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A coated rigid elliptical inclusion loaded by a couple in the presence of uniform interfacial and hoop stresses

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ABSTRACT

We consider a confocally coated rigid elliptical inclusion, loaded by a couple and introduced into a remote uniform stress field. We show that uniform interfacial and hoop stresses along the inclusion–coating interface can be achieved when the two remote normal stresses and the remote shear stress each satisfy certain conditions. Our analysis indicates that: (i) the uniform interfacial tangential stress depends only on the area of the inclusion and the moment of the couple; (ii) the rigid-body rotation of the rigid inclusion depends only on the area of the inclusion, the coating thickness, the shear moduli of the composite and the moment of the couple; (iii) for given remote normal stresses and material parameters, the coating thickness and the aspect ratio of the inclusion are required to satisfy a particular relationship; (iv) for prescribed remote shear stress, moment and given material parameters, the coating thickness, the size and aspect ratio of the inclusion are also related. Finally, a harmonic rigid inclusion emerges as a special case if the coating and the matrix have identical elastic properties.

1. Introduction

It is well known that a uniform distribution of interfacial normal and tangential stresses is optimal in that it will eliminate any stress peaks at the interface between an inclusion and the surrounding material [1]. Furthermore, a uniform distribution of hoop stress along the edge of a hole or inclusion is also known to be ideal in the design of what Cherepanov refers to as "equally strong outlines of holes" [2]. The design of inclusions with constant interfacial and hoop stresses has attracted considerable attention in the literature (see, for example, [1,3–7]). In particular, confocal elliptical interfaces can be used to achieve the design objective of uniform interfacial and hoop stresses for a three-phase elliptical inclusion [1]. In all these previous investigations, the inclusion itself is free of any external loading. It is of great practical and theoretical interest to ask whether, if a rigid inclusion were loaded, for example, by a couple, it would still be possible to achieve uniform interfacial and hoop stresses along the inclusion boundary. This question forms the basis of the present study.

In this paper, we consider a rigid elliptical inclusion, loaded by a couple moment and bonded to an infinite elastic matrix through a coating consisting of two confocal elliptical interfaces. We develop one condition on remote normal stresses and another on remote shear stress that ensure that the interfacial and hoop stresses along the inclusion–coating interface are

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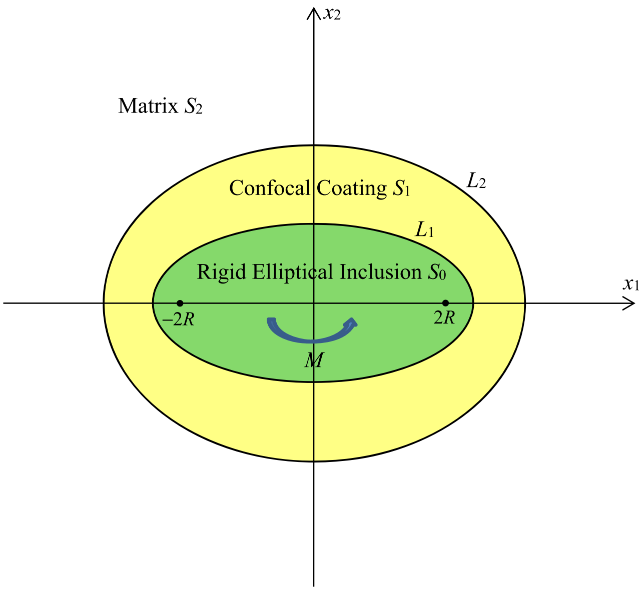

Fig. 1. A confocally coated rigid elliptical inclusion loaded by a couple.

uniformly distributed. For prescribed remote normal stresses and material parameters, a relationship is established between the coating thickness and the aspect ratio of the inclusion. In the case of prescribed remote shear stress, couple moment and material parameters, we establish a relationship among the coating thickness, the size, and the aspect ratio of the inclusion.

