=EA\left[\varepsilon(x)-\xi_1 l_c^2\varepsilon''(x)\right] \tag{48}$$

Note that the consideration of Eqs. (47) and (48) shows that the uniform local solution for both the stress and the strain cannot be a solution to this nonlocal model, except in the local case where $\xi_1=1$. The kernel of this nonlocal model is obtained from:

$$G(x,y)=\xi_1\delta(x,y)+\xi_2F(x,y) \quad \text{with } F(x,y)=\frac{1}{2l_c}e^{-\frac{|x-y|}{l_c}} \text{ and } \xi_1=\left(\frac{a}{l_c}\right)^2 \tag{49}$$

The solution to this problem is given by Benvenuti and Simone [21] for nonlocal axial bars or by Khodabakhshi and Reddy [36] or Wang et al. [37] for nonlocal beams. Such a model has been also considered by Pisano and Fuschi [38] – see also the recent discussion of Pisano and Fuschi [39].

The solution for uniform stress is clearly affected by the small-length-scale terms. It is possible to check in this case that the normalization criterion is not fulfilled in the finite domain:

$$\int_0^L G(x,y)\,dy \neq 1 \tag{50}$$

3.7. Some alternative kernels – a spatial-dependent combination of local and nonlocal measures

The following nonlocal measure has been first proposed by Borino et al. [19] for nonlocal damage mechanics applications (see also the discussion in [40]), and has been used quite recently by Koutsoumaris et al. [41] for nonlocal elastic beams. The model can be presented in the following form:

$$N(x)=EA[\xi_1\varepsilon+\xi_2\overline{\varepsilon}] \quad \text{with } \xi_1+\xi_2=1 \text{ and } \overline{\varepsilon}=\left[1-\int_0^L F(x,y)\,dy\right]\varepsilon+\int_0^L F(x,y)\varepsilon(y)\,dy \tag{51}$$

If now we specify $\xi_2=1$, the nonlocal kernel is obtained:

$$G(x,y)=\left[1-\int_0^L F(x,y)\,dy\right]\delta(x,y)+F(x,y) \quad \text{with } F(x,y)=\frac{1}{2l_c}e^{-\frac{|x-y|}{l_c}} \tag{52}$$

In this case, the normalization criterion is effectively checked. This problem has been numerically investigated by Koutsoumaris et al. [41] for beam problems, but no analytical solutions have been found up to now.

This model can be also expressed in the following form:

$$N(x)=EA\left[\xi_1(x)\varepsilon+\overline{\varepsilon}\right] \quad \text{with } \overline{\varepsilon}=\int_0^L F(x,y)\varepsilon(y)\,dy,\ \xi_1(x)=1-\int_0^L F(x,y)\,dy \text{ and } F(x,y)=\frac{1}{2l_c}e^{-\frac{|x-y|}{l_c}} \tag{53}$$

The coefficient $\xi_1(x)$ can be easily calculated as:

$$\xi_1(x)=\frac{1}{2}\left[e^{-\frac{x}{l_c}}+e^{-\frac{L-x}{l_c}}\right] \tag{54}$$

This model can be cast in a differential form:

$$N(x)-l_c^2N''(x)=EA\left[(1+\xi_1)\varepsilon(x)-l_c^2\left[\xi_1\varepsilon(x)\right]''\right] \tag{55}$$

It is easy to check from Eq. (55) that:

$$\varepsilon(x)=\varepsilon_0 \quad \Rightarrow \quad N(x)-l_c^2N''(x)=EA\varepsilon_0 \tag{56}$$

However, from the boundary conditions to be checked by Eq. (45), we find that:

$$\left[\frac{N'(0)}{EA}-(\xi_1\varepsilon)'(0)\right]-\frac{1}{l_c}\left[\frac{N(0)}{EA}-\xi_1\varepsilon(0)\right]=\left[\frac{N'(L)}{EA}-(\xi_1\varepsilon)'(L)\right]+\frac{1}{l_c}\left[\frac{N(L)}{EA}-\xi_1\varepsilon(L)\right]=0 \tag{57}$$

Again, we note that the uniform local solution given by Eq. (42) for both the stress and the strain is not compatible with these boundary conditions.

