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# Blow-up of solutions to a quasilinear wave equation for high initial energy

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ABSTRACT

This paper deals with blow-up solutions to a nonlinear hyperbolic equation with variable exponent of nonlinearities. By constructing a new control function and using energy inequalities, the authors obtain the lower bound estimate of the $L^2$ norm of the solution. Furthermore, the concavity arguments are used to prove the nonexistence of solutions; at the same time, an estimate of the upper bound of blow-up time is also obtained. This result extends and improves those of [1,2].



## 1. Introduction

In this paper, we consider the following quasilinear hyperbolic problem

$$\begin{cases} u_{tt} - \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) - \Delta u_t = |u|^{q(x)-2}u, & (x,t) \in \Omega \times (0,T) := Q_T \\ u(x,t) = 0, & (x,t) = \partial\Omega \times (0,T) := \Gamma_T \\ u(x,0) = u_0(x), \;\; u_t(x,0) = u_1(x), & x \in \Omega \end{cases} \tag{1.1}$$

where $\Omega \subset R^N (N \geqslant 1)$ is a bounded domain, $\partial\Omega$ is Lipschitz continuous. It will be assumed throughout the paper that the exponents $p(x), q(x)$ satisfy the following conditions.

$$2 \leqslant p^- \leqslant p(x) \leqslant p^+ < \infty,\; 1 < q^- \leqslant q(x) \leqslant q^+ < \infty$$

Problem (1.1) models many physical problems such as viscoelastic fluids, electro-rheological fluids, processes of filtration through a porous media, fluids with temperature-dependent viscosity, etc. The interested readers may refer to [3–5] and the references therein. When $p, q$ are fixed constants, many authors discussed the existence of solutions, finite-time blow-up of solutions for low initial energy and arbitrarily high initial energy as well as some estimates of a lower bound for blow-up times – the interested readers may refer to [6–14]. When $p, q$ are continuous functions, S.N. Antontsev in [1,15] discussed the blowing-up properties of solutions to the initial and homogeneous boundary value problem of quasilinear wave equations involving the $p(x,t)$-Laplacian operator and a negative initial energy. Guo–Gao of [2] proved that the solution blew up in finite time for positive initial energy. Later, Guo, in [16], applied the interpolation inequality and energy inequalities to obtain an estimate of the lower bound for the blow-up time when the source is super-linear. In addition, Messaoudi and

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Talahmeh in [17,18] discussed blow-up properties of solutions to the nonlinear wave equations with weak damping terms and a $p(.)$-Laplacian operator. However, there are few works that discuss blow-up properties of solutions for high initial energy. In fact, dealing with such problems, one has to face some difficulties:

• how can one give an estimate of the lower bound of the norm $\|\nabla u\|_{L^{p(.)}(\Omega)}$? In fact, due to the presence of strong damping term $\Delta u_t$, the technique used in [17] is not applicable, so, we have to find some new methods or techniques;

• how can one establish the quantitative relationship between the term $\int_\Omega |u|^{q(.)}\mathrm{d}x$ and the initial energy? Since the initial energy is arbitrary, we can not apply the technique used in [2] to give the quantitative relation between $\int_\Omega |u|^{q(.)}\mathrm{d}x$ and $E(0)$.

In this paper, we construct a new control function and apply energy estimate inequalities to bypass the first difficulty above. Furthermore, by modifying the functional constructed in [2] and utilizing the quantitative relationship between the term $\int_\Omega |u|^2\mathrm{d}x$ and the initial energy, we prove that the solution blows up in finite time for arbitrary positive initial energy. In particular, it is worth pointing out that our results extend and improve those of [1,2]. For the existence of solutions, we may refer to [1,2,15].

Define the energy functional as the following:

$$E(t)=\frac{1}{2}\int_\Omega |u_t|^2\mathrm{d}x+\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\mathrm{d}x-\int_\Omega \frac{1}{q(x)}|u|^{q(x)}\mathrm{d}x$$

First of all, due to $p_t=q_t=0$ and $a(x,t)=b(x,t)=1$, we follow the line of the proof of Lemma 2.1 in [1] or Lemma 1.1 of [2] to obtain the energy functional $E(t)$, which satisfies the following identity.

