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Model order reduction for dynamical systems: A geometric approach

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The aim of this paper is to ask the question as whether it is possible, for a given dynamical system defined by a vector field over a finite dimensional inner product space, to construct a reduced-order model over a finite dimensional manifold. In order to give a positive answer to this question, we prove that if the manifold under consideration is an immersed submanifold of the vector space, considered as ambient manifold, then it is possible to construct explicitly a reduced-order vector field over this submanifold. In particular, we found that the reduced-order vector field satisfies the variational principle of Dirac–Frenkel and that we can formulate the Proper Orthogonal Decomposition under this framework. Finally, we propose a local-point estimator of the time-dependent error between the original vector field and the reduced-order one.

1. Introduction

Model reduction applied to a dynamical system (described by an ordinary differential equation) allows one to extract the most significant features of this system, representing them in a reduced system of coordinates. The goal of this approach is to construct a computational low-cost procedure that reproduces the dominant physical mechanisms of the original model. The interested reader is referred to the following review papers and books [1–4].

One of the more widely used model reduction technique is the Proper Orthogonal Decomposition (POD). Its main goal is to obtain a lower dimensional approximation of a given dynamical system, as follows. Let be an ordinary differential equation (ODE)

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = X(t,\mathbf{u}), \quad \mathbf{u}(0) = \mathbf{u}_0 \tag{1}$$

for $t \in [0, t_f]$, with $\mathbf{u}, \mathbf{u}_0 \in \mathbb{R}^n$ and $X : [0, t_f] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$. Consider next the solutions to (1) at m-time points $\{\mathbf{u}(t_1), \ldots, \mathbf{u}(t_m)\}$ collected in the $n \times m$-matrix $A = [\mathbf{u}(t_1) - \overline{\mathbf{x}} \cdots \mathbf{u}(t_m) - \overline{\mathbf{x}}]$ and where $\overline{\mathbf{x}} = \frac{1}{r}\sum_{i=1}^m \mathbf{u}(t_i)$ is the mean of these observations. POD seeks a r-dimensional subspace S of $\mathbb{R}^n$ $(r \le n)$ and the corresponding projection matrix $\Pi_S \in \mathbb{R}^{n\times n}$, so that $\|A - \Pi_S A\|$ is minimized over all k-dimensional subspaces. The projection matrix corresponding to the optimal subspace S is obtained as $\Pi_S = ZZ^\intercal$, where the matrix $Z \in \mathbb{R}^{n\times r}$ consists of the columns of the singular vectors corresponding

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to the r largest singular values obtained from A. In a coordinate system embedded in S, the projection of a point $\mathbf{u}$ onto S is represented by $\boldsymbol{\xi} = Z^\intercal(\mathbf{u} - \overline{\mathbf{x}}) \in \mathbb{R}^r$. In particular, if $\mathbf{x} \in \overline{\mathbf{x}} + S$ then $\mathbf{x} - \overline{\mathbf{x}} = Z\boldsymbol{\xi}$ for some $\boldsymbol{\xi} \in \mathbb{R}^r$. A POD-based reduced model that approximates the original problem (1) can then be constructed by the following rule. For any point $\mathbf{x} = Z\boldsymbol{\xi} \in S$, compute the vector-field $X(t, \overline{\mathbf{x}} + Z\boldsymbol{\xi}) \in \mathbb{R}^n$ and take the projection $Z^\intercal X(t, \overline{\mathbf{x}} + Z\boldsymbol{\xi}) \in \mathbb{R}^r$ onto the subspace S. Therefore, we obtain

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} = Z^\intercal X(t, \overline{\mathbf{x}} + Z\boldsymbol{\xi}), \quad \xi(0) = Z^\intercal \mathbf{u}_0 \tag{2}$$

The dynamical system (2) allows an efficient (typically low-dimensional) representation of the key system behaviour. This framework appears useful in a wide variety of applications.

The Dirac–Frenkel variational principle is a well-known tool in the numerical treatment of equations of quantum dynamics. It was originally proposed by Dirac and Frenkel in 1930 to approximately solve the time-dependent Schrödinger equation. It assumes the existence of a vector field over a configuration space represented by a Hilbert space. This configuration space contains an immersed submanifold, the so-called Hartree manifold, and the reduced-order model is then obtained by projecting the vector field at each point of the submanifold onto its tangent space (see [1,5]). It allows also one to introduce the so-called geometric numerical integration methods for differential equations (see VI.9 in [6]).

A similar approach is used in the so-called *dynamical low-rank approximation* for time-dependent data matrices and tensors [7,8]. The reduced model is obtained by using the Dirac–Frenkel variational principle, over a manifold of matrices (respectively, tensors) of fixed rank (respectively, tensor rank).

To the authors' knowledge, there is no proof, in a general setting, that the reduced-order dynamical system is a vector field, that is, a differentiable map between the immersed submanifold and its tangent bundle. This fact implies that the existence and uniqueness of the solutions to the reduced dynamical system is not ensured. The main result of this paper is to give a positive answer to the above question. In particular, given a vector field defined over a finite dimensional inner product space, we will prove the following. Assume that the manifold chosen to construct the reduced-order dynamical system is an immersed submanifold of the vector space, considered as ambient manifold. Then we will show that it is possible to construct explicitly a reduced-order vector field over this submanifold.

The paper is organized as follows. In the next section, we give some preliminary definitions. In Section 3, we state and prove the main result of this paper. We also give some examples and we propose a point estimator of the time-dependent error between the original vector field and the reduced order one. Finally, in Section 4, some conclusions are given.

2. Preliminary definitions

A differentiable manifold can be seen as a configuration space used to describe a particular physical system. The most obvious examples are related to mechanical systems for the study of the movements of a pendulum or of a system of solids. It may equally well be used to model the evolution of a chemical system where the parameters are the temperature and the concentrations of various species. One of main characteristics of these abstract objects is the property to describe a neighbourhood on each point in the configuration space by using a set of (local) coordinates into an open set of a particular finite-dimensional normed space. This neighbourhood and its corresponding set of local coordinates are known as a chart, and the whole set of charts constitutes an atlas for the manifold. The atlas can be used to endow the manifold with a topology. Since we need to perform infinitesimal variations in our configuration space, a compatibility condition between two different coordinates systems is needed.

Along this paper, we will consider a manifold as a pair $(\mathbb{M}, \mathcal{A})$ where $\mathbb{M}$ is a subset of some finite-dimensional vector space V and $\mathcal{A}$ is an atlas representing the local coordinate system of $\mathbb{M}$. We recall the definition of an atlas associated with a set $\mathbb{M}$.