3.8. Peridynamic model – kernel with inverse distance

Semi-analytical solutions are available and are expressed in integral form in [10] and [16] for the static analysis of infinite bars with the relative displacement-based (or peridynamic) model. Recently, Nishawala and Ostoja-Starzewski [18] obtained an explicit analytical solution for a homogeneous elastic bar with various distributed loading, based on the following peridynamic model with the inverse distance interaction function:

$$N'(x)=EA\int_{x-a}^{x+a}H(x,y)\big[u(y)-u(x)\big]\,dy \quad \text{with } H(x,y)=\frac{2}{a^2|y-x|} \tag{58}$$

and the normal force is calculated by integration:

$$N(x)=N(0)+\frac{2EA}{a^2}\int_{y_1=0}^{x}\int_{y_2=y_1-a}^{y_1+a}\frac{u(y_2)-u(y_1)}{|y_2-y_1|}\,dy_2\,dy_1 \tag{59}$$

It can be checked that the normalization criterion is verified, at least if the kernel interaction is extended beyond the limit of the finite bar:

$$\int_{-\infty}^{+\infty}H(x,y)(y-x)^2\,dy=\int_{x-a}^{x+a}H(x,y)(y-x)^2\,dy=\int_{x-a}^{x+a}\frac{2}{a^2}|y-x|\,dy=2 \tag{60}$$

Note that the normalization criterion cannot be checked in the vicinity of the boundaries, if the kernel is truncated at the boundaries, as suggested by Nishawala and Ostoja-Starzewski [18]. Consequently, Nishawala and Ostoja-Starzewski [18] obtained a non-uniform stress state for the homogeneous bar under pure tension, by adding some distributed forces at the boundary to correct this effect.

To avoid this effect due to the truncated nonlocal measure, and following the methodology introduced for the strain-based nonlocal problem, the displacement field can be extended outside the domain to be compatible with the natural and essential boundary conditions of the bar problem, which can be related to the extended layer concept introduced by Silling [9] or Macek and Silling [11].

As the kernel is truncated over a distance defined by a, the extended domain is concerned by $x\in[-a;0]$ and $x\in[L;L+a]$.

One possibility is to take the continuation rule:

$$u^-(x)=u^+(x) \quad \text{for } x\le 0 \quad \text{and} \quad u^-(x)=u^+(x) \quad \text{for } x\ge L \tag{61}$$

The following local-type solution is solution to the problem:

$$u(x)=\frac{\sigma_0}{E}x \quad \text{for } x\in[-a;L+a] \tag{62}$$

It can be checked that this solution verified the nonlocal governing equation (58), i.e. $N'(x)=0$ for $x\in[0;L]$, and the boundary conditions:

$$u(0)=0 \quad \text{and} \quad N(L)=N(0)=\sigma_0 A \tag{63}$$

due to the normal force integration along the bar:

$$\frac{2EA}{a^2}\int_{y_1=0}^{L}\int_{y_2=y_1-a}^{y_1+a}\frac{u(y_2)-u(y_1)}{|y_2-y_1|}\,dy_2\,dy_1=\frac{2\sigma_0 A}{a^2}\int_{y_1=0}^{L}\int_{y_2=y_1-a}^{y_1+a}\frac{y_2-y_1}{|y_2-y_1|}\,dy_2\,dy_1=0 \tag{64}$$

The natural boundary condition of this problem would not have been verified with a truncated kernel, as observed by Nishawala and Ostoja-Starzewski [18].

