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On the influence of a magnetic field on the separation of the boundary layer of a non-Newtonian MHD medium ☆

Sur l'influence d'un champ magnétique sur la séparation de la couche limite d'un milieu MHD non newtonien

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A B S T R A C T

We study the behavior of a magnetohydrodynamic (MHD) stationary boundary layer in a framework modified according to O.A. Ladyzhenskaya. We estimate the shift of a separation point under the influence of a magnetic field.

R É S U M É

Nous étudions le comportement d'une couche limite stationnaire magnétohydrodynamique (MHD) dans le cadre modifié par O.A. Ladyzhenskaya. Nous estimons le déplacement du point de séparation sous l'influence d'un champ magnétique.

1. Introduction

To describe boundary layers of fluids, mathematicians use the Prandtl system of equations, which is obtained from Navier–Stokes system by rescaling. The solvability of boundary value problems as well as initial boundary value problems for boundary layers of Newtonian fluids, and separation of boundary layers were investigated in [1], [2] and [3]. Similar topics for non-Newtonian fluids and liquid crystals were studied in [4], [5], [6], [7], [8]. Nonuniqueness of solutions to the Navier–Stokes system leads to several modifications of this system. We consider the O.A. Ladyzhenskaya modification,

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Fig. 1. The separation of a boundary layer.

where a nonlinear viscosity was suggested also for Newtonian fluids [9], [10]. The theory of boundary layer for the O.A. Ladyzhenskaya fluids was developed in [11].

In this paper, we study a boundary layer of magnetic fluid of O.A. Ladyzhenskaya. We show the influence of magnetic field on the separation of the boundary layer (Fig. 1).

2. Settings

In the two-dimensional case, the stationary system of modified magnetic hydrodynamical boundary layer has the form

$$\begin{cases} \nu \dfrac{\partial}{\partial y}\left(\left(1+k\left(\dfrac{\partial u}{\partial y}\right)^2\right)\dfrac{\partial u}{\partial y}\right) - u\dfrac{\partial u}{\partial x} - v\dfrac{\partial u}{\partial y} + B^2(U-u) = -U\dfrac{\mathrm{d}U}{\mathrm{d}x}, \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0, \end{cases} \tag{1}$$

where ν denotes the kinematic viscosity, k is a small positive constant, the density ρ and the medium conduction σ are equal to one, $B(x)$, $U(x)$ are given functions. The function $U(x)$, the pressure $p(x)$, and the electromagnetic field $B(x)$ satisfy

$$U\frac{\mathrm{d}U}{\mathrm{d}x} = -\frac{\mathrm{d}p}{\mathrm{d}x} - E\frac{B}{\sigma} - B^2 U.$$

We consider equations (1) in $D = \{0 < x < X,\ 0 < y < \infty\}$ with the boundary conditions

$$\begin{gathered} u(0,y) = u_0(y), \quad u(x,0) = 0, \quad v(x,0) = v_0(x), \\ u(x,y) \to U(x) \text{ as } y \to +\infty. \end{gathered} \tag{2}$$

Here the functions $u_0(x)$, $v_0(x)$ are given.

Definition 2.1. It is said that the functions $u(x,y)$ and $v(x,y)$ are a classical solution to problem (1), (2), if they satisfy the following properties: u and v are continuous in $\overline{D}$, have in D continuous derivatives appearing in Eq. (1); and satisfy pointwise equations (1) and conditions (2).

To prove the existence and the uniqueness results, we need some more restrictions to given functions. Assume that the following conditions hold true.

Conditions I:

- $U(x) > 0$; $u_0(y) > 0$ as $y > 0$;
- $u_0(0) = 0$, $u_0' > 0$, $u_0(y) \to U(0)$ as $y \to \infty$;
- $\dfrac{\mathrm{d}U}{\mathrm{d}x}$, $v_0(x)$, $B(x)$ are differentiable on $[0, X]$;
- $u_0(y)$, $u_0'(y)$, $u_0''(y)$ are bounded as $0 \le y < \infty$ and satisfy the Hölder condition;
- in $(0,0)$ the following compatibility condition holds true as $y \to 0$:

$$\begin{gathered} \nu\frac{\partial}{\partial y}\left(\left(1+k\left(\frac{\partial u_0}{\partial y}\right)^2\right)\frac{\partial u_0}{\partial y}\right) - v_0(0)\frac{\partial u_0}{\partial y} + B^2(0)(U(0)-u_0(y)) + \\ + U(0)\frac{\mathrm{d}U(0)}{\mathrm{d}x} = O(y^2). \end{gathered}$$