Definition 2.1. Let $\mathbb{M}$ be a set. An atlas of class C^p $(p \geq 0)$ or analytic on $\mathbb{M}$ is a family of charts with some indexing set A, namely $\{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$, having the following properties (see [9]):

AT1 $\{U_\alpha\}_{\alpha \in A}$ is a covering of $\mathbb{M}$, that is, $U_\alpha \subset \mathbb{M}$ for all $\alpha \in A$ and $\cup_{\alpha \in A} U_\alpha = \mathbb{M}$;

AT2 for each $\alpha \in A, (U_\alpha, \varphi_\alpha)$ stands for a bijection $\varphi_\alpha : U_\alpha \to W_\alpha$ of U_α onto an open set W_α of a finite-dimensional normed space $(X_\alpha, \|\cdot\|_\alpha)$, and for any α and β the set $\varphi_\alpha(U_\alpha \cap U_\beta)$ is open in X_α;

AT3 finally, if we let $U_\alpha \cap U_\beta = U_{\alpha,\beta}$ and $\varphi_\alpha(U_{\alpha,\beta}) = U_{\alpha,\beta}$, the transition mapping $\varphi_\beta \circ \varphi_\alpha^{-1} : U_{\alpha,\beta} \to U_{\beta,\alpha}$ is a diffeomorphism of class C^p $(p \geq 0)$ or is analytic.

Since different atlases can give the same manifold, we say that two atlases are *compatible* if each chart of one atlas is compatible with the charts of the other one in the sense of AT3. One verifies that the relation of compatibility between atlases is an equivalence relation.

Definition 2.2. An equivalence class of atlases of class C^p on $\mathbb{M}$, also denoted by $\mathcal{A}$, is said to define a structure of a C^p-manifold on $\mathbb{M}$, and hence we say that $(\mathbb{M}, \mathcal{A})$ is a finite-dimensional manifold. In a similar way, if an equivalence class of atlases is given by analytic maps, then we say that $(\mathbb{M}, \mathcal{A})$ is an analytic finite-dimensional manifold.

Now, we introduce the definition of tangent vector to a manifold. It is related to the notion of velocity vector to a curve lying in the configuration space.

Let $(\mathbb{M}, \mathcal{A})$ be a manifold of class $\mathcal{C}^p$ $(p \geq 1)$ or analytic. Let m be a point of $\mathbb{M}$. We consider triples (U, φ, v) where (U, φ) is a chart at m and v is an element of the vector space in which $\varphi(U)$ lies. We say that two of such triples (U, φ, v) and (V, ψ, w) are *equivalent* if the derivative of $\psi \circ \varphi^{-1}$ at $\varphi(m)$ maps v on w. Thanks to the chain rule, it is an equivalence relation. An equivalence class of such triples is called a *tangent vector of* $\mathbb{M}$ *at* m.

Definition 2.3. The set of such tangent vectors is called the *tangent space of* $\mathbb{M}$ *at* m and it is denoted by $\mathbb{T}_m(\mathbb{M})$.

Each chart (U, φ) determines a bijection of $\mathbb{T}_m(\mathbb{M})$ on a finite-dimensional normed space, namely the equivalence class of (U, φ, v) corresponds to the vector v.

The whole set of tangent spaces in a given manifold, called the tangent bundle, plays an important role to describe the velocity fields over the configuration space.

Definition 2.4. Let $(\mathbb{M}, \mathcal{A})$ be a manifold of class $\mathcal{C}^p$ $(p \geq 1)$. The set

$$\mathbb{TM} = \bigcup_{m \in \mathbb{M}} \mathbb{T}_m(\mathbb{M}) = \{(m, X_m) : m \in \mathbb{M} \text{ and } X_m \in \mathbb{T}_m(\mathbb{M})\}$$

is called the tangent bundle of $\mathbb{M}$.

To define an atlas for the tangent bundle, we consider the projection $\pi : \mathbb{TM} \longrightarrow \mathbb{M}$ over the first component, and we can construct for a given $(m, X_m) \in \mathbb{TM}$ a local chart by taking $(U_m, \varphi) \in \mathcal{A}$ and using the fact that there exists a bijection

$$\psi_m : \pi^{-1}(U_m) = \{(m', X_{m'}) : m' \in U_m \text{ and } X'_m \in \mathbb{T}_{m'}(\mathbb{M})\} \longrightarrow U_m \times \mathbb{T}_m(\mathbb{M})$$

It can be shown that $\mathcal{A}^* = \{(U_m, \psi_m) : m \in \mathbb{M}\}$ is an atlas for $\mathbb{TM}$.

Our next step is to recall the definition of the differential of a morphism between manifolds. It gives a linear map between the tangent spaces of the involved manifolds.

Definition 2.5. Let $(\mathbb{M}, \mathcal{A})$ and $(\mathbb{N}, \mathcal{B})$ be two $\mathcal{C}^p$-manifolds $(p \geq 1)$. A morphism F from the manifold $(\mathbb{M}, \mathcal{A})$ to the manifold $(\mathbb{N}, \mathcal{B})$ is a map $F : (\mathbb{M}, \mathcal{A}) \longrightarrow (\mathbb{N}, \mathcal{B})$ where we take into account the representation of F by using both atlas. More precisely, assume that $F(x) = y$, then take $(U, \varphi) \in \mathcal{A}$ a chart in $\mathbb{M}$ at x and $(W, \psi) \in \mathcal{B}$ a chart in $\mathbb{N}$ at $F(x)$, since we usually perform calculus by means of the parametric representation of both manifolds; in practice, we use the map $\psi \circ F \circ \varphi^{-1} : \varphi(U) \longrightarrow \psi(W)$ where $(\psi \circ F \circ \varphi^{-1})(\varphi(x)) = \psi(y)$. Assume that $F : (\mathbb{M}, \mathcal{A}) \to (\mathbb{N}, \mathcal{B})$ is a $\mathcal{C}^p$ morphism, i.e.

$$\psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(W)$$

is a $\mathcal{C}^p$-differentiable map. For $x \in X$, we define

$$\mathrm{T}_x F : \mathbb{T}_x(\mathbb{M}) \longrightarrow \mathbb{T}_{F(x)}(\mathbb{N}), \quad v \mapsto [(\psi \circ F \circ \varphi^{-1})'(\varphi(x))]v$$

Finally, we will introduce the notion of vector field in a manifold. It represents a global velocity field over a particular configuration space. It is defined allocating on each point of the configuration space a velocity vector compatible with its local coordinate system.