**Lemma 1.1.** *Suppose that $u\in L^{q(x)}(Q_T)\cap L^\infty(0,T;W_0^{1,p(x)}(\Omega))$, $u_t\in L^2(0,T;H^1(\Omega))$ is a solution to Problem (1.1), then $E(t)$ satisfies the following identity:*

$$E(t)+\int_0^t\int_\Omega |\nabla u_s|^2\mathrm{d}x\,\mathrm{d}s=E(0),\ t\geqslant 0 \tag{1.2}$$

*if the following conditions are fulfilled*

$$\begin{cases} 2\leqslant p^-\leqslant p(x)\leqslant p^+<\infty,\ 1<q^-\leqslant q(x)\leqslant q^+<p^{-*} \\ p^{-*}=\begin{cases}\frac{Np^-}{N-p^-}, & \text{if } 1<p^-<N\\ \infty, & \text{if } p^-\geqslant N\end{cases} \\ u_0\in W_0^{1,p(x)}(\Omega)\cap L^{q(x)}(\Omega),\ u_1\in L^2(\Omega)\end{cases} \tag{1.3}$$

Let $\lambda_1$ be the first eigenvalue of the following problem

$$\begin{cases}-\Delta\psi=\lambda\psi, & x\in\Omega\\ \psi=0, & x\in\partial\Omega\end{cases}$$

and set $B_1=\min\{\lambda_1,\frac{p^+(q^-+2)}{2(q^--p^+)}\}$.

**Lemma 1.2.** *If all the conditions of Lemma 1.1 remain true and $q^->p^+$, then the solution to Problem (1.1) satisfies the following inequality*

$$\int_\Omega uu_t\mathrm{d}x\geqslant \mathrm{e}^{M_0t}\left[\int_\Omega u_0u_1\mathrm{d}x-\frac{q^-}{M_0}E(0)-\frac{|\Omega|}{M_0}\right]+\frac{q^-}{M_0}E(t)+\frac{|\Omega|}{M_0},\ \ t>0 \tag{1.4}$$

*where $M_0=\frac{4(q^--p^+)q^-B_1}{p^+(B_1+4q^-)}>0$.*

**Proof.** Define $F(t)=\int_\Omega uu_t\mathrm{d}x$. Then the definition of the weak solution shows that

$$
\begin{aligned}
F'(t) &= \int_\Omega |u_t|^2\mathrm{d}x + \int_\Omega uu_{tt}\mathrm{d}x \\
&= \int_\Omega |u_t|^2\mathrm{d}x + \int_\Omega \Big[-|\nabla u|^{p(x)} - \nabla u\nabla u_t + |u|^{q(x)}\Big]\mathrm{d}x \\
&\geqslant \int_\Omega |u_t|^2\mathrm{d}x - \int_\Omega |\nabla u|^{p(x)}\mathrm{d}x - \int_\Omega \nabla u\nabla u_t\mathrm{d}x + q^-\Big[\int_\Omega \Big[\frac{|u_t|^2}{2}+\frac{|\nabla u|^{p(x)}}{p(x)}\Big]\mathrm{d}x - E(t)\Big] \\
&= \frac{q^-+2}{2}\int_\Omega |u_t|^2\mathrm{d}x + \int_\Omega \frac{q^- - p(x)}{p(x)}|\nabla u|^{p(x)}\mathrm{d}x - q^-E(t) - \int_\Omega \nabla u\nabla u_t\mathrm{d}x
\end{aligned} \tag{1.5}
$$

The Cauchy–Schwarz inequality implies that

$$
\Big|\int_\Omega \nabla u\nabla u_t\mathrm{d}x\Big| \leqslant \frac{(q^- - p^+)B_1}{p^+(B_1+4q^-)}\int_\Omega |\nabla u|^2\mathrm{d}x + \frac{(B_1+4q^-)p^+}{4(q^- - p^+)B_1}\int_\Omega |\nabla u_t|^2\mathrm{d}x \tag{1.6}
$$

Combining (1.5) and (1.6) with (1.2), we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Big[F(t) - \frac{(B_1+4q^-)p^+}{4(q^- - p^+)B_1}E(t)\Big] &= F'(t) - \frac{(B_1+4q^-)p^+}{4(q^- - p^+)B_1}E'(t) \\
&\geqslant \frac{q^-+2}{2}\int_\Omega |u_t|^2\mathrm{d}x + \frac{q^- - p^+}{p^+}\int_\Omega |\nabla u|^{p(x)}\mathrm{d}x - q^-E(t) - \frac{(q^- - p^+)B_1}{p^+(B_1+4q^-)}\int_\Omega |\nabla u|^2\mathrm{d}x
\end{aligned} \tag{1.7}
$$