The system (1) with conditions (2) can be rewritten as a quasilinear differential equation by means of the Mises transform. To this end, we introduce new independent variables $x = x$, $\psi = \psi(x, y)$, where

$$u = \frac{\partial \psi}{\partial y}, \quad v - v_0(x) = -\frac{\partial \psi}{\partial x}, \quad \psi(x, 0) = 0,$$

and a new unknown function $w(x, \psi) = u^2(x, y)$. Then

$$u = \sqrt{w}; \qquad y = \int_0^\psi \frac{\mathrm{d}\psi}{\sqrt{w(x,\psi)}};$$

$$\frac{\partial u}{\partial x} = \frac{1}{2\sqrt{w}}\frac{\partial w}{\partial x} + \frac{1}{2\sqrt{w}}\frac{\partial w}{\partial \psi}\frac{\partial \psi}{\partial x};$$

$$\frac{\partial u}{\partial y} = \frac{1}{2\sqrt{w}}\frac{\partial w}{\partial y} = \frac{1}{2\sqrt{w}}\frac{\partial w}{\partial \psi}\frac{\partial \psi}{\partial y} = \frac{1}{2}\frac{\partial w}{\partial \psi};$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\Big(\frac{1}{2}\frac{\partial w}{\partial \psi}\Big) = \frac{1}{2}\frac{\partial}{\partial \psi}\Big(\frac{\partial w}{\partial \psi}\Big)\frac{\partial \psi}{\partial y} = \frac{\sqrt{w}}{2}\frac{\partial^2 w}{\partial \psi^2}.$$

Substituting this expressions into the first equation of the system (1), we get

$$\nu\frac{\sqrt{w}}{2}\frac{\partial^2 w}{\partial \psi^2}\left(1 + 3k\Big(\frac{1}{2}\frac{\partial w}{\partial \psi}\Big)^2\right) - \sqrt{w}\Big(\frac{1}{2\sqrt{w}}\frac{\partial w}{\partial x} + \frac{1}{2\sqrt{w}}\frac{\partial w}{\partial \psi}\frac{\partial \psi}{\partial x}\Big) -$$

$$-\frac{1}{2}\Big(v_0(x) - \frac{\partial \psi}{\partial x}\Big)\frac{\partial w}{\partial \psi} + B^2(U - \sqrt{w}) = -U\frac{\partial U}{\partial x}.$$

Finally, the new equation has the form

$$\nu\sqrt{w}\left(1 + \frac{3}{4}k\Big(\frac{\partial w}{\partial \psi}\Big)^2\right)\frac{\partial^2 w}{\partial \psi^2} - \frac{\partial w}{\partial x} - v_0\frac{\partial w}{\partial \psi} + 2\,B^2(U - \sqrt{w}) = -2\,U\frac{\partial U}{\partial x} \tag{3}$$

in domain $G = \{0 < x < X, \quad 0 < \psi < \infty\}$ with conditions

$$w(0,\psi) = w_0(\psi),\ w(x,0) = 0,\ w(x,\psi) \to U^2(x) \quad \text{as} \quad \psi \to \infty. \tag{4}$$

The function $w_0(\psi)$ is defined from the equation

$$w_0\Big(\int_0^y u_0(\eta)\mathrm{d}\eta\Big) \equiv u_0^2(y).$$

The above conditions in terms of the new unknown function become the following ones.

Conditions II:

- $w_0(\psi) > 0$, as $\psi > 0$;
- $w_0(0) = 0,\ w_0'(0) > 0$;
- $w_0(\psi) \to U^2(x)$ as $\psi \to \infty$;
- $\dfrac{\mathrm{d}U}{\mathrm{d}x}$, $v_0(x)$, $B(x)$ are continuously differentiable as $x \in [0, X]$;
- $w_0(\psi)$, $w_0'(\psi)$, $w_0''(\psi)$ are bounded as $0 \le \psi < \infty$ and Hölder continuous;
- in $(0,0)$, the compatibility condition holds true as $\psi \to 0$:

$$\begin{aligned}\mu(\psi) =& \nu\sqrt{w_0(\psi)}\left(\left(1 + \frac{3}{4}k\Big(\frac{\partial w_0(\psi)}{\partial \psi}\Big)^2\right)\frac{\partial^2 w_0(\psi)}{\partial \psi^2}\right) - v_0(0)\frac{\partial w_0(\psi)}{\partial \psi} + \\ &+ 2B^2(0)(U(0) - \sqrt{w_0(\psi)}) + 2U(0)\frac{\mathrm{d}U(0)}{\mathrm{d}x} = O(\psi).\end{aligned} \tag{5}$$

3. Existence and uniqueness

The following assertions hold true.