Let $(\mathbb{M}, \mathcal{A})$ be a $\mathcal{C}^p$-manifold $(p \geq 2)$, a $\mathcal{C}^{p-1}$-vector field on $\mathbb{M}$ is a $\mathcal{C}^{p-1}$-morphism

$$X : (\mathbb{M}, \mathcal{A}) \longrightarrow (\mathbb{TM}, \mathcal{A}^*)$$

such that

$$(\pi \circ X)(m) = m$$

holds for all $m \in \mathbb{M}$. Let $m \in \mathbb{M}$ and take $(U(m), \psi_m)$ be a local chart in $\mathbb{M}$, such that $\psi_m(U(m))$ is an open set in $\mathbb{R}^M$. Then $U(m)$ is also a manifold and its tangent bundle is trivial, that is, $\mathbb{T}(U(m)) = U(m) \times \mathbb{R}^M$ and the vector field X on U is a map $X|_{U(m)} : U(m) \longrightarrow U(m) \times \mathbb{R}^M$ of the form $X(m') = (m', X_{U(m)}(m'))$, the map $X_{U(m)}$ is called *the principal part of* X. However, having a separate notation for the principal part turns out to be an unnecessary burden. By abuse of notation, in linear spaces we shall write a vector field simply as a map $X : U(m) \longrightarrow \mathbb{R}^M$ and shall mean the vector field $m' \mapsto (m', X(m'))$. When it is necessary to be careful with the distinction, we shall be.

If $X : (\mathbb{M}, \mathcal{A}) \longrightarrow (\mathbb{TM}, \mathcal{A}^*)$ is a $\mathcal{C}^p$-vector field $(p \geq 1)$ on $\mathbb{M}$, an integral curve of X is a $\mathcal{C}^p$-morphism $(p \geq 1)$ $\boldsymbol{\gamma} : J \longrightarrow \mathbb{M}$, from an open interval $J \subset \mathbb{R}$ to $\mathbb{M}$, such that $\boldsymbol{\gamma}'(t) = X(\boldsymbol{\gamma}(t))$ for all $t \in J$. If $0 \in J$, then the point $x = \boldsymbol{\gamma}(0) \in \mathbb{M}$ is called the starting point of $\boldsymbol{\gamma}$. The theorem of existence and uniqueness of integral curves is the following (see [9, IV §2 Theorem 2.6]).

Theorem 2.6. *Let $(\mathbb{M}, \mathcal{A})$ be a $\mathcal{C}^p$-manifold $(p \geq 2)$, X be a $\mathcal{C}^{p-1}$-vector field on $\mathbb{M}$ and x be a point in $\mathbb{M}$. Then there exists one and only one maximal integral curve $\boldsymbol{\gamma}$ of X with starting point x.*

3. Reduced-order models on immersed manifolds

Assume that given $X : V \longrightarrow V$, a $\mathcal{C}^p$-vector field $(p \geq 1)$ on V, where $(V, \|\cdot\|)$ is a finite-dimensional inner product space, we want to construct a reduced-order model of the dynamical system

$$\dot{\mathbf{v}} = X(\mathbf{v}), \quad \mathbf{v}(0) = \mathbf{v}_0$$

To this end we will consider, a $\mathcal{C}^{p+1}$-manifold $(\mathbb{M}, \mathcal{A})$, where $\mathbb{M}$ is a subset of V. The inner product space V is an analytic manifold modelled by itself taking into account the trivial atlas $\mathcal{A}_{\text{trivial}} = \{(V, id_V)\}$, where $id_V : V \longrightarrow V$. It is well known that the trivial atlas endows V with a natural manifold structure. Since the standard inclusion map

$$\mathfrak{i} : \mathbb{M} \longrightarrow V$$

given by $\mathfrak{i}(\mathbf{v}) = \mathbf{v}$ is injective, we shall study $\mathfrak{i}$ as a morphism between manifolds. To this end, we recall the definition of an immersion between manifolds.

Definition 3.1. Let $F : (\mathbb{M}, \mathcal{A}) \to (\mathbb{N}, \mathcal{B})$ be a morphism between manifolds and let $x \in \mathbb{M}$. We shall say that F is an *immersion at* x if there exists an open neighbourhood U_x of x in $\mathbb{M}$ such that the restriction of F to U_x induces an isomorphism from U_x onto a submanifold of Y. We say that F is an *immersion* if it is an immersion at each point of X.

For manifolds, we have the following criterion for immersions (see Theorem 3.5.7 in [10]).

Proposition 3.2. *Let $(\mathbb{M}, \mathcal{A})$ and $(\mathbb{N}, \mathcal{B})$ be two manifolds of class $\mathcal{C}^p$ $(p \geq 1)$. Let $F : (\mathbb{M}, \mathcal{A}) \to (\mathbb{N}, \mathcal{B})$ be a $\mathcal{C}^p$ morphism and $x \in \mathbb{M}$. Then F is an immersion at x if and only if $\mathrm{T}_x F$ is injective.*

A concept related to an immersion between manifolds is introduced in the following definition.

Definition 3.3. Assume that $(\mathbb{M}, \mathcal{A})$ and $(\mathbb{N}, \mathcal{B})$ are manifolds and let $f : (\mathbb{M}, \mathcal{A}) \to (\mathbb{N}, \mathcal{B})$ be a $\mathcal{C}^p$ morphism. If f is an injective immersion, then $f(\mathbb{M})$ is called an *immersed submanifold of* $\mathbb{N}$.

From now on, we will assume that $(\mathbb{M}, \mathcal{A})$ is $\mathcal{C}^p$-manifold $(p \geq 2)$ and that the standard inclusion map

$$\mathfrak{i} : (\mathbb{M}, \mathcal{A}) \longrightarrow (V, \mathcal{A}_{\text{trivial}})$$

is a $\mathcal{C}^p$-immersion, that is, $\mathfrak{i}$ is a $\mathcal{C}^p$-differentiable morphism, and the linear map

$$\mathrm{T}_{\mathbf{v}}\mathfrak{i} : \mathbb{T}_{\mathbf{v}}(\mathbb{M}) \longrightarrow V$$

is injective and $\mathrm{T}_{\mathbf{v}}\mathfrak{i}(\mathbb{T}_{\mathbf{v}}(\mathbb{M}))$ is a linear subspace of V, for each $\mathbf{v} \in \mathbb{M}$. In consequence,

$$\mathrm{T}_{\mathbf{v}}\mathfrak{i} : \mathbb{T}_{\mathbf{v}}(\mathbb{M}) \longrightarrow \mathrm{T}_{\mathbf{v}}\mathfrak{i}(\mathbb{T}_{\mathbf{v}}(\mathbb{M}))$$