2. Complex variable formulation

In a Cartesian coordinate system $Ox_1x_2x_3$, in the case of plane elastostatics, the stresses $(\sigma_{11}, \sigma_{22}, \sigma_{12})$, displacement components (u_1, u_2) and the stress functions (ϕ_1, ϕ_2) can be expressed in terms of two analytic functions $\varphi(z)$ and $\psi(z)$ of the complex variable $z = x_1 + \mathrm{i}x_2$ as [8]

$$\begin{aligned}&\sigma_{11}+\sigma_{22}=2\big[\varphi'(z)+\overline{\varphi'(z)}\big],\\&\sigma_{22}-\sigma_{11}+2\mathrm{i}\sigma_{12}=2\big[\bar{z}\varphi''(z)+\psi'(z)\big],\end{aligned}\tag{1}$$

$$\begin{aligned}&2\mu(u_1+\mathrm{i}u_2)=\kappa\varphi(z)-z\overline{\varphi'(z)}-\overline{\psi(z)}\\&\phi_1+\mathrm{i}\phi_2=\mathrm{i}\big[\varphi(z)+z\overline{\varphi'(z)}+\overline{\psi(z)}\big]\end{aligned}\tag{2}$$

where $\kappa = 3 - 4\nu$ for plane strain, $\kappa = (3 - \nu)/(1 + \nu)$ for plane stress, μ and ν $(0 \le \nu \le 1/2)$ are the shear modulus and Poisson's ratio, respectively. In addition, the stresses are related to the stress functions through [9]

$$\begin{aligned}&\sigma_{11}=-\phi_{1,2}, \qquad \sigma_{12}=\phi_{1,1}\\&\sigma_{21}=-\phi_{2,2}, \qquad \sigma_{22}=\phi_{2,1}\end{aligned}\tag{3}$$

3. General solution

Consider a rigid elliptical inclusion bonded to an infinite elastic matrix through a confocal coating. Let S_0, S_1 and S_2 denote the rigid inclusion, the coating and the matrix, respectively, all of which are perfectly bonded across two confocal elliptical interfaces L_1 and L_2, the common foci of which are located at $z = \pm 2R$ $(R > 0)$ on the real axis, as shown in Fig. 1. The rigid inclusion is loaded by a couple of moment M and the matrix is subjected to remote uniform in-plane stresses $(\sigma_{11}^{\infty}, \sigma_{22}^{\infty}, \sigma_{12}^{\infty})$. In what follows, the subscripts 1 and 2 will be used to identify the respective quantities in S_1 and S_2.

We first introduce the following conformal mapping function [8]

$$z=\omega(\xi)=R\left(\xi+\frac{1}{\xi}\right), \qquad \xi=\omega^{-1}(z)=\frac{z}{2R}+\sqrt{\frac{z^2}{4R^2}-1}\tag{4}$$

which maps the segment $[-2R, 2R]$ onto the unit circle in the ξ-plane and the two interfaces L_1 and L_2 onto two coaxial circles with radii R_1 and R_2. Thus S_1 and S_2 are mapped onto $R_1 \le |\xi| \le R_2$ and $|\xi| \ge R_2$, respectively. For the sake of convenience and without loss of generality, we write $\varphi_i(\xi) = \varphi_i(\omega(\xi))$, $\psi_i(\xi) = \psi_i(\omega(\xi))$, $i = 1, 2$.

In the image ξ-plane, the boundary value problem has the form

$$\kappa_1\varphi_1(\xi)-\frac{\omega(\xi)}{\overline{\omega'(\xi)}}\overline{\varphi_1'(\xi)}-\overline{\psi_1(\xi)}=2\mathrm{i}\mu_1\varpi\,\omega(\xi),\quad |\xi|=R_1 \tag{5a}$$

$$\begin{aligned}&\varphi_2(\xi)+\frac{\omega(\xi)}{\overline{\omega'(\xi)}}\overline{\varphi_2'(\xi)}+\overline{\psi_2(\xi)}=\varphi_1(\xi)+\frac{\omega(\xi)}{\overline{\omega'(\xi)}}\overline{\varphi_1'(\xi)}+\overline{\psi_1(\xi)}\\ &\kappa_2\varphi_2(\xi)-\frac{\omega(\xi)}{\overline{\omega'(\xi)}}\overline{\varphi_2'(\xi)}-\overline{\psi_2(\xi)}=\Gamma\kappa_1\varphi_1(\xi)-\frac{\Gamma\omega(\xi)}{\overline{\omega'(\xi)}}\overline{\varphi_1'(\xi)}-\Gamma\overline{\psi_1(\xi)},\quad |\xi|=R_2\end{aligned} \tag{5b}$$

