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Disk in a groove with friction: An analysis of static equilibrium and indeterminacy

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ABSTRACT

This note studies the statics of a rigid disk placed in a V-shaped groove with frictional walls and subjected to gravity and a torque. The two-dimensional equilibrium problem is formulated in terms of the angles that contact forces form with the normal to the walls. This approach leads to a single trigonometric equation in two variables whose domain is determined by Coulomb's law of friction. The properties of solutions (existence, uniqueness, or indeterminacy) as functions of groove angle, friction coefficient and applied torque are derived by a simple geometric representation. The results modify some of the conclusions by other authors on the same problem.

1. Introduction

The problem of the equilibrium of a rigid disk wedged in a V-shaped groove with frictional walls is of interest in the study of granular packing, as it represents an elementary model of particle contact. In [1], the disk was considered to be held by gravity in an inclined groove; in other studies [2,3], the groove was vertical, but the disk was subjected to more general forces. Both configurations lead to an indeterminate problem, because four unknowns (the components of contact forces) are involved, but only three equations are available.

For the frictional vertical groove, McNamara et al. [2] gave a detailed analytical discussion of the disk equilibrium under the action of gravity and a torque. The presence of indeterminacy was related to the applied forces, the groove angle, and the friction coefficient; some of the results of [2] were also summarized by Stamm [4], pp. 12–14.

The aim of this note is to propose a different and possibly simpler treatment of the configuration discussed by McNamara et al. [2], by using the xy-components and angles of contact forces rather than their normal/tangential projections. This choice leads to a trigonometric equation in two variables, whose solutions are discussed with the aid of a geometrical scheme.

2. Equilibrium equations

A homogeneous rigid disk of radius r, center O and weight $m\mathbf{g}$ lies in a groove of aperture angle 2θ; the contacts A and B between the disk and the groove walls have friction, with static coefficient $\mu < 1$. In addition to gravity, the disk is acted upon by a torque $M > 0$ (torque and angles are taken positive in the counterclockwise direction). Disk equilibrium

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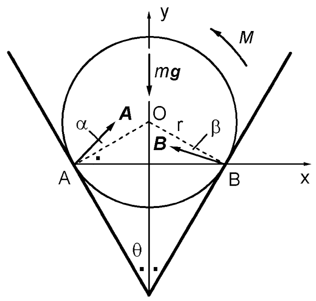

Fig. 1. Scheme of disk in a groove with frictional walls.

requires that the resultant force and total moment, for instance about O, be zero. In the xy reference system of Fig. 1, the components of the contact forces **A** and **B** must then obey the linear equations

$$A_x + B_x = 0 \tag{1a}$$

$$A_y + B_y - mg = 0 \tag{1b}$$

$$(-A_y + B_y) r \cos\theta + M = 0 \tag{1c}$$

Equations (1b) and (1c) determine the values of A_y and B_y; by introducing the non-dimensional, normalized torque

$$\tau = \frac{M}{mgr\cos\theta} \tag{2}$$

which is positive, since in a groove $0 < \theta < \pi/2$, the solution for A_y and B_y can be written in the form

$$A_y = \frac{1}{2} mg(1+\tau) \tag{3a}$$

$$B_y = \frac{1}{2} mg(1-\tau) \tag{3b}$$

The components A_x and B_x appear only in Equation (1a), which yields $B_x = -A_x$, but the value of A_x cannot be established, so that the problem is in general indeterminate. By converting the components of the four-dimensional contact forces used in [2] into their xy equivalents, it can be proved that the undetermined coefficient a_0 introduced there is the same as $2A_x$.