Moreover, by virtue of $p^- \geqslant 2$ and the inequality $\int_\Omega |v|^2\mathrm{d}x \leqslant \frac{1}{\lambda_1}\int_\Omega |\nabla v|^2\mathrm{d}x$ *for* $v \in H_0^1(\Omega)$, it is easy to verify that

$$
\begin{aligned}
\int_\Omega |\nabla u|^{p(x)}\mathrm{d}x &= \int_{\{x\in\Omega:|\nabla u|\leqslant 1\}} |\nabla u|^{p(x)}\mathrm{d}x + \int_{\{x\in\Omega:|\nabla u|\geqslant 1\}} |\nabla u|^{p(x)}\mathrm{d}x \\
&\geqslant \int_{\{x\in\Omega:|\nabla u|\geqslant 1\}} |\nabla u|^2\mathrm{d}x \geqslant \int_\Omega |\nabla u|^2\mathrm{d}x - \int_{\{x\in\Omega:|\nabla u|\leqslant 1\}} |\nabla u|^2\mathrm{d}x \\
&\geqslant \int_\Omega |\nabla u|^2\mathrm{d}x - |\Omega| \geqslant \lambda_1\int_\Omega |u|^2\mathrm{d}x - |\Omega| \geqslant B_1\int_\Omega |u|^2\mathrm{d}x - |\Omega|
\end{aligned} \tag{1.8}
$$

So, we apply Inequalities (1.7) and (1.8) to obtain that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Big[F(t) - \frac{q^-}{M_0}E(t) - \frac{|\Omega|}{M_0}\Big] &\geqslant \frac{q^-+2}{2}\int_\Omega |u_t|^2\mathrm{d}x - |\Omega| - q^-E(t) + \frac{M_0}{B_1}\int_\Omega |\nabla u|^2\mathrm{d}x \\
&\geqslant \frac{q^-+2}{2}\int_\Omega |u_t|^2\mathrm{d}x - |\Omega| - q^-E(t) + M_0\int_\Omega |u|^2\mathrm{d}x \\
&\geqslant M_0\Big[\int_\Omega |u_t|^2\mathrm{d}x - \frac{q^-}{M_0}E(t) - \frac{|\Omega|}{M_0} + \int_\Omega |u|^2\mathrm{d}x\Big] \\
&\geqslant M_0\Big[F(t) - \frac{q^-}{M_0}E(t) - \frac{|\Omega|}{M_0}\Big]
\end{aligned} \tag{1.9}
$$

Inequality (1.9) indicates that

$$
F(t) \geqslant \mathrm{e}^{M_0t}\Big[F(0) - \frac{q^-}{M_0}E(0) - \frac{|\Omega|}{M_0}\Big] + \frac{q^-}{M_0}E(t) + \frac{|\Omega|}{M_0} \qquad \square
$$

Our main results are as follows.

**Theorem 1.1.** *Assume that all the conditions of Lemma 1.1 and the following conditions are satisfied,*

$$(H_1)\ q^- > p^+ > p^- \geqslant 2$$
$$(H_2) \int_\Omega u_0 u_1 \mathrm{d}x \geqslant \frac{p^+(B_1+4q^-)}{4(q^- - p^+)B_1}\left[E(0)+\frac{|\Omega|}{q^-}\right] > 0$$
$$(H_3) \min\left\{\left(B_1\int_\Omega |u_0|^2\mathrm{d}x\right)^{\frac{p^-}{2}}, \left(B_1\int_\Omega |u_0|^2\mathrm{d}x\right)^{\frac{p^+}{2}}\right\} > p^+(1+|\Omega|)^{\frac{p^+(2-3p^-)}{p^-}} E(0)$$

*then the solution to Problem (1.1) blows up in finite time.*

**Proof. Case 1.** For all $t \geqslant 0$, we first assume that $E(t) \geqslant 0$. We will divide the proof into three steps.

**Step 1.** According to $(H_2)$ and Inequality (1.4), we observe that the following facts remain true

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u|^2\mathrm{d}x = 2\int_\Omega u u_t \mathrm{d}x \geqslant 0,\ t \geqslant 0 \tag{1.10}$$

which implies that the term $\int_\Omega |u|^2\mathrm{d}x$ is nondecreasing with respect to the time variable.