3.9. Peridynamic model – kernel with discrete distance

Let us now consider the following peridynamic model (based on single discrete symmetrical interactions) for a finite bar with some extended compatible boundary conditions:

$$N'(x) = EA \int_{-\infty}^{+\infty} H(x, y)\big[u(y) - u(x)\big]\,\mathrm{d}y \quad \text{with } H(x, y) = \frac{\delta(x+a, y) - 2\delta(x, y) + \delta(x-a, y)}{a^2} \tag{65}$$

which is equivalent to:

$$N' = EA\frac{u(x+a) - 2u(x) + u(x-a)}{a^2} = 0 \quad \text{with } N = \sigma_0 A \tag{66}$$

which is a functional equation that should be valid for $x \in [0; L]$. This functional equation needs to define the displacement function for $x \in [-a; L+a]$, which is somehow related to the extended layer concept introduced by Silling [9] or Macek and Silling [11].

By integration of Eq. (66), the normal force of this peridynamic model can be calculated from:

$$N(x) = EA \int_{x-a}^{x} \frac{u(y+a) - u(y)}{a^2}\,\mathrm{d}y \tag{67}$$

or equivalently from:

$$N(x) = EA \int_{x}^{x+a} \frac{u(y) - u(y-a)}{a^2}\,\mathrm{d}y \tag{68}$$

or also using a "centred scheme":

$$N(x) = EA \int_{x-\frac{a}{2}}^{x+\frac{a}{2}} \frac{u(y+a/2) - u(y-a/2)}{a^2}\,\mathrm{d}y \tag{69}$$

The domain is extended outside its physical definition $x \in [-a; 0]$ and $x \in [L; L+a]$, following the same method used for finite difference problems using some fictitious points.

Again, the homogeneous solution Eq. (62) is valid for this nonlocal peridynamic bar, and it can be checked that the natural and essential boundary conditions are checked:

$$u(0) = 0 \quad \text{and} \quad N(L) = EA \int_{L-a}^{L} \frac{u(y+a) - u(y)}{a^2}\,\mathrm{d}y = \sigma_0 A \tag{70}$$

As already pointed out in the paper, and as mentioned by Silling [9] or Macek and Silling [11], the static boundary condition needs to include the behaviour of the bar outside its initial domain, using some extension properties, as used in finite difference method by the concept of fictitious nodes.

The solution to the functional equation (66) with the boundary conditions – Eq. (70) – partially expressed in integral format is the local displacement solution, given by Eq. (62).

The proof comes from the difference formulation associated with the functional problem, and expressed by:

$$EA\frac{u_{i+1} - 2u_i + u_{i-1}}{a^2} = 0 \quad \text{with } u_0 = 0 \tag{71}$$

The solution to this linear second-order difference equation can be built from the difference equation:

$$u_{i+1} - 2u_i + u_{i-1} = 0 \quad \Rightarrow \quad u_i = \alpha i + \beta \tag{72}$$

Introducing the fixed boundary condition leads to

$$u_0 = 0 \quad \Rightarrow \quad u_i = \alpha i \quad \Rightarrow \quad u(x) = \alpha\frac{x}{a} \tag{73}$$

The last boundary condition $N(L) = \sigma_0 A$ gives $\alpha = \frac{\sigma_0 a}{E}$. Of course, these results, valid for the peridynamic model based on the kernel given by Eq. (25), are also valid for the strain-based nonlocal model with the kernel given by Eq. (6), due to the correspondence rule.

4. Dynamic case

4.1. Loading configuration

We are now studying the vibration of a fixed–fixed uniform nonlocal bar, where the integral nonlocality can be related to a strain-based or to a relative displacement-based (or peridynamic) model.

The principle of virtual work which includes the inertia effects can be written as:

$$\int_0^L N\delta u' + \rho A\ddot{u}\delta u \,\mathrm{d}x = 0 \tag{74}$$

which means, after an integration by part, that

$$\int_0^L -N'\delta u + \rho A\ddot{u}\delta u \,\mathrm{d}x + [N\delta u]_0^L = 0 \tag{75}$$