Although the contact forces **A** and **B** cannot be uniquely specified, they must still comply with Coulomb's law of friction. This law sets a maximum magnitude γ to the angles of inclination α and β of these forces to the respective wall normal. The value $\mu = \tan\gamma$ is the static friction coefficient: here the assumption $\mu < 1$ entails $\gamma < \pi/4$. As suggested by Fig. 1, here both α and β must be positive or zero. In fact, for negative α (or β) the frictional force **A** (or **B**) would produce a moment about O with the same sense as M, in contradiction with friction's oppositional nature (see, e.g., rule #1 in Goodman and Warner [5], page 286). Hence Coulomb's law yields the inequalities:

$$0 \le \tan\alpha \le \mu, \qquad 0 \le \tan\beta \le \mu \tag{4}$$

We now note that B_x is certainly ≤ 0, since $0 \le \beta \le \gamma$ (Fig. 1); from equation (1a), we have $A_x = -B_x = |B_x|$, hence $A_x \ge 0$. The relation between α, β and the components of **A**, **B** then takes the form (see Fig. 1)

$$\tan(\theta + \alpha) = A_y / A_x \tag{5a}$$

$$\tan(\theta - \beta) = B_y / |B_x| \tag{5b}$$

under the assumption that $A_x \ne 0$ (the case $A_x = 0$ is considered in section 4.1). By taking the ratio of equation (5b) to (5a) and recalling that $|B_x| = A_x$, we can eliminate the unknown component A_x. Then we replace in this ratio the respective expressions for A_y and B_y given in (3a) and (3b), and finally get

$$\frac{\tan(\theta - \beta)}{\tan(\theta + \alpha)} = k \tag{6}$$

where

$$k = \frac{1-\tau}{1+\tau} \tag{7}$$

In equation (6) the contact forces at A and B appear through the angles α and β, and the forces imposed on the disk are condensed into the parameter k of (7), with τ given by (2). The absolute value of k is always less than 1; k is near unity

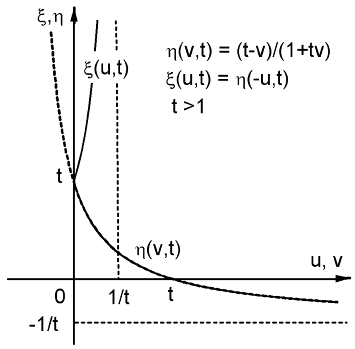

Fig. 2. Plot of the functions $\xi(u,t)$ and $\eta(v,t)$ for given $t>1$.

for $\tau << 1$, and approaches -1 for large τ. The correspondence τ-k of (7) is one to one, and is inverted by exchanging τ and k. The single equation (6) for α and β, associated with the limitations (4) for $\tan\alpha$ and $\tan\beta$, describes the equilibrium conditions of the disk.

3. Solution method

To take into account the restrictions (4) on the variables α and β of equation (6), we first use the angle addition formula for the tangent functions appearing there. By setting $t = \tan\theta$ ($t > 0$ since $\theta < \pi/2$), $u = \tan\alpha$, $v = \tan\beta$, we obtain

$$\tan(\theta+\alpha) = \frac{t+u}{1-tu} = \xi(u,t) \tag{8a}$$

$$\tan(\theta-\beta) = \frac{t-v}{1+tv} = \eta(v,t) \tag{8b}$$

having introduced the functions $\xi(u,t)$ and $\eta(v,t)$. Thus equation (6), with k given by (7), and the conditions (4) lead to this problem in the unknowns u and v

$$\frac{\eta(v,t)}{\xi(u,t)} = k \tag{9a}$$

$$0 \le u \le \mu, \quad 0 \le v \le \mu \tag{9b}$$

For a given t, a couple of values $(u,\ v)$ that satisfy equation (9a) and the inequalities (9b) can be seen as coordinates of a "solution point" in the uv-plane. This geometric view suggests a method to find the related solution set; preliminarily, the graphs of $\xi(u,t)$ and $\eta(v,t)$ are examined.