$$\varphi_2(\xi)=\frac{R(\sigma_{11}^\infty+\sigma_{22}^\infty)}{4}\xi+O(1),\,\psi_2(\xi)=\frac{R(\sigma_{22}^\infty-\sigma_{11}^\infty+2\mathrm{i}\sigma_{12}^\infty)}{2}\xi+O(1),\quad |\xi|\to\infty \tag{5c}$$

where ϖ is the (to be determined) rigid-body rotation of the inclusion and $\Gamma=\mu_2/\mu_1$.

In order to ensure that the interfacial and hoop stresses along the inclusion–coating interface L_1 are uniform, the analytic function $\varphi_1(\xi)$ should take the following form

$$\varphi_1(\xi)=X\left(\xi+\frac{1}{\xi}\right),\quad R_1\le|\xi|\le R_2 \tag{6}$$

where X is a complex number to be determined. We can thus write $X=X'+\mathrm{i}X''$ identifying X' and X'' as the real and imaginary parts of X.

Using the expression for $\varphi_1(\xi)$ from Eq. (6) and imposing the interface condition on the inner interface $|\xi|=R_1$, from Eq. (5a) we obtain

$$\psi_1(\xi)=(\kappa_1\bar{X}-X+2\mathrm{i}R\mu_1\varpi)\left(\frac{R_1^2}{\xi}+\frac{\xi}{R_1^2}\right),\quad R_1\le|\xi|\le R_2 \tag{7}$$

Similarly, using Eqs. (6) and (7) and imposing the two interface conditions on the outer interface $|\xi|=R_2$ from Eq. (5b), we arrive at

$$\begin{aligned}\varphi_2(\xi)&=\frac{X(\Gamma\kappa_1+1)+\bar{X}(1-\Gamma)}{\kappa_2+1}\left(\xi+\frac{1}{\xi}\right)+\frac{(1-\Gamma)(\kappa_1X-\bar{X}-2\mathrm{i}R\mu_1\varpi)}{\kappa_2+1}\left(\rho\xi+\frac{1}{\rho\xi}\right)\\ \psi_2(\xi)&=\frac{X(\kappa_2+\Gamma)+\bar{X}(\kappa_2-\Gamma\kappa_1)}{\kappa_2+1}\left(\frac{R_2^2}{\xi}+\frac{\xi}{R_2^2}\right)-\frac{(\Gamma+\kappa_2)(X-\kappa_1\bar{X}-2\mathrm{i}R\mu_1\varpi)}{\kappa_2+1}\left(\frac{R_1^2}{\xi}+\frac{\xi}{R_1^2}\right)\\ &\quad-\frac{\frac{R_2^2}{\xi}+\frac{\xi}{R_2^2}}{1-\frac{1}{\xi^2}}\varphi_2'(\xi),\quad |\xi|\ge R_2\end{aligned} \tag{8}$$

where $\rho=R_1^2/R_2^2$ $(0\le\rho\le1)$ is a measure of the relative thickness of the coating.

Using Eq. (8) and imposing the asymptotic conditions from Eq. (5c) as well as the moment balance for the circular disk $|z|=R_0\to\infty$, we finally obtain

$$\varpi=\frac{MR_1^2[\Gamma+\rho(1-\Gamma)]}{4\pi\mu_2R^2(R_1^4-1)}=\frac{M[\Gamma+\rho(1-\Gamma)]}{4\mu_2A},\qquad X''=\frac{2\rho R\mu_1\varpi(1-\Gamma)}{(\kappa_1+1)[\Gamma+\rho(1-\Gamma)]}=\frac{M\rho R_1^2(1-\Gamma)}{2\pi\Gamma(\kappa_1+1)R(R_1^4-1)} \tag{9}$$