Theorem 3.1 *(Existence). Suppose that the functions $U(x)$, $u_0(y)$, $v_0(x)$ satisfy the conditions **I**. In the domain D for some $X>0$, the problem (1), (2) has a solution $u(x)$, $v(x)$, satisfying the following properties: $u(x)$ is continuous and bounded in D; $u>0$ as $y>0$; $\dfrac{\partial u}{\partial y}>m>0$ as $0\le y\le y_0$; m, $y_0=\mathrm{const}$; $\dfrac{\partial u}{\partial y}$, $\dfrac{\partial^2 u}{\partial y^2}$ are bounded and continuous in any finite subdomain of D. The solution to problem (1), (2) does exist for any $X>0$ if*

$$U\left(\frac{\mathrm{d}U}{\mathrm{d}x}+B^2\right)\ge 0 \text{ and } v_0\le 0 \quad \text{or} \quad U\left(\frac{\mathrm{d}U}{\mathrm{d}x}+B^2\right)\ge \alpha=\mathrm{const}>0.$$

Theorem 3.2 *(Existence). Suppose that the functions $U(x)$, $w_0(\psi)$, $v_0(x)$ satisfy the conditions **II**. Then problem (3), (4) has a solution $w(x,\psi)$ in domain $G=\{0<x<X,\ 0<\psi<\infty\}$ for some X. This solution satisfies the following properties: $w(x,\psi)$ is bounded in $\overline{G}$, $w(x,\psi)>0$ as $\psi>0$,*

$$\left|\frac{\partial w}{\partial \psi}\right|\le M, \qquad \left|\sqrt{w}\frac{\partial^2 w}{\partial \psi^2}\right|\le M \quad \text{in} \quad G,$$

$$\left|\frac{\partial w}{\partial x}\right|\le M\psi^{1-\beta}, \qquad \frac{\partial w}{\partial \psi}\ge m>0 \quad \text{as} \quad 0\le\psi\le\psi_1, \quad 0<\beta<\frac{1}{2},$$

where the positive constants M, m, ψ_1 depend only on X, U, w_0, v_0.

If $U\left(\dfrac{\mathrm{d}U}{\mathrm{d}x}+B^2\right)\ge 0$ and $v_0(x)\le 0$ or $U\left(\dfrac{\mathrm{d}U}{\mathrm{d}x}+B^2\right)>0$, then such a solution does exist in G for any $X>0$.

Theorem 3.3 *(Uniqueness). Assume that the functions $u(x,y)$, $v(x,y)$ satisfy the system (1) in the domain $D=\{0<x<X,\ 0<y<\infty\}$, are continuous in $\overline{D}$ and satisfy the conditions (2); moreover, the inequalities $0<u<C$ as $y>0$,*

$$k_1 y\le u\le k_2 y \quad \text{as} \quad 0<y<y_0, \tag{6}$$

$$\frac{\partial^2 u}{\partial y^2}\le k_3 \quad \text{in} \quad D \tag{7}$$

hold true, where C, k_j, y_0 are positive constants. Then the solution $u(x,y)$, $v(x,y)$ to the problem (1), (2), satisfying the conditions above, is unique. If $U\left(\dfrac{\mathrm{d}U}{\mathrm{d}x}+B^2\right)\ge 0$ and $v_0\le 0$, then the conditions (6), (7) can be omitted. If $U\left(\dfrac{\mathrm{d}U}{\mathrm{d}x}+B^2\right)\ge 0$, and v_0 is an arbitrary function, then the condition (6) can be omitted (i.e. we only need the condition (7)).