is a linear isomorphism for each $\mathbf{v} \in \mathbb{M}$. Since V is an inner product space, for each $\mathbf{v} \in \mathbb{M} \subset V$ we have that $V = \mathrm{T}_{\mathbf{v}}\mathfrak{i}(\mathbb{T}_{\mathbf{v}}(\mathbb{M})) \oplus \mathrm{T}_{\mathbf{v}}\mathfrak{i}(\mathbb{T}_{\mathbf{v}}(\mathbb{M}))^{\perp}$, where $\mathrm{T}_{\mathbf{v}}\mathfrak{i}(\mathbb{T}_{\mathbf{v}}(\mathbb{M}))^{\perp}$ is the orthogonal complement of the linear subspace $\mathrm{T}_{\mathbf{v}}\mathfrak{i}(\mathbb{T}_{\mathbf{v}}(\mathbb{M}))$. It allows us to write each velocity vector $X(\mathbf{v}) = X(\mathfrak{i}(\mathbf{v})) \in V$ for $\mathbf{v} \in \mathbb{M}$ as

$$X(\mathfrak{i}(\mathbf{v})) = \Pi_{\mathbf{v}}(X(\mathfrak{i}(\mathbf{v}))) + (id_V - \Pi_{\mathbf{v}})(X(\mathfrak{i}(\mathbf{v}))) \tag{3}$$

where $\Pi_{\mathbf{v}} : V \longrightarrow V$ is the orthogonal projection onto $\mathrm{T}_{\mathbf{v}}\mathfrak{i}(\mathbb{T}_{\mathbf{v}}(\mathbb{M}))$. Thus, $\Pi_{\mathbf{v}}(X(\mathfrak{i}(\mathbf{v}))) \in \mathrm{T}_{\mathbf{v}}\mathfrak{i}(\mathbb{T}_{\mathbf{v}}(\mathbb{M}))$ for all $\mathbf{v} \in \mathbb{M}$, that is, $\mathrm{T}_{\mathbf{v}}\mathfrak{i}^{-1}(\Pi_{\mathbf{v}}(X(\mathfrak{i}(\mathbf{v})))) \in \mathbb{T}_{\mathbf{v}}(\mathbb{M})$. A natural question arising in this context is as to whether the associated morphism

$$\widehat{X} : \mathbb{M} \longrightarrow \mathrm{T}\mathbb{M}, \quad \mathbf{v} \mapsto \widehat{X}(\mathbf{v}) := (\mathbf{v}, \mathrm{T}_{\mathbf{v}}\mathfrak{i}^{-1}(\Pi_{\mathbf{v}}(X(\mathfrak{i}(\mathbf{v}))))) \tag{4}$$

where $\mathrm{T}\mathbb{M}$ is the tangent bundle of $\mathbb{M}$, satisfies the conditions of Theorem 2.6 and, in consequence, for each $\mathbf{v}_0 \in \mathbb{M}$, the differential equation

$$\dot{\mathbf{v}} = \mathrm{T}_{\mathbf{v}}\mathfrak{i}^{-1}(\Pi_{\mathbf{v}}(X(\mathfrak{i}(\mathbf{v}))), \quad \mathbf{v}(0) = \mathbf{v}_0 \tag{5}$$

is well posed on $\mathbb{M}$. We want to point out that the velocity $\dot{\mathbf{v}}$ in (5) satisfies the so-called *Variational Principle of Dirac–Frenkel* (see [1]), that is,

$$\dot{\mathbf{v}} \in \arg \min_{\dot{\mathbf{z}} \in \mathbb{T}_{\mathbf{v}}(\mathbb{M})} \|\mathrm{T}_{\mathbf{v}}\, \mathrm{i}(\dot{\mathbf{z}}) - X(\mathrm{i}(\mathbf{v}))\| \tag{6}$$

Since, without loss of generality, we may assume that $V = \mathbb{R}^n$ and $\|\cdot\|$ is the Euclidean norm, by using [11, Corollary p. 183] (see also [12, Remark 1 p. 124]), we obtain from (6) that (5) is equivalent to

$$\dot{\mathbf{v}} = \mathrm{T}_{\mathbf{v}}\, \mathrm{i}^{+}(X(\mathrm{i}(\mathbf{v}))), \quad \mathbf{v}(0) = \mathbf{v}_0 \tag{7}$$

where $\mathrm{T}_{\mathbf{v}}\, \mathrm{i}^{+}$ denotes the Moore–Penrose pseudo-inverse of $\mathrm{T}_{\mathbf{v}}\, \mathrm{i}$. We point out that, in contrast to matrix inversion, the map from $\mathbb{R}^{r\times n}$ to $\mathbb{R}^{n\times r}$ given by $A \mapsto A^{+}$ is not continuous (see [13, Example 4.1]). However, if we consider the set $\mathcal{M}_s(\mathbb{R}^{n\times r}) := \{A \in \mathbb{R}^{n\times r} : \mathrm{rank}(A) = s\}$, where $s \in \{1, 2, \ldots, r\}$, it can be shown that the map $A \mapsto A^{+}$ is continuous from $\mathcal{M}_s(\mathbb{R}^{n\times r})$ to $\mathbb{R}^{r\times n}$ (see [13, Theorem 4.2]).

We illustrate the above construction with the following two examples.

Example 1 *(Proper Orthogonal Decomposition).* Suppose the original dynamical system under consideration in $V = \mathbb{R}^n$ is given by the time-dependent differential equation

$$\dot{\mathbf{u}} = X(t, \mathbf{u}), \quad \mathbf{u}(0) = \mathbf{u}_0 \tag{8}$$