The boundary conditions are obtained for the fixed–fixed bar from:

$$u(0) = 0 \quad \text{and} \quad u(L) = 0 \tag{76}$$

and the balance equation is given by:

$$N' = \rho A\ddot{u} \tag{77}$$

Considering the harmonic motion $u(t) = u\mathrm{e}^{\mathrm{j}\omega t}$ where $\mathrm{j} = \sqrt{-1}$ and ω is the angular frequency of vibration, the balance equation reads:

$$N' + \omega^2\rho A u = 0 \tag{78}$$

4.2. Eringen's differential model for finite domain

For finite domains, the strain-based nonlocal problem to be solved is the following integro-differential eigenvalue problem:

$$\frac{\mathrm{d}}{\mathrm{d}x} EA \int_0^L G(x, y)\frac{\partial u}{\partial y}(y)\,\mathrm{d}y + \omega^2\rho A u = 0 \quad \text{with } u(0) = u(L) \tag{79}$$

The kernel of Eringen's differential model depends on the compatible natural boundary condition. Due to Eq. (78), the fixed–fixed boundary conditions can be expressed in term of static equivalent boundary conditions:

$$u(0) = u(L) \quad \Rightarrow \quad N'(0) = N'(L) = 0 \tag{80}$$

so that Eringen's differential model can be inverted in this case in the following form (see also Eq. (40)):

$$N - l_c^2 N'' = EAu' \quad \Rightarrow \quad N(x) = EA\int_0^L G(x, y)\frac{\partial u}{\partial y}\,\mathrm{d}y \quad \text{with} \begin{cases} G(x, y) = \frac{1}{l_c}\frac{\cosh(\frac{L-y}{l_c})}{\sinh(\frac{L}{l_c})}\cosh(\frac{x}{l_c}) & \text{if } x \le y \\ G(x, y) = \frac{1}{l_c}\frac{\cosh(\frac{L-x}{l_c})}{\sinh(\frac{L}{l_c})}\cosh(\frac{y}{l_c}) & \text{if } x \ge y \end{cases} \tag{81}$$

The integro-differential Eq. (79) can be equivalently cast in a single second-order differential equation:

$$\left(EA - \omega^2\rho A l_c^2\right)u'' + \omega^2\rho A u = 0 \quad \text{with } u(0) = u(L) \tag{82}$$

The dimensionless parameters can be introduced as:

$$\overline{u} = \frac{u}{L}, \quad \overline{x} = \frac{x}{L}; \quad \beta = \rho\frac{\omega^2 L^2}{E} \quad \text{and} \quad \mu = \frac{l_c^2}{L^2} \tag{83}$$

The nonlocal vibration equation is then obtained from:

$$(1 - \beta\mu)\overline{u}'' + \beta\overline{u} = 0 \quad \text{with } \overline{u}(0) = \overline{u}(1) \tag{84}$$

where the derivatives are now expressed with respect to the dimensionless spatial variable $\bar{x}$. For the fixed–fixed nonlocal rod, the natural modes and the natural frequencies are obtained as (see [42]):

$$u(x) = U \sin\left(\frac{n\pi x}{L}\right) \quad \text{and} \quad \beta_n = \frac{(n\pi)^2}{1+\mu(n\pi)^2} \leq (n\pi)^2 \quad \text{with } \beta_n = \rho\frac{\omega_n^2 L^2}{E} \text{ and } \mu = \frac{l_c^2}{L^2} \tag{85}$$

It is shown that the small-length-scale term μ tends to soften the natural frequencies of this nonlocal bar, a conclusion that is in agreement with the results obtained for nonlocal beams [43]. It is worth mentioning that such a nonlocal kernel may be considered as a good approximation of lattice model with direct neighbouring interaction, as shown by Challamel et al. [24] for instance.

It may be also commented that the exponential kernel considered in Eq. (36) cannot be used because the stress-based boundary conditions in Eq. (37) coupled with the natural and essential boundary conditions would lead to an over-constrained problem without any solution. The same incompatibility was found for the bar in pure tension.