3.1. The functions $\xi(u,t)$ and $\eta(v,t)$

Fig. 2 displays the graph of the functions $\xi(u,t)$ and $\eta(v,t)$ as defined in equations (8a) and (8b), for a fixed $t > 1$, in the relevant ranges of u, v; the plot for $t < 1$ is analogous, but with $t < 1/t$. Both ξ and η are strictly monotone and hence invertible functions u and v, respectively; their definition shows that $\xi(u,t) = \eta(-u,t)$. Equation (5a) defines $\tan(\theta+\alpha)$ as the ratio of two positive quantities, hence $\xi(u,t) = \tan(\theta+\alpha) > 0$; therefore, only values of u smaller than $1/t$ are allowed [see (8a) and Fig. 2]. For a given t, the couple of invertible functions $\xi(u,t)$, $\eta(v,t)$ establishes a one-to-one transformation of points in the uv-plane into points of the $\xi\eta$-plane. Thus, the solution set S to the problem defined by equations (9a) and (9b) is mapped bijectively to a set Σ in the $\xi\eta$-plane. The identification of Σ then specifies S, i.e. it establishes the existence, non-existence, and indeterminacy of solutions for the original problem; here Σ is identified by simple analytic geometry.

3.2. Geometric interpretation

The mapping defined by the functions $\xi(u,t)$, $\eta(v,t)$ transforms lines parallel to the axes in the uv-plane into lines parallel to the $\xi\eta$-axes, since constant u (or v) implies constant ξ (or η). Thus, the square of side μ in the uv-plane, as defined by the inequalities (9b), will in general go into a rectangle R in the $\xi\eta$-plane (Fig. 3). The square's vertices at $(0,0)$ and (μ,μ) are mapped to those of R at $\Omega \equiv (t,t)$ and $Q \equiv [\xi(\mu,t), \eta(\mu,t)]$, which are sufficient to specify R. The points (if any) of R that obey the condition (9a) $\eta/\xi = k$ (or, equivalently, that lie on the straight line $\eta = k\xi$) constitute the set Σ mentioned before and correspond to solution points in the uv-plane.

Fig. 3 suggests that, for $\eta(\mu,t) \ge 0$, the existence of intersections between the line $\eta = k\xi$ and R can be established by comparing k with the slope k_2 of the straight line from the origin to Q:

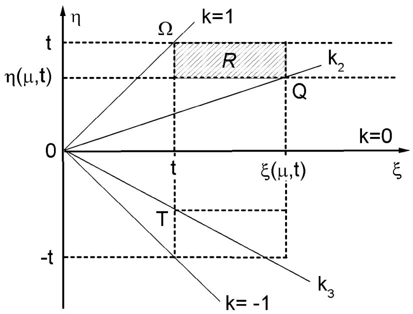

Fig. 3. Diagram used to characterize the solutions to the equilibrium problem. The hatched rectangle R is the $\xi\eta$-transform for $\eta(\mu,t)>0$ of the square $0 \le u \le \mu$, $0 \le v \le \mu$ in the uv-plane. When $\eta(\mu,t)<0$, R extends down to T. The straight lines from the origin are used to specify the solution set.

$$k_2 = \eta(\mu,t)/\xi(\mu,t) \tag{10}$$

If $k > k_2$, the line $\eta = k\xi$ has a segment in common with R and the problem has an infinite number of solutions, i.e. is indeterminate. For $k = k_2$ the intersection occurs only at Q, so that there is a unique solution; for $k < k_2$, there are no intersections and hence no solutions. The explicit form of k_2 can be given by means of equations (8a) and (8b), with $u = v = \mu$, but it is more interesting for later discussions to write out the expression for the corresponding $\tau = \tau_2$, as obtained by inverting the relation (7). With some algebra, we get

$$\tau_2 = \frac{1-k_2}{1+k_2} = \frac{2\mu}{1+\mu^2} \times \frac{1+t^2}{2t} \tag{11}$$

In this expression, we can recognize the sine double-angle formula as a function of the tangent, with $t = \tan\theta$ and $\mu = \tan\gamma$, hence τ_2 may be given the alternative form

$$\tau_2 = \sin(2\gamma)/\sin(2\theta) \tag{12}$$

For $\eta(\mu,t) < 0$, R extends below the ξ axis and the appropriate comparison is between k and the slope k_3 of the straight line from the origin to T (see Fig. 3)

$$k_3 = \eta(\mu,t)/t \tag{13}$$

4. Equilibrium conditions

Fig. 2 indicates that t and $1/t$ represent special values of the variables u and v, which are restricted to the interval $(0,\mu)$ (see equation (9b)). Therefore, the comparison of relative magnitudes of t, $1/t$ and μ is relevant, and leads to three possible locations of μ with respect to t and $1/t$. Equivalently, t can be compared with μ and $1/\mu$, also for easier reference to the results of McNamara et al. [2]. We recall that $t = \tan\theta$ gives a measure of the groove's aperture, which is referred to in the heading of the next subsections.