Let $\mathbb{M} := S + \overline{\mathbf{x}} \subset \mathbb{R}^n$ be the best r–dimensional approximating affine subspace, where S is a linear subspace of $\mathbb{R}^n$ defined by $S = \{Z\boldsymbol{\xi} : \boldsymbol{\xi} \in \mathbb{R}^r\}$ where $Z \in \mathbb{R}^{n\times r}$ is such that $Z^{\mathsf{T}} Z = id_r$. Assume that $\mathbf{u}_0 = \overline{\mathbf{x}} + \mathbf{x}_0 \in S$. Let $\Pi_S := Z Z^{\mathsf{T}} \in \mathbb{R}^{n\times n}$ be the orthogonal projection onto the linear subspace S. The local coordinate of $\mathbf{x} + \overline{\mathbf{x}} \in S + \overline{\mathbf{x}}$, will be $\boldsymbol{\xi} := \varphi(\mathbf{x} + \overline{\mathbf{x}}) \in \mathbb{R}^r$ if and only if $Z\boldsymbol{\xi} + \overline{\mathbf{x}} = \mathbf{x} + \overline{\mathbf{x}} = \varphi^{-1}(\boldsymbol{\xi})$, that is, $\boldsymbol{\xi} = Z^{\mathsf{T}}\mathbf{x}$. Thus, (S, φ) where $\mathbf{x} + \overline{\mathbf{x}} \mapsto \varphi(\mathbf{x} + \overline{\mathbf{x}}) := Z^{\mathsf{T}}\mathbf{x}$ is a chart, $\mathcal{A} = \{(S, \varphi)\}$ is an atlas for the manifold $S + \overline{\mathbf{x}}$, and the tangent space of $S + \overline{\mathbf{x}}$ at $\mathbf{x} + \overline{\mathbf{x}}$ is $\mathbb{T}_{\mathbf{x}+\overline{\mathbf{x}}}(S + \overline{\mathbf{x}}) = \mathbb{R}^r = \{\boldsymbol{\xi} : \boldsymbol{\xi} \in \mathbb{R}^r\}$. Now, $\mathrm{T}_{\mathbf{x}+\overline{\mathbf{x}}}\, \mathrm{i} : S \longrightarrow \mathbb{R}^n$ is given by $\mathrm{T}_{\mathbf{x}+\overline{\mathbf{x}}}\, \mathrm{i}\,(\boldsymbol{\xi}) = [(\mathrm{i} \circ \varphi^{-1})'(\varphi(\mathbf{x} + \overline{\mathbf{x}}))](\boldsymbol{\xi}) = Z\boldsymbol{\xi}$, and hence $(\mathrm{T}_{\mathbf{x}+\overline{\mathbf{x}}}\, \mathrm{i})^{+}(\mathbf{x}) = Z^{\mathsf{T}}\mathbf{x}$. From (7), we obtain the time-dependent differential equation on $\overline{\mathbf{x}} + S$ given by

$$\dot{\boldsymbol{\xi}} = Z^{\mathsf{T}} X(t, \overline{\mathbf{x}} + Z\boldsymbol{\xi}), \quad \boldsymbol{\xi}(0) = Z^{\mathsf{T}}\mathbf{x}_0$$

Then $\widehat{X}(\boldsymbol{\xi}) := (\boldsymbol{\xi}, Z^{\mathsf{T}} X(t, \overline{\mathbf{x}} + Z\boldsymbol{\xi}))$ is the reduced-order model of (8) in the manifold $S + \overline{\mathbf{x}}$. Observe that $\boldsymbol{\xi} \in \mathbb{R}^r$ and $\mathbf{u} \in \mathbb{R}^n$.

In the above example, it is clear that the reduced-order model is also a vector field because the map $\mathrm{T}_{\mathbf{x}+\overline{\mathbf{x}}}\, \mathrm{i} = Z$ is independent of the point $\mathbf{x} + \overline{\mathbf{x}}$. Unfortunately, it is not true for other manifolds in general as we show in the next example.

Example 2. Let us consider the northern hemisphere of the manifold $\mathbb{S}^2 = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x}^{\mathsf{T}}\mathbf{x} = 1\}$, that is, $\mathbb{M} = \{(x, y, z) \in \mathbb{S}^2 : z > 0\}$. In this case, we consider the chart (U, φ) given by the open set $U = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} \subset \mathbb{R}^2$ and the map $\varphi : \mathbb{M} \longrightarrow U$, where $\varphi(x, y, z) = (x, y)$, clearly $\varphi^{-1}(x, y) = (x, y, \sqrt{1 - x^2 - y^2})$. Then $\mathbb{M}$ as an immersed manifold in $\mathbb{R}^3$ is also described by $\mathbf{x} = (\mathrm{i} \circ \varphi^{-1})(x, y) = (x, y, \sqrt{1 - x^2 - y^2})$, defined over the open set U. Then

$$\mathrm{T}_{\mathbf{x}}\, \mathrm{i} = (\mathrm{i} \circ \varphi^{-1})'(x, y) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{-x}{\sqrt{1-x^2-y^2}} & \frac{-y}{\sqrt{1-x^2-y^2}} \end{bmatrix}$$

is a full-rank matrix for all $(x, y) \in U$ and hence its pseudo-inverse is

$$\mathrm{T}_{\mathbf{x}}\, \mathrm{i}^{+} = \begin{bmatrix} (1 - x^2) & -xy & -x\sqrt{1 - x^2 - y^2} \\ -xy & (1 - y^2) & -y\sqrt{1 - x^2 - y^2} \end{bmatrix}$$

Assume that $X : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is a vector field in $\mathbb{R}^3$. Then (7) appears as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} (1 - x^2) & -xy & -x\sqrt{1 - x^2 - y^2} \\ -xy & (1 - y^2) & -y\sqrt{1 - x^2 - y^2} \end{bmatrix} X(x, y, \sqrt{1 - x^2 - y^2}) \tag{9}$$

where $[x(0) \quad y(0)]^{\mathsf{T}} \in U$. In this case (9) defines a vector field, namely $\widehat{X}_{(U,\varphi)}$, on $U \subset \mathbb{S}^2$ and (U, φ) is a chart for the atlas $\mathcal{A} = \{(U_a^b, \varphi_a^b) : a \in \{x, y, z\}, b \in \{+, -\}\}$ of $\mathbb{S}^2$ given by the open sets

$$U_x^{+} := \{(x, y, z) : x > 0\}, \;\; U_y^{+} := \{(x, y, z) : y > 0\}, \;\; U_z^{+} := \{(x, y, z) : z > 0\}$$
$$U_x^{-} := \{(x, y, z) : x < 0\}, \;\; U_y^{-} := \{(x, y, z) : y < 0\}, \;\; U_z^{-} := \{(x, y, z) : z < 0\}$$

and where $\varphi_a^b(x,y,z) \in \mathbb{R}^2$ is obtained by removing $a \in \{x,y,z\}$ from $(x,y,z) \in \mathbb{R}^3$. We can proceed in a similar way for each chart $(U_a^b, \psi_a^b) \in \mathcal{A}$, obtaining a vector field $\widehat{X}_{(U_a^b,\psi_a^b)}$. However, this procedure does not ensure the existence of a vector field $\widehat{X}$ on $\mathbb{S}^2$, constructed by using $\widehat{X}_{(U_a^b,\psi_a^b)}$ for $(U_a^b, \psi_a^b) \in \mathcal{A}$. Observe that the existence of $\mathbf{x} \in \mathbb{S}^2$ satisfying that $\mathbf{x} \in U_a^b \cap U_{a'}^{b'}$ implies that a possible definition of $\widehat{X}(\mathbf{x})$ will depend on the choice of the chart.