4.3. Exponential kernel with finite domain – combination of nonlocal and local measures

The dynamic behaviour of a nonlocal bar composed of local and nonlocal strain measures (which has been shown to be equivalent to a nonlocal strain gradient model) was considered by Challamel et al. [29], Song et al. [44], Challamel [45] or Lim et al. [30] from the wave dispersion properties. Recently, Li et al. [46] studied the vibration of such a nonlocal bar, using the combination of local and nonlocal strain measures. In this case, the nonlocal strain measure is given by Eq. (39).

Using Eq. (48) coupled to the balance equation (78) leads to the normal force expression:

$$N = \left(EA - \omega^2 \rho A l_c^2\right)u' - \xi_1 l_c^2 EAu''' \tag{86}$$

The fourth-order governing differential equation is finally obtained:

$$-\xi_1 l_c^2 EAu^{(4)} + \left(EA - \omega^2 \rho A l_c^2\right)u'' + \omega^2 \rho Au = 0 \tag{87}$$

This is a higher-order elasticity theory with a fourth-order spatial derivative similar to what was observed for gradient elasticity theories of axial bar (see [46–49]), which can be presented as a generalization of the nonlocal equation (82):

$$-\xi_1 \mu \bar{u}^{(4)} + (1-\beta\mu)\bar{u}'' + \beta\bar{u} = 0 \quad \text{with } \bar{u}(0) = \bar{u}(1) = \beta\bar{u}(0) + \xi_1\bar{u}''(0) = \beta\bar{u}(1) + \xi_1\bar{u}''(1) = 0 \tag{88}$$

where the higher-order boundary conditions $\bar{\varepsilon}'(0) = \bar{\varepsilon}'(1) = 0$ has been used. The solution to this fourth-order differential equation may be expressed as:

$$\bar{u}(\bar{x}) = C_0\cos(\lambda_1\bar{x}) + C_1\sin(\lambda_1\bar{x}) + C_2\cosh(\lambda_2\bar{x}) + C_3\sinh(\lambda_2\bar{x})$$
$$\text{with } \lambda_1 = \sqrt{\frac{\sqrt{(1-\beta\mu)^2 + 4\beta\xi_1\mu} - 1 + \beta\mu}{2\xi_1\mu}} \text{ and } \lambda_2 = \sqrt{\frac{\sqrt{(1-\beta\mu)^2 + 4\beta\xi_1\mu} + 1 - \beta\mu}{2\xi_1\mu}} \tag{89}$$

Injecting the four boundary conditions give the vibration mode and the natural frequency of this nonlocal bar:

$$\bar{u}(\bar{x}) = C_1\sin(n\pi\bar{x}) \quad \text{and} \quad \beta_n = (n\pi)^2\frac{1+\xi_1\mu(n\pi)^2}{1+\mu(n\pi)^2} \leq (n\pi)^2 \tag{90}$$

which is also the value reported by Li et al. [46]. It is shown that the small length scale term μ also tends to soften the natural frequencies of this nonlocal bar, as $\xi_1 \in [0; 1]$ a conclusion in agreement with the results presented for nonlocal beams [28,35,45].

4.4. Eringen's differential model for infinite domain

If we now consider the Eringen's differential model – Eq. (8) – for an infinite medium, we assume a continuous kinematic field outside the domain, so that the natural vibration mode is valid not only for $x \in [0; L]$, but also outside the domain of definition:

$$u(x) = U\sin\left(n\pi\frac{x}{L}\right) \quad \text{for } x \in [-\infty; +\infty] \tag{91}$$

It can be checked that this trigonometric solution verifies the following integrodifferential equation:

$$\frac{d}{dx}EA\int_{-\infty}^{+\infty}\frac{1}{2l_c}e^{-\frac{|x-y|}{l_c}}\frac{\partial u}{\partial y}(y)\,dy + \omega^2\rho Au = 0 \tag{92}$$

which is equivalent to the differential Eringen model applied to the infinite periodic problem. Introducing Eq. (91) in Eq. (92) gives:

$$\frac{EA}{2l_c^2}U\frac{n\pi}{L}\left[-\int_{-\infty}^{+x}\mathrm{e}^{\frac{-x+y}{l_c}}\cos\left(n\pi\frac{y}{L}\right)\mathrm{d}y+\int_{x}^{+\infty}\mathrm{e}^{\frac{x-y}{l_c}}\cos\left(n\pi\frac{y}{L}\right)\mathrm{d}y\right]+\omega^2\rho AU\sin\left(n\pi\frac{x}{L}\right)=0 \tag{93}$$