$$\sigma_{12}^\infty=\frac{M[\Gamma+\kappa_2+\rho^2(1-\Gamma)]}{2\pi R^2(R_1^4-1)(\kappa_2+1)} \tag{10}$$

$$\begin{aligned}\sigma_{11}^\infty+\sigma_{22}^\infty&=\frac{4X'[\Gamma(\kappa_1-1)+2+\rho(1-\Gamma)(\kappa_1-1)]}{R(\kappa_2+1)}\\ \sigma_{22}^\infty-\sigma_{11}^\infty&=\frac{2X'\{2\rho[\kappa_2-1-\Gamma(\kappa_1-1)]+(\Gamma+\kappa_2)(\kappa_1-1)+\rho^2(\Gamma-1)(\kappa_1-1)\}}{RR_1^2(\kappa_2+1)}\end{aligned} \tag{11}$$

where A is the area of the elliptical inclusion S_0.

The necessary and sufficient condition for the existence of the real coefficient X' simultaneously satisfying the two conditions in Eq. (11) is

$$\frac{\sigma_{22}^\infty-\sigma_{11}^\infty}{\sigma_{11}^\infty+\sigma_{22}^\infty}R_1^2=\frac{2\rho[\kappa_2-1-\Gamma(\kappa_1-1)]+(\Gamma+\kappa_2)(\kappa_1-1)+\rho^2(\Gamma-1)(\kappa_1-1)}{2[\Gamma(\kappa_1-1)+2+\rho(1-\Gamma)(\kappa_1-1)]} \tag{12}$$

which is found to be in agreement with Eq. (3.7) or Eq. (4.3) given in Ru [1]. Furthermore, the interfacial normal stress σ_{nn} and interfacial tangential stress σ_{nt} are uniformly distributed along the inclusion–coating interface L_1 as follows:

$$
\begin{aligned}
\sigma_{nn} &= \frac{(\kappa_1+1)X'}{R} = \frac{(\kappa_1+1)(\kappa_2+1)(\sigma_{11}^\infty+\sigma_{22}^\infty)}{4[\Gamma(\kappa_1-1)+2+\rho(1-\Gamma)(\kappa_1-1)]} \\
\sigma_{nt} &= -\frac{MR_1^2}{2\pi R^2(R_1^4-1)} = -\frac{M}{2A} = -\frac{R_1^2(\kappa_2+1)\sigma_{12}^\infty}{\Gamma+\kappa_2+\rho^2(1-\Gamma)}, \quad z\in L_1
\end{aligned} \tag{13}
$$

It is deduced from Eqs. (6) and (9) that the constant mean stress and uniform rigid body rotation in the coating are given by

$$
\begin{aligned}
\sigma_{11}+\sigma_{22} &= \frac{4X'}{R} = \frac{(\kappa_2+1)(\sigma_{11}^\infty+\sigma_{22}^\infty)}{\Gamma(\kappa_1-1)+2+\rho(1-\Gamma)(\kappa_1-1)} \\
\varpi_0 &= \frac{1}{2}(u_{2,1}-u_{1,2}) = \frac{(\kappa_1+1)X''}{2\mu_1 R} = \frac{M\rho R_1^2(1-\Gamma)}{4\pi\mu_2 R^2(R_1^4-1)} = \frac{\varpi\rho(1-\Gamma)}{\Gamma+\rho(1-\Gamma)}, \quad z\in S_1
\end{aligned} \tag{14}
$$

Consequently, from Eqs. (6), (7) and (13), it follows that the hoop stress σ_{tt} is uniformly distributed along the inclusion–coating interface L_1 on the coating side as follows:

$$
\sigma_{tt} = \frac{(3-\kappa_1)X'}{R} = \frac{(3-\kappa_1)(\kappa_2+1)(\sigma_{11}^\infty+\sigma_{22}^\infty)}{4[\Gamma(\kappa_1-1)+2+\rho(1-\Gamma)(\kappa_1-1)]}, \quad z\in L_1 \tag{15}
$$