The next section is devoted to state and prove the main result of this paper.

3.1. Statement and proof of main result

Let $n = \dim V$ and $r = \dim \mathrm{T}_{\mathbf{v}}(\mathbb{M})$ for all $\mathbf{v} \in \mathbb{M}$. Clearly, $r < n$. We recall the definition of the non-compact Stiefel manifold of $\mathbb{R}^{n\times r}$ denoted by $\mathcal{M}_r(\mathbb{R}^{n\times r}) = \{A \in \mathbb{R}^{n\times r} : \mathrm{rank}\,(A) = r\}$, which is an open set in $\mathbb{R}^{n\times r}$ and hence a manifold. Assume that $\mathfrak{i}$ is a $\mathcal{C}^p$-immersion ($p \ge 2$). Then for each $\mathbf{v} \in \mathbb{M}$ the linear map $\mathrm{T}_{\mathbf{v}}\,\mathfrak{i} : \mathbb{T}_{\mathbf{v}}(\mathbb{M}) \longrightarrow V$ can be identify with a matrix, also denoted by $\mathrm{T}_{\mathbf{v}}\,\mathfrak{i}$, in $\mathbb{R}^{n\times r}$. Hence, we can write its Moore–Penrose pseudo-inverse as

$$\mathrm{T}_{\mathbf{v}}\,\mathfrak{i}^+ := (\mathrm{T}_{\mathbf{v}}\,\mathfrak{i}^\intercal\, \mathrm{T}_{\mathbf{v}}\,\mathfrak{i})^{-1}\, \mathrm{T}_{\mathbf{v}}\,\mathfrak{i}^\intercal \in \mathbb{R}^{r\times n}$$

It allows us to introduce a map $\mathrm{MP} : \mathbb{M} \longrightarrow \mathbb{R}^{r\times n}$ defined by $\mathrm{MP}(\mathbf{v}) = \mathrm{T}_{\mathbf{v}}\,\mathfrak{i}^+$. Our first result is the following

Lemma 3.4. *Let V be a finite dimensional inner product space and $(\mathbb{M}, \mathcal{A})$ be a $\mathcal{C}^p$-manifold ($p \ge 2$) such that $\mathbb{M} \subset V$. Assume that the standard inclusion map $\mathfrak{i} : \mathbb{M} \longrightarrow V$ is an $\mathcal{C}^p$ immersion. Then the map MP is a $\mathcal{C}^{p-1}$-morphism between manifolds.*

Proof. Given $\mathbf{v} \in \mathbb{M}$ take $(U, \varphi) \in \mathcal{A}$ be such that $\mathbf{v} \in U$. Since the standard inclusion map is an immersion, by Proposition 3.2, we known that $\mathrm{T}_{\mathbf{v}}\mathfrak{i}$ is injective. Hence $\mathrm{T}_{\mathbf{v}}\,\mathfrak{i} \in \mathbb{R}^{n\times r}$ has $\mathrm{rank}\,(\mathrm{T}_{\mathbf{v}}\mathfrak{i}) = r$, that is,

$$\mathrm{T}_{\mathbf{v}}\mathfrak{i} = [(\mathfrak{i} \circ \varphi^{-1})'(\varphi(\mathbf{v}))] \in \mathcal{M}_r(\mathbb{R}^{n\times r})$$

and $(\mathfrak{i} \circ \varphi^{-1})' : \varphi(U) \longrightarrow \mathbb{R}^{n\times r}$ is a $\mathcal{C}^{p-1}$ map ($p \ge 2$). Since $\mathcal{M}_r(\mathbb{R}^{n\times r})$ is open in $\mathbb{R}^{n\times r}$, there exists an open set $W \subset \varphi(U)$, where $\varphi(\mathbf{v}) \in W$, and such that $(\mathfrak{i} \circ \varphi^{-1})'(W) \subset \mathcal{M}_r(\mathbb{R}^{n\times r})$. We point out that since $\mathrm{T}_{\mathbf{v}}\mathfrak{i} \in \mathcal{M}_r(\mathbb{R}^{n\times r})$, then $\mathrm{T}_{\mathbf{v}}\mathfrak{i}\,\mathrm{T}_{\mathbf{v}}\mathfrak{i}^+$ is the orthogonal projection onto $\mathrm{T}_{\mathbf{v}}\mathfrak{i}(\mathbb{T}_{\mathbf{v}}(\mathbb{M}))$. In consequence, we obtain that $[(\mathfrak{i}\circ\varphi^{-1})'(\varphi(\mathbf{u}))][(\mathfrak{i}\circ\varphi^{-1})'(\varphi(\mathbf{u}))]^+$ is the orthogonal projection onto $[(\mathfrak{i} \circ \varphi^{-1})'(\varphi(\mathbf{u}))](\mathrm{T}_{\mathbf{u}}U) = \mathrm{T}_{\mathbf{u}}\mathfrak{i}(\mathrm{T}_{\mathbf{u}}U) = \mathrm{T}_{\mathbf{u}}\mathfrak{i}(\mathrm{T}_{\mathbf{u}}\mathbb{M})$ for all $\mathbf{u} \in \varphi^{-1}(W) \subset U \subset \mathbb{M}$. Hence,

$$[(\mathfrak{i} \circ \varphi^{-1})'(\varphi(\mathbf{u}))]^+ = \mathrm{T}_{\mathbf{u}}\mathfrak{i}^+$$

and it maps $\mathrm{T}_{\mathbf{u}}\mathfrak{i}(\mathrm{T}_{\mathbf{u}}\mathbb{M})$ to $\mathrm{T}_{\mathbf{u}}\mathbb{M}$. Thus, we have that the map

$$(\mathrm{MP} \circ \varphi^{-1}) : W \subset \varphi(U) \longrightarrow \mathbb{R}^{r\times n}, \quad \varphi(\mathbf{u}) \mapsto \mathrm{T}_{\mathbf{u}}\mathfrak{i}^+ = [(\mathfrak{i} \circ \varphi^{-1})'(\varphi(\mathbf{u}))]^+$$

is well defined as a morphism. Since, the map from $\mathcal{M}_r(\mathbb{R}^{n\times r})$ to $\mathbb{R}^{r\times n}$ given by $Z \mapsto Z^+ := (Z^\intercal Z)^{-1} Z^\intercal$ is analytic, then the lemma follows. □

The main result of this paper is the following.