It can be shown by integration by part that:

$$\left(1+\frac{L^2}{(n\pi)^2l_c^2}\right)\int_{-\infty}^{+x}\mathrm{e}^{\frac{-x+y}{l_c}}\cos\left(n\pi\frac{y}{L}\right)\mathrm{d}y=\frac{L}{n\pi}\sin\left(n\pi\frac{x}{L}\right)+\frac{L^2}{(n\pi)^2l_c}\cos\left(n\pi\frac{x}{L}\right)\quad\text{and}$$
$$\left(1+\frac{L^2}{(n\pi)^2l_c^2}\right)\int_{x}^{+\infty}\mathrm{e}^{\frac{x-y}{l_c}}\cos\left(n\pi\frac{y}{L}\right)\mathrm{d}y=-\frac{L}{n\pi}\sin\left(n\pi\frac{x}{L}\right)+\frac{L^2}{(n\pi)^2l_c}\cos\left(n\pi\frac{x}{L}\right) \tag{94}$$

and then Eq. (94) shows that $\beta_n=\frac{(n\pi)^2}{1+\mu(n\pi)^2}$ which is the value reported for Eringen's differential model.

In other words, for this dynamic problem, Eringen's differential model is equivalent to an integral approach with a finite kernel based on Eq. (79) for the finite bar, or an integral approach with an infinite kernel given by Eq. (92) using a continualization of the kinematic field outside the finite domain of the bar. It is also possible, using the correspondence principle between the strain-based and the peridynamic nonlocal models, to derive the same solution valid for the peridynamic bar for the extended infinite periodic bar with the exponential kernel given by Eq. (26).

4.5. Peridynamic model – kernel with discrete distance

Finally, the lattice-based peridynamic model based on the discrete kernel of Eq. (23) can be formulated by the functional equation:

$$EA\frac{u(x+a)-2u(x)+u(x-a)}{a^2}+\omega^2\rho Au(x)=0\quad\text{with } u(0)=u(L)=0 \tag{95}$$

and the continuation rule – Eq. (33) – is followed.

Then, the problem can be converted, for some solution points of the problem, into a second-order linear difference eigenvalue problem, expressed by:

$$EA\frac{u_{i+1}-2u_i+u_{i-1}}{a^2}+\omega^2\rho Au_i=0\quad\text{with } u_0=u_p=0 \tag{96}$$

where the integer parameter p is defined such as $pa=L$. This problem is analogous to the vibration problem of a string with concentrated masses, as already solved by Lagrange [50,51] for the string of the axial lattice. The vibration solution can be also expressed in sinusoidal form (see also [24]), analogous to Eq. (91), but now presented in its discrete version, $u(x)=U\sin(n\pi\frac{i}{p})$ for $i\in\{0;1;2\ldots p-1;p\}$. More generally, it can be shown that Eq. (91) is the continuous solution to the functional difference equation. Note, however, that Eq. (91) is a continuous solution, instead of the discrete solution induced by the difference eigenvalue problem. The Lagrange solution is obtained from the resolution of the linear second-order difference equation:

$$\beta_n=4p^2\sin^2\left(\frac{n\pi}{2p}\right)\quad\text{for } n\in\{1;2\ldots p-1;p\} \tag{97}$$

which also leads to the softening effect, due to the approximated formulae $\beta_n=\frac{(n\pi)^2}{1+(n\pi)^2\frac{a^2}{12L^2}}+\ldots$. It is shown that the small length scale term $\mu=a^2/(12L^2)$ also tends to soften the natural frequencies of this nonlocal bar, a conclusion in agreement with what we previously observed for strain-based or relative displacement-based nonlocal models. We then obtained an approximated formula for the frequency of this fixed–fixed peridynamic bar. One also recognized in the approximated formulae the natural frequency equation of a nonlocal bar based on Eringen's differential model, as shown by Eq. (85).