**Step 2.** By Hölder inequality and Corollary 3.34 in [4], we have

$$\int_\Omega |\nabla u|^2\mathrm{d}x \leqslant (1+|\Omega|)^{3-\frac{2}{p^-}}\|\nabla u\|^2_{p(.)} \tag{1.11}$$

Further, using the inequality $\min\{\|u\|^{p^-}_{p(.)}, \|u\|^{p^+}_{p(.)}\} \leqslant \int_\Omega |u|^{p(.)}\mathrm{d}x \leqslant \max\{\|u\|^{p^-}_{p(.)}, \|u\|^{p^+}_{p(.)}\}$ and Inequality (1.11), we get

$$\begin{aligned}
\int_\Omega |\nabla u|^{p(.)}\mathrm{d}x &\geqslant \min\{\|\nabla u\|^{p^-}_{p(.)}, \|\nabla u\|^{p^+}_{p(.)}\} \\
&\geqslant \min\left\{(1+|\Omega|)^{2-3p^-}\|\nabla u\|_2^{p^-}, (1+|\Omega|)^{\frac{2p^+}{p^-}-3p^+}\|\nabla u\|_2^{p^+}\right\} \\
&\geqslant (1+|\Omega|)^{\frac{p^+(2-3p^-)}{p^-}} \min\left\{\|\nabla u\|_2^{p^-}, \|\nabla u\|_2^{p^+}\right\} \\
&\geqslant (1+|\Omega|)^{\frac{p^+(2-3p^-)}{p^-}} \min\left\{(\sqrt{B_1}\|u\|_2)^{p^-}, (\sqrt{B_1}\|u\|_2)^{p^+}\right\}
\end{aligned} \tag{1.12}$$

Obviously, applying $\|u(.,t)\|_2 \geqslant \|u_0(.)\|_2$ and (1.12), it is not difficult to prove

$$\int_\Omega |\nabla u|^{p(.)}\mathrm{d}x \geqslant (1+|\Omega|)^{\frac{p^+(2-3p^-)}{p^-}} \min\left\{(\sqrt{B_1}\|u_0\|_2)^{p^-}, (\sqrt{B_1}\|u_0\|_2)^{p^+}\right\} \tag{1.13}$$

**Step 3.** Define

$$L(t) = \frac{1}{2}\int_\Omega |u|^2\mathrm{d}x + \frac{1}{2}\int_0^t\int_\Omega |\nabla u|^2\mathrm{d}x\mathrm{d}\tau - \frac{t}{2}\|\nabla u_0\|_2^2 + \beta(t+t_0)^2$$

where $\beta > 0, t_0 > \max\left\{\frac{\|\nabla u_0\|_2^2}{\beta}, \frac{\|u_1 u_0\|_1}{2\beta}\right\}$ will be determined later. A direct computation shows that

$$L'(t) = \int_\Omega u u_t \mathrm{d}x + \int_0^t\int_\Omega \nabla u \nabla u_\tau \mathrm{d}x\mathrm{d}\tau + 2\beta(t+t_0)$$
$$L''(t) = \int_\Omega u_t u_t \mathrm{d}x + \int_\Omega u u_{tt}\mathrm{d}x + \int_\Omega \nabla u \nabla u_t \mathrm{d}x + 2\beta$$

According to the definition $E(t)$, the expression of $L''(t)$, and Lemma 1.2, we have

$$\begin{aligned} L''(t) &\geqslant \frac{q^-+2}{2}\|u_t\|_2^2+\frac{q^-}{p^+}\int_\Omega |\nabla u|^{p(x)}\mathrm{d}x+q^-\int_0^t\int_\Omega |\nabla u_\tau|^2\mathrm{d}x\mathrm{d}\tau-q^-E(0)+2\beta \\ &\geqslant \frac{q^-+2}{2}\Big[\|u_t\|_2^2+\int_0^t\int_\Omega |\nabla u_\tau|^2\mathrm{d}x\mathrm{d}\tau+4\beta\Big] \end{aligned}$$

where

$$\beta=\frac{q^-(1+|\Omega|)^{\frac{p^+(2-3p^-)}{p^-}}}{2(q^-+1)p^+}\min\left\{\Big(B_1\int_\Omega |u_0|^2\mathrm{d}x\Big)^{\frac{p^-}{2}},\Big(B_1\int_\Omega |u_0|^2\mathrm{d}x\Big)^{\frac{p^+}{2}}\right\}-\frac{q^-E(0)}{2(q^-+1)}>0$$