4.1. Large aperture: $t > 1/\mu$

The initial assumption $\mu < 1$ yields in this case $t > 1/\mu > 1$ and the plot of Fig. 2 applies. Moreover, the condition $t > 1/\mu$ is equivalent to $\mu > 1/t$, but u must be restricted to the interval $(0, 1/t)$ (see section 3.1). For $u \to 1/t$, $\xi(u,t)$ approaches infinity, and we can write $Q \equiv [\infty, \eta(\mu,t)]$, with $\eta(\mu,t) > 0$, since $\mu < t$ (see Fig. 2). In this case, R becomes a stripe starting from Ω and extending to $+\infty$, with thickness $t - \eta(\mu,t)$. The definition (10) now yields $k_2 = 0$, and the discussion of section 3.2 shows that, for $k > 0$ ($\tau < 1$), the problem is indeterminate; for $k = 0$ ($\tau = 1$), it has just one solution and for $k < 0$ ($\tau > 1$) it does not admit solutions.

The unique solution of the equilibrium problem for $\tau = \tau_1 = 1$ yields a special set of contact forces. Equations (3a) and (3b) give $A_y = mg$ and $B_y = 0$: we show that $B_x = 0$ as well. In fact, if $B_x \neq 0$, $\mathbf{B}$ would be parallel to the x-axis, hence $\beta = \theta$ and $v = t$; here $t > 1/\mu > \mu$, from which $v > \mu$, but this result is in contrast with the condition (9b) $v \le \mu$. Therefore, $B_x = 0$ and by (1a) we get $A_x = 0$, with the conclusion that $\mathbf{A} = -m\mathbf{g}$ and $\mathbf{B} = 0$: in this case, equilibrium is maintained by the contact force at A, which balances both the disk weight and applied torque (see equation (14) of [2]).

4.2. Intermediate aperture: $\mu \le t \le 1/\mu$

In this case, R is the hatched rectangle of Fig. 3, with $\eta(\mu,t) > 0$ (for $t \neq \mu$) and finite $\xi(\mu,t)$ (for $t \neq 1/\mu$) (see equations (8a) and (8b) with $u = v = \mu$). As discussed in section 3.2, for $k > k_2$ ($\tau < \tau_2$) the problem is indeterminate, for $k = k_2$ it has a unique solution and for $k < k_2$ ($\tau > \tau_2$) is without solutions. When $t = \mu$ or $t = 1/\mu$ we have $\eta(\mu,t) = 0$ or $\xi(\mu,t) = \infty$, respectively; in either case $k_2 = 0$, $\tau_2 = \tau_1 = 1$ (see equation (11)) and the discussion of section 4.1 applies.

4.3. Small aperture: $t < \mu$

For $t < \mu$, we have $\eta(\mu, t) < 0$, finite $\xi(\mu, t)$, and R extends below the ξ axis; the solutions behave as in the previous section, but with k_3 as the discriminating parameter. It is interesting to examine the case $k_3 \le -1$, when k can be arbitrarily close to -1 and solutions for arbitrarily large τ exist (see comments after equation (7)). Using equations (8b) and (13), we obtain

$$k_3 = \frac{t-\mu}{(1+t\mu)\,t} \le -1 \tag{14}$$

which yields

$$2t/(1-t^2) \le \mu \tag{15}$$

In terms of θ and the friction angle γ, this inequality becomes $\tan(2\theta) \le \tan\gamma$, which leads to the simple relation

$$2\theta \le \gamma \tag{16}$$

Therefore, equilibrium for arbitrary τ is possible only if the aperture angle 2θ of the groove does not exceed the Coulomb friction angle γ of its walls. Since $\gamma < \pi/4$ ($\mu = \tan\gamma < 1$), we see that in any case 2θ must be smaller than $\pi/4$. The condition (16) of unrestricted disk equilibrium is also found to hold when only one wall the groove has friction [6]. In that configuration, there are only three unknown force components and the equilibrium problem is determinate.