Theorem 3.5. *Let V be a finite dimensional inner product space and $(\mathbb{M}, \mathcal{A})$ be a $\mathcal{C}^p$-manifold ($p \ge 2$) such that $\mathbb{M} \subset V$. Assume that the standard inclusion map $\mathfrak{i} : \mathbb{M} \longrightarrow V$ is an $\mathcal{C}^p$ immersion. If $X : V \longrightarrow V$ is a $\mathcal{C}^{p-1}$-vector field on V, then*

$$\widehat{X}(\mathbf{v}) := (\mathbf{v}, \mathrm{MP}(\mathbf{v})X(\mathfrak{i}(\mathbf{v}))) = \left(\mathbf{v}, \mathrm{T}_{\mathbf{v}}\,\mathfrak{i}^+ X(\mathfrak{i}(\mathbf{v}))\right) \tag{10}$$

is a $\mathcal{C}^{p-1}$-vector field on $\mathbb{M}$ such that

$$T_{\mathbf{v}}\,\mathfrak{i}(\mathrm{MP}(\mathbf{v})X(\mathfrak{i}(\mathbf{v}))) = \Pi_{\mathbf{v}}(X(\mathfrak{i}(\mathbf{v}))) \tag{11}$$

holds for all $\mathbf{v} \in \mathbb{M}$.

Proof. Without loss of generality, we may assume that $V = \mathbb{R}^n$. From Lemma 3.4, the map MP is a $\mathcal{C}^{p-1}$-morphism ($p \ge 2$). In consequence, the map

$$W \subset \varphi(U) \longrightarrow \mathbb{R}^{r\times n} \times \mathbb{R}^n, \quad \mathbf{u} \mapsto (\mathrm{MP}(\mathbf{v}), X(\mathfrak{i}(\mathbf{u})))$$

is also a $\mathcal{C}^{p-1}$-morphism. Let us consider the evaluation map

$$\mathfrak{eval} : \mathbb{R}^{r\times n} \times \mathbb{R}^n \longrightarrow \mathbb{R}^r$$

given by $\mathfrak{eval}(L, \mathbf{u}) := L\mathbf{u}$. It is clearly a $\mathcal{C}^\infty$-bilinear map and

$$\mathrm{MP}(\mathbf{v})X(\mathrm{i}(\mathbf{v})) = \mathfrak{eval}\,(\mathrm{MP}(\mathbf{v}), X(\mathrm{i}(\mathbf{v})))$$

is a $\mathcal{C}^{p-1}$-morphism. Thus, we conclude that $\widehat{X}$ is a vector field on $\mathbb{M}$. Finally, (11) follows from the fact that $\Pi_\mathbf{v} = T_\mathbf{v}\,\mathrm{i}\,T_\mathbf{v}\,\mathrm{i}^+ = \mathrm{T}_\mathbf{v}\,\mathrm{i}\,\mathrm{MP}(\mathbf{v})$. □

Remark 1. The above theorem remains true for time-dependent vector fields $X : \mathbb{R} \times V \longrightarrow V$.

Theorem 3.5 says us that the reduced-order model given by (10) is a true dynamical system on the manifold $\mathbb{M}$, and from Theorem 2.6 we have ensured the existence and uniqueness of solutions for that reduced model.

3.2. A local-point estimate for the time-dependent error

In practice, we need to consider, for a given initial condition $\mathbf{v}_0 \in \mathbb{M}$, a local chart system $(U, \varphi) \in \mathcal{A}$, such that $\mathbf{v}_0 \in U$. Denote by $\boldsymbol{\xi} = \varphi(\mathbf{v})$, where $\mathbf{v} \in U$, the local coordinates on the manifold $\mathbb{M}$. Then we can write the dynamical system associated with (10) in $\varphi(U)$ as

$$\dot{\boldsymbol{\xi}} = \mathrm{MP}(\varphi^{-1}(\boldsymbol{\xi}))X((\mathrm{i} \circ \varphi^{-1})(\boldsymbol{\xi}))), \quad \boldsymbol{\xi}(0) = \varphi(\mathbf{v}_0) \tag{12}$$

The differential equation (12) is the reduced-order model in $\mathbb{M}$ of the dynamical system in V given by

$$\dot{\mathbf{v}} = X(\mathbf{v}), \quad \mathbf{v}(0) = \mathbf{v}_0 \in \mathbb{M} \tag{13}$$

To describe the time-dependent error, we take into account

$$\mathbf{e}(t) = \mathbf{v}(t) - (\mathrm{i} \circ \varphi^{-1})(\boldsymbol{\xi}(t))$$

where $\mathbf{v}(t)$ is the solution to (13) and $\boldsymbol{\xi}(t)$ the solution to (12). Clearly, $\mathbf{e}(0) = \mathbf{0} \in V$, and if $\mathbb{M}$ is an invariant manifold for the vector field X, that is, $X(\mathbf{v}) \in \mathrm{T}_\mathbf{v}\,\mathrm{i}(\mathbb{T}_\mathbf{v}(\mathbb{M}))$ for all $\mathbf{v} \in \mathbb{M}$, then $\mathbf{e}(t) = \mathbf{0}$ for any time t. Now, we introduce $\widehat{\mathbf{v}}(t) := (\mathrm{i} \circ \varphi^{-1})(\boldsymbol{\xi}(t)) \in \mathbb{M}$, and from (11), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\mathbf{v}}(t) = \mathrm{T}_{\widehat{\mathbf{v}}(t)}\,\mathrm{i}(\dot{\boldsymbol{\xi}}(t)) = \Pi_{\widehat{\mathbf{v}}(t)}(X(\widehat{\mathbf{v}}(t)))$$

Recall that $\Pi_{\widehat{\mathbf{v}}(t)}$ is the orthogonal projection onto $\mathrm{T}_{\widehat{\mathbf{v}}(t)}\,\mathrm{i}(\mathbb{T}_{\widehat{\mathbf{v}}(t)}(\mathbb{M}))$. Since $\mathbf{v}(t) = \mathbf{v}_0 + \int_0^t X(\mathbf{v}(s))ds$ and $\widehat{\mathbf{v}}(t) = \mathbf{v}_0 + \int_0^t \Pi_{\widehat{\mathbf{v}}(s)}(X(\widehat{\mathbf{v}}(s)))ds$ we can write

$$\begin{aligned}
\mathbf{e}(t) &= \int_0^t \left(X(\mathbf{v}(s)) - \Pi_{\widehat{\mathbf{v}}(s)}(X(\widehat{\mathbf{v}}(s)))\right) \mathrm{d}s \\
&= \int_0^t \left(X(\mathbf{v}(s)) - \Pi_{\widehat{\mathbf{v}}(s)}(X(\mathbf{v}(s)))\right) \mathrm{d}s + \int_0^t \left(\Pi_{\widehat{\mathbf{v}}(s)}(X(\mathbf{v}(s))) - \Pi_{\widehat{\mathbf{v}}(s)}(X(\widehat{\mathbf{v}}(s)))\right) \mathrm{d}s \\
&= \int_0^t \left(id_V - \Pi_{\widehat{\mathbf{v}}(s)}\right)(X(\mathbf{v}(s))\,\mathrm{d}s + \int_0^t \Pi_{\widehat{\mathbf{v}}(s)}\,(X(\mathbf{v}(s)) - X(\widehat{\mathbf{v}}(s))\,\mathrm{d}s
\end{aligned}$$