In view of the fact that $\sigma_{12}^\infty \neq 0$ (see Eq. (10)), the remote principal stresses are inclined at an angle to the principal axes of the elliptical inclusion. It is seen from the above results that: (i) the uniform interfacial tangential stress depends only on the area of the inclusion and the moment of the couple; (ii) the rigid-body rotation of the rigid inclusion is dependent only on the area of the inclusion, the relative coating thickness ρ, the shear moduli of the composite and the moment; (iii) ϖ and ϖ_0 have the same sign when the coating is stiffer than the matrix ($\Gamma<1$) and opposite signs when the coating is more compliant than the matrix ($\Gamma>1$); (iv) the magnitude of the interfacial tangential stress is always greater than that of the remote shear stress if the coating is stiffer than the matrix, although it can be greater than, equal to, or smaller than that of the remote shear stress if the coating is more compliant than the matrix; (v) for given remote normal stresses and material parameters, the relative coating thickness and the aspect ratio of the inclusion here characterized by R_1 should satisfy the relationship given in Eq. (12); (vi) for prescribed remote shear stress, couple moment and given material parameters, the relative coating thickness, the size of the inclusion (characterized by RR_1) and the aspect ratio of the inclusion should satisfy the relationship given in Eq. (10).

In the case when the elliptical inclusion contracts to a line inclusion ($R_1\to 1$), the denominator of the right-hand side of Eq. (10) will become zero. This implies that it is impossible to constrain a rigid line inclusion to continue to maintain uniform interfacial and hoop stresses along its boundary L_1 when both the remote shear stress and the moment take finite values. At the other extreme, when the elliptical inclusion becomes circular (as $R_1\to\infty$) (in which case the coating becomes an annulus), we deduce from Eqs. (10) and (12) that $\sigma_{11}^\infty=\sigma_{22}^\infty$, $\sigma_{12}^\infty=0$, which indicates that the remote stress field should now be hydrostatic in order to ensure that the interfacial and hoop stresses along the circular interface L_1 remain uniform.

4. A harmonic rigid inclusion

When the materials comprising the coating and the matrix are identical ($\Gamma=1$ and $\kappa_1=\kappa_2$), we have from Eqs. (9)–(15) that

$$
\varpi = \frac{M}{4\mu_1 A}, \qquad \varpi_0 = 0 \tag{16}
$$

$$
\sigma_{12}^\infty = \frac{M}{2\pi R^2(R_1^4-1)}, \qquad \frac{\sigma_{22}^\infty-\sigma_{11}^\infty}{\sigma_{11}^\infty+\sigma_{22}^\infty} = \frac{\kappa_1-1}{2R_1^2} \tag{17}
$$

$$
\sigma_{nn} = \frac{(\kappa_1+1)(\sigma_{11}^\infty+\sigma_{22}^\infty)}{4}, \qquad \sigma_{tt} = \frac{(3-\kappa_1)(\sigma_{11}^\infty+\sigma_{22}^\infty)}{4}, \qquad \sigma_{nt} = -\frac{M}{2A}, \quad z\in L_1 \tag{18}
$$

In this case,

$$
\varphi_1(z) = \varphi_2(z) = \frac{\sigma_{11}^\infty+\sigma_{22}^\infty}{4}z, \quad z\in S_1\cup S_2 \tag{19}
$$

which implies that the trace of the original stress field in the matrix is unaltered following the introduction of the loaded (by a couple) rigid inclusion. Thus the loaded rigid inclusion now satisfies the harmonic condition of Bjorkman and Richards [3,10–12].

5. Conclusions

We investigate the possibility of uniform interfacial and hoop stresses for a loaded (by a couple) rigid elliptical inclusion with confocal coating inserted into an infinite matrix subjected to a uniform stress field at infinity. Using complex variable methods, we have shown that constant interfacial and hoop stresses along the inclusion–coating interface can indeed be realized when the two conditions specified in Eqs. (10) and (12) are satisfied. The presence of the couple loading the inclusion means that the interfacial tangential stress at the inclusion–coating interface and the remote shear stress are no longer zero. When the coating and the matrix have identical elastic properties, our analysis identifies a harmonic rigid inclusion.

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