5. Discussion and conclusions

Most of the results obtained here agree with those of McNamara et al. [2], as can be recognized by converting the aperture half-angle θ into their slope angle $\phi = \pi/2 - \theta$, and removing the torque normalization of equation (2) by multiplying τ by the factor $mgr\cos\theta = mgr\sin\phi$. The non-normalized torque is denoted here by $\bar{\tau}$. Thus, we see that the value $\tau_1 = 1$ becomes $\bar{\tau}_1 = mgr\sin\phi$, as given in equation (13) of [2], and the expression for τ_2 of (11) turns into $\bar{\tau}_2 = mgr\mu\sec\phi/(1+\mu^2)$ (see equation (16) of [2]). The results of sections 4.1 and 4.2 concerning indeterminacy, uniqueness of the solution, or its nonexistence are the same as those of [2].

However, for small groove aperture ($t < \mu$, and hence $\theta < \gamma$) McNamara et al. [2] found disk equilibrium for any torque, whereas here the condition of equilibrium for any τ is the stronger inequality $2\theta \le \gamma$ of equation (16). A possible explanation of this disagreement is given below.

According to the authors of reference [2], when $t < \mu$, the equilibrium is determined by two inequalities, which are given in their equations (9a) and (9c), and with the present symbols read

$$2A_x \ge mg(1+\tau)/\xi(\mu, t) \tag{17a}$$

$$2A_x \ge mg(1-\tau)/\eta(\mu, t) \tag{17b}$$

These inequalities set a lower bound for $a_0 = 2A_x$ and can therefore be satisfied by making $2A_x$ sufficiently large; this argument led to the conclusion that, for $t < \mu$, there is equilibrium for an arbitrary value of the torque τ.

However, $2A_x$ is also constrained by an *upper* bound, which must be satisfied for any τ as well. In fact, from equations (3a), (5a) and (8a), we get $2A_x = mg(1+\tau)/\xi(u, t)$; Fig. 2 shows that $\xi(u, t) \ge t$, hence $2A_x \le mg(1+\tau)/t$. By combining this result with (17b), we get the condition $(1-\tau)/\eta(\mu, t) \le (1+\tau)/t$, or

$$\eta(\mu, t)/t \le \frac{1-\tau}{1+\tau} \tag{18}$$

because $\eta(\mu, t)$ is negative for $t < \mu$. The inequality (18) will hold for any τ if $\eta(\mu, t)/t \le -1$, since $(1-\tau)/(1+\tau)$ is always larger than -1. The condition $\eta(\mu, t)/t \le -1$ is seen to coincide with equation (14) if $\eta(\mu, t)$ is written explicitly through (8b), with $v = \mu$. In summary, when $t < \mu$, the equilibrium condition of [2] departs from that of section 4.3, because in that study the proper upper bound to $a_0 = 2A_x$ was not added to the lower bounds of (17a) and (17b).

Moreover, according to McNamara et al. [2], if $t < \mu$ (i.e. $\theta < \gamma$) both static and moving solutions coexist for $\tau > \tau_2$, i.e. it cannot be decided whether the disk is in equilibrium or not. This ambiguous outcome also seems to arise from the conclusion that for $\theta < \gamma$ there is equilibrium for any torque. But for $\gamma/2 < \theta < \gamma$ and sufficiently large τ, the disk cannot be in equilibrium, i.e. there is no static solution. Equilibrium exists for any value of τ only for $\theta \le \gamma/2$ (see equation (16)); in either range of θ values, the disk state is definite.

This study has provided an analysis of the equilibrium problem for a disk in a groove with frictional walls within the framework of the rigid-body model and Coulomb's law of friction. The use of angles as variables provided a simplification of the discussion and led to a geometric method to visualize the existence, uniqueness, or indeterminacy of solutions. By this procedure, the reason for some conflicting results obtained by other authors was explained, and a new condition of unrestricted equilibrium was given.

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