The present solution is also valid for strain-based nonlocal bar based on the kernel exposed in Eq. (6). In a certain sense, Lagrange [50,51] indirectly obtained the first analytical solution to a peridynamic problem, by solving a lattice problem based on a difference equation, whose continualization constitutes a particular strain-based or relative displacement-based (or peridynamic) nonlocal problem.

We may also comment that the sinusoidal function has been found as a common solution to various integral strain-based or peridynamic nonlocal models. This property, however, would not have been verified for the vibration of a peridynamic bar with the model considered by Nishawala and Ostoja-Starzewski [18], who used an alternative kernel.

5. Conclusions

Strain-based or relative displacement-based (also referred to as peridynamic) integral nonlocal models need to account for the finiteness of the structural element for engineering applications. This paper aims to classify different nonlocal models, with respect to their mathematical properties and their ability to preserve some fundamental features such as the invariance of the nonlocal operator for uniform field. It is shown in this paper that such a fundamental requirement is preserved for:

- an unmodified infinite kernel for finite domain by extending the domain outside its boundary. This method is used, for instance, for solving the finite difference equation using the so-called fictitious point method. As already outlined by Maugin for instance ([52]; see also [53]), lattice difference equations may be related to the finite difference formulation of the partial differential equations associated with the local continuum mechanics problem. In that spirit, it is not surprising that integral nonlocal models that may capture the small-scale effects in lattice mechanics have similar features in terms of mechanical resolution. Strain-based or relative displacement-based nonlocal models are similar from this point of view;
- a modified finite kernel valid inside the domain by combining a local measure with a nonlocal one. The nonlocal measure should be compatible with the natural and essential boundary conditions of the problem. It has also to respect the invariance of the homogeneous strain state for uniform stress state.

Analytical solutions valid for finite bars have been derived for nonlocal integral models including strain-based and peridynamic models, both in static and in dynamic conditions. It has been shown for both families of integral models (peridynamic or strain-based nonlocal integral models) that softening occurs in presence of non-homogeneous strain states, with relevant nonlocal strain measures. These analytical solutions can be used as benchmark solutions in the computational validation of nonlocal structural problems. The conclusions of this paper, especially the ones associated with the softening effect induced by the nonlocal scale effects are valid for the simple one-dimensional cases considered in this paper, with the considered kernels, and should be investigated more generally for two-dimensional and three-dimensional nonlocal media.

Appendix A

Starting from the introduction of the integral model in the finite domain, it is not difficult to show that:

$$N(x) = EA\int_0^L \frac{1}{2l_c}e^{-\frac{|x-y|}{l_c}}\varepsilon(y)\,dy = EA\left[\int_0^x \frac{1}{2l_c}e^{-\frac{x-y}{l_c}}\varepsilon(y)\,dy + \int_x^L \frac{1}{2l_c}e^{\frac{x-y}{l_c}}\varepsilon(y)\,dy\right] \tag{98}$$

By derivation and using Leibniz's rule for differentiation, the derivative of the normal force is equal to:

$$N'(x) = \frac{EA}{l_c}\left[-\int_0^x \frac{1}{2l_c}e^{-\frac{x-y}{l_c}}\varepsilon(y)\,dy + \int_x^L \frac{1}{2l_c}e^{\frac{x-y}{l_c}}\varepsilon(y)\,dy\right] \tag{99}$$