In consequence, the error speed

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{e}(t) = \left(id_V - \Pi_{\widehat{\mathbf{v}}(t)}\right)(X(\mathbf{v}(t)) + \Pi_{\widehat{\mathbf{v}}(t)}\,(X(\mathbf{v}(t)) - X(\widehat{\mathbf{v}}(t)))$$

can be decomposed into two orthogonal components. The first one represents the normal error speed, and it is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{e}_\perp(t) := \left(id_V - \Pi_{\widehat{\mathbf{v}}(t)}\right)(X(\mathbf{v}(t)) \in \mathrm{T}_{\widehat{\mathbf{v}}(t)}\mathrm{i}(\mathbb{T}_{\widehat{\mathbf{v}}(t)}(\mathbb{M}))^\perp$$

and the second is associated with the tangent error speed and is defined as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{e}_\parallel(t) := \Pi_{\widehat{\mathbf{v}}(t)}\,(X(\mathbf{v}(t)) - X(\widehat{\mathbf{v}}(t))) \in \mathrm{T}_{\widehat{\mathbf{v}}(t)}\mathrm{i}(\mathbb{T}_{\widehat{\mathbf{v}}(t)}(\mathbb{M}))$$

Recall that $\mathbf{e}(t) = \mathbf{v}(t) - \widehat{\mathbf{v}}(t)$. Since $X \in \mathcal{C}^p$-vector field $(p \geq 1)$ on V, by using the Taylor's expansion, we have

$$X(\mathbf{v}(t)) - X(\widehat{\mathbf{v}}(t)) = X(\widehat{\mathbf{v}}(t) + \mathbf{e}(t)) - X(\widehat{\mathbf{v}}(t)) = X'(\widehat{\mathbf{v}}(t))\mathbf{e}(t) + \mathcal{O}(\|\mathbf{e}(t)\|^2)$$

Then,

$$\Pi_{\widehat{\mathbf{v}}(t)}\left(X(\mathbf{v}(t)) - X(\widehat{\mathbf{v}}(t))\right) = \Pi_{\widehat{\mathbf{v}}(t)}\left(X(\widehat{\mathbf{v}}(t) + \mathbf{e}(t)) - X(\widehat{\mathbf{v}}(t))\right) \approx \Pi_{\widehat{\mathbf{v}}(t)}\left(X'(\widehat{\mathbf{v}}(t))\mathbf{e}(t)\right)$$

Finally, we obtain the following expression

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{e}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{e}_{\|}(t) + \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{e}_{\perp}(t) \approx \Pi_{\widehat{\mathbf{v}}(t)}\left(X'(\widehat{\mathbf{v}}(t))\mathbf{e}(t)\right) + \left(id_V - \Pi_{\widehat{\mathbf{v}}(t)}\right)(X(\mathbf{v}(t))$$

It allows us to propose a definition of a local point estimate $\widehat{\mathbf{e}}(t)$ for the time-dependent error $\mathbf{e}(t)$ as follows.

Definition 3.6. Let be $\mathbf{v}_0 \in \mathbb{M}$, $(U, \varphi) \in \mathcal{A}$ be such that $\mathbf{v}_0 \in U$ and $\widehat{\mathbf{v}}(t) = (\mathrm{i} \circ \varphi^{-1})(\boldsymbol{\xi}(t))$ with $\boldsymbol{\xi}(t)$, which solves (12). We define the local point estimate error $\widehat{\mathbf{e}}(t)$ of (13) in $(U, \varphi) \in \mathcal{A}$ as the solution to the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\mathbf{e}}(t) = \Pi_{\widehat{\mathbf{v}}(t)}(X'(\widehat{\mathbf{v}}(t))\widehat{\mathbf{e}}(t)) + \left(id_V - \Pi_{\widehat{\mathbf{v}}(t)}\right)(X(\widehat{\mathbf{v}}(t))), \quad \widehat{\mathbf{e}}(0) = \mathbf{0} \tag{14}$$

In the Example 1, we known that $\Pi_{\widehat{\mathbf{v}}(t)} = \Pi_S = ZZ^\intercal \in \mathrm{R}^{n\times n}$ for all time t, and hence (14) for a time independent vector field $X : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\mathbf{e}}(t) = \Pi_S X'(\widehat{\mathbf{v}}(t))\,\widehat{\mathbf{e}}(t) + (id_{\mathbb{R}^n} - \Pi_S)\,X(\widehat{\mathbf{v}}(t)), \quad \widehat{\mathbf{e}}(0) = \mathbf{0}$$

In particular, for a linear vector field $X(\mathbf{v}) = A\mathbf{v}$, where $A \in \mathbb{R}^{N\times N}$, we obtain the linear differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\mathbf{e}}(t) = \Pi_S A\,\widehat{\mathbf{e}}(t) + (id_{\mathbb{R}^n} - \Pi_S)\,A\widehat{\mathbf{v}}(t), \quad \widehat{\mathbf{e}}(0) = \mathbf{0}$$

Finally, by using the method of variation of parameters, we have the following expression for the point estimate error

$$\widehat{\mathbf{e}}(t) = \int_0^t \mathrm{e}^{(t-s)\Pi_S A}\,(id_{\mathbb{R}^n} - \Pi_S)\,A\widehat{\mathbf{v}}(s)\,\mathrm{d}s$$

which allows us to give the following bound

$$\|\widehat{\mathbf{e}}(t)\| \leq \left(\sup_{0\leq s\leq t} \|(id_{\mathbb{R}^n} - \Pi_S)\,A\widehat{\mathbf{v}}(s)\|\right)(\mathrm{e}^{\|\Pi_S A\|t} - 1)$$

4. Conclusions

In this paper, we give a constructive approach under a geometric framework of the reduced-order model of a vector field defined over a finite-dimensional inner product space. Moreover, a local point estimate for the time dependent error is given. Establishing the properties of this proposed estimate is part of our future research and will be published elsewhere.

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