Following the lines of the proof of Theorem 1.1 of [2], we have

$$L(t)L''(t)-\frac{q^-+2}{4}(L'(t))^2\geqslant 0$$

where $q^->2$, which implies

$$(L^{1-\frac{q^-+2}{4}}(t))''\leqslant 0,\ \ \textit{for}\ \ t>0$$

Noting that $L^{1-\frac{q^-+2}{4}}(0)>0$, $(L^{1-\frac{q^-+2}{4}})'(0)<0$, then

$$L^{1-\frac{q^-+2}{4}}(T^*)=0,\ \ \textit{for some}\ \ T^*\in(0,\frac{-L^{1-\frac{q^-+2}{4}}(0)}{(L^{1-\frac{q^-+2}{4}})'(0)})$$

**Case 2.** There exists $t_1>0$ such that $E(t_1)<0$. Noting that $E(0)>0$ and considering the continuity of $E(t)$, we know that there exists $t_2\in(0,t_1)$ such that $E(t_2)=0$. In addition, we apply the monotonicity of $E(t)$ to obtain $E(t)\geqslant 0,\ 0<t\leqslant t_2$. Similar as the proof of Case 1, we may prove that the solution to Problem (1.1) blows up before the time $t_1$. □

It is worth pointing out that the principle significance of the condition $(H_3)$ is that it allows us to establish an explicit upper bound of the blow-up time. In fact, if this condition is removed, we also prove the nonexistence of solutions.

**Theorem 1.2.** *Assume that the exponents $p(x), q(x)$ and the initial data $u_0(x), u_1(x)$ satisfy the conditions of Lemma 1.1 and the following conditions*

$$(H_4)\ q^->p^+>p^-\geqslant 2$$
$$(H_5)\int_\Omega u_0u_1\mathrm{d}x>\frac{p^+(B_1+4q^-)}{4(q^--p^+)B_1}\Big[E(0)+\frac{|\Omega|}{q^-}\Big]>0$$

*then the solution to Problem (1.1) blows up in finite time.*

**Proof. Case 1.** For all $t\geqslant 0$, we first assume that $E(t)\geqslant 0$. According to $(H_5)$ and to Inequality (1.4), it is easy to check that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u|^2\mathrm{d}x=2\int_\Omega uu_t\mathrm{d}x\geqslant 2\mathrm{e}^{M_0t}H(0),\ t\geqslant 0 \tag{1.14}$$

where $H(0)=\int_\Omega u_0u_1\mathrm{d}x-\frac{p^+(B_1+4q^-)}{4(q^--p^+)B_1}\Big[E(0)+\frac{|\Omega|}{q^-}\Big]>0$. Assume by contradiction that the solution $u$ is global. Then, it is easily seen that

$$\begin{aligned} \|u(.,t)\|_2 &=\|u_0\|_2+2\int_0^t\int_\Omega u(.,\tau)u_\tau(.,\tau)\mathrm{d}x\mathrm{d}\tau\geqslant\|u_0\|_2+2\int_0^t \mathrm{e}^{M_0\tau}H(0)\mathrm{d}\tau \\ &=\|u_0\|_2+\frac{2H(0)}{M_0}\Big(\mathrm{e}^{M_0t}-1\Big) \end{aligned} \tag{1.15}$$

On the other hand, by Lemma 1.1, Minkowski inequality and Hölder inequality, we have

$$\begin{aligned}\|u(.,t)\|_2 &\leqslant \|u_0\|_2 + \|u(.,t)-u_0\|_2 \leqslant \|u_0\|_2 + \Big\| \int_0^t u_\tau \mathrm{d}\tau \Big\|_2 \leqslant \|u_0\|_2 + \int_0^t \|u_\tau\|_2 \mathrm{d}\tau \\ &\leqslant \|u_0\|_2 + \frac{1}{\sqrt{B_1}} \int_0^t \|\nabla u_\tau\|_2 \mathrm{d}\tau \leqslant \|u_0\|_2 + \frac{\sqrt{t}}{\sqrt{B_1}} \Big( \int_0^t \int_\Omega |\nabla u_\tau|^2 \mathrm{d}x\mathrm{d}\tau \Big)^{\frac{1}{2}} \\ &\leqslant \|u_0\|_2 + \frac{\sqrt{t}}{\sqrt{B_1}} \Big( E(0) - E(t) \Big)^{\frac{1}{2}} \leqslant \|u_0\|_2 + \sqrt{\frac{tE(0)}{B_1}}\end{aligned} \tag{1.16}$$

which contradicts Inequality (1.15).

**Case 2.** There exists $t_1 > 0$ such that $E(t_1) < 0$. This proof is similar to the argument of the second case of Theorem 1.1. We omit it here. □

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