By combining both equation (98) and equation (99), the integral definition of the normal force in the finite domain implies a constraint on both the normal force and its derivative at the boundaries of the domain (as also shown by Benvenuti and Simone [21]):

$$N'(0) - \frac{1}{l_c}N(0) = 0 \quad \text{and} \quad N'(L) + \frac{1}{l_c}N(L) = 0 \tag{100}$$

which may violate the natural boundary conditions of the mechanical problem to be studied. Furthermore, the uniform strain field $\varepsilon(x) = \varepsilon_0$ leads to the non-uniform normal force field:

$$N(x) = EA\varepsilon_0\int_0^L \frac{1}{2l_c}e^{-\frac{|x-y|}{l_c}}\,dy = EA\left[1 - \frac{1}{2}\left(e^{-\frac{x}{l_c}} + e^{\frac{x-L}{l_c}}\right)\right] \tag{101}$$

Lignola et al. [54] recently suggest adding a stress-based boundary term in the integral definition of the normal force. This additional boundary term could be chosen as:

$$N(x) = EA\int_0^L \frac{1}{2l_c}e^{-\frac{|x-y|}{l_c}}\varepsilon(y)\,dy + g\big(N(x), x, l_c\big) \quad \text{with } g\big(N(x), x, l_c\big) = \frac{1}{2}\left[N(0)e^{-\frac{x}{l_c}} + N(L)e^{\frac{x-L}{l_c}}\right] \tag{102}$$

One easily verifies that the additional term automatically fulfils the following second-order differential equation:

$$g(x) - l_c^2 g''(x) = 0 \tag{103}$$

so that both nonlocal elastic laws equation (98) and equation (102) verify Eringen's differential law:

$$N(x) - l_c^2 N''(x) = EA\varepsilon(x) \tag{104}$$

The advantage of the nonlocal law of equation (102) is that the uniform strain field $\varepsilon(x) = \varepsilon_0$ is associated with a uniform normal force field $N(x) = EA\varepsilon_0$, as shown by the normal force equation valid for the homogeneous strain field:

$$N(x) - \frac{1}{2}\left[N(0)\mathrm{e}^{-\frac{x}{l_c}} + N(L)\mathrm{e}^{\frac{x-L}{l_c}}\right] = EA\varepsilon_0\left[1 - \frac{1}{2}\left(\mathrm{e}^{-\frac{x}{l_c}} + \mathrm{e}^{\frac{x-L}{l_c}}\right)\right] \tag{105}$$

However, this integral definition of the normal force is also associated with some constraints on the normal force. Indeed, it can be easily shown that:

$$g'(0) - \frac{1}{l_c}g(0) = \frac{-N(0)}{l_c} \quad \text{and} \quad g'(L) + \frac{1}{l_c}g(L) = \frac{N(L)}{l_c} \tag{106}$$

Applying the combination of the normal force derivative and the normal force to Eq. (5) gives by superposition:

$$N'(0) - \frac{1}{l_c}N(0) = -\frac{1}{l_c}N(0) \quad \text{and} \quad N'(L) + \frac{1}{l_c}N(L) = \frac{1}{l_c}N(L) \tag{107}$$

The nonlocal integral definition of the normal force in equation (102) is then equivalent to the normal force boundary conditions:

$$N'(0) = 0 \quad \text{and} \quad N'(L) = 0 \tag{108}$$

which may also violate the natural boundary conditions of the mechanical problem to be studied. Even if this measure leaves the uniform strain field invariant, it may be as well in contradiction with the imposed natural boundary conditions of the problem. Note that equation (102) can be equivalently written as:

$$N = EA\int_0^L F(x,y)\varepsilon(y)\,\mathrm{d}y \quad \text{with} \quad \begin{cases} F(x,y) = \frac{1}{l_c}\frac{\cosh\left(\frac{L-y}{l_c}\right)}{\sinh\left(\frac{L}{l_c}\right)}\cosh\left(\frac{x}{l_c}\right) & \text{if } x \le y \\ F(x,y) = \frac{1}{l_c}\frac{\cosh\left(\frac{L-x}{l_c}\right)}{\sinh\left(\frac{L}{l_c}\right)}\cosh\left(\frac{y}{l_c}\right) & \text{if } x \ge y \end{cases} \tag{109}$$

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