Theorem 3.4 *(Uniqueness). The solution $w(x,\psi)$ to the problem (3), (4), continuous and bounded in $\overline{G}$, satisfying the inequalities $k_1\psi\le w(x,\psi)\le k_2\psi$ as $\psi\le\psi_1$; $w(x,\psi)\ge a>0$ as $\psi\ge\psi_1$; $\sqrt{w}\frac{\partial^2 w}{\partial\psi^2}\le M$, is unique in $\overline{G}$, where k_1, k_2, ψ_1, M are positive constants.*

The proof of the theorems is based on the following statements.

Lemma 3.5. *Assume that in $G=\{0<x<X,\ 0<\psi<\infty\}$, there exists a solution $w(x,\psi)$ to the problem (3), (4), with the following properties: the function $w(x,\psi)$ is bounded in $\overline{G}$; $w(x,\psi)>0$ as $\psi>0$. There exist constants M, m, ψ_1, only dependent on X, u_0, v_0, and $p(x)$, such that*

$$\left|\frac{\partial w}{\partial \psi}\right|\le M, \qquad \left|\sqrt{w}\,\frac{\partial^2 w}{\partial \psi^2}\right|\le M, \qquad (x,\psi)\in G; \tag{8}$$

moreover,

$$\left|\frac{\partial w}{\partial x}\right|\le M\psi^{1-\beta}, \qquad \frac{\partial w}{\partial \psi}\ge m>0 \text{ as } 0\le\psi\le\psi_1,\ 0<\beta<\frac{1}{2}.$$

Then in $D=\{0<x<X,\ 0<y<\infty\}$, there exists a solution $u(x,y)$, $v(x,y)$ to the problem (1), (2), satisfying the following properties: the function $u(x,y)$ is continuous and bounded in $\overline{D}$, $u>0$ as $y>0$; $\frac{\partial u}{\partial y}>m_1>0$ as $0<y<y_0$ (m_1 and y_0 are some constants); $\frac{\partial u}{\partial y}$ and $\frac{\partial^2 u}{\partial y^2}$ are continuous and bounded in D; $\frac{\partial u}{\partial x}$, v, $\frac{\partial v}{\partial y}$ are continuous and bounded in any finite subdomain of $\overline{D}$.

Consider the domain $G_\varepsilon = \left\{0 < x < X,\ 0 < \psi < \dfrac{1}{\varepsilon}\right\}$. Let

$$\Gamma_\varepsilon = \{0 < x < X,\ \psi = 0\} \cup \left\{x = 0,\ 0 < \psi < \frac{1}{\varepsilon}\right\} \cup \left\{0 < x < X,\ \psi = \frac{1}{\varepsilon}\right\}.$$

In the domain G_ε, we consider Eq. (3) with conditions:

$$\begin{gathered} w(x,0) = w_0(\varepsilon)\exp\left(\frac{\mu(\varepsilon)x}{w_0(\varepsilon)}\right), \qquad w(0,\psi) = w_0(\varepsilon+\psi), \\ w\left(x, \frac{1}{\varepsilon}\right) = w_0\left(\varepsilon + \frac{1}{\varepsilon}\right)\exp\left(\frac{\mu(\varepsilon+1/\varepsilon)x}{w_0(\varepsilon+1/\varepsilon)}\right). \end{gathered} \tag{9}$$

Lemma 3.6. *If the problem (3), (9) has a positive solution $w_\varepsilon(x,\psi)$ in the domain G_ε, then there exist numbers $X > 0$ and $\varepsilon_0 > 0$ such that, for any $\varepsilon > \varepsilon_0$, the inequality*

$$w_\varepsilon(x,\psi) \ge w_\varepsilon(x,0) + f(\psi)(1 + \mathrm{e}^{-\alpha x}), \quad (x,\psi) \in G_\varepsilon \tag{10}$$

holds true, where $\alpha > 0$, and the function $f(\psi)$ satisfies $f(\psi) = A_1\psi^{4/3} + A_2\psi$ as $\psi \le 1$ and $f(1) \le f(\psi) \le A_3$ as $\psi > 1$; in addition, $|f'(\psi)| \le A_4$, $|f''(\psi)| \le A_5$ as $\psi > 1$. Here A_i are positive constants.

If $U\left(\dfrac{\mathrm{d}U}{\mathrm{d}x} + B^2\right) \ge 0$ and $v_0(x) \le 0$, then in the domain G_ε, the a priori estimate

$$w_\varepsilon(x,\psi) \ge w_\varepsilon(x,0) + f(\psi)\,\mathrm{e}^{-\alpha x} \tag{11}$$

holds for any $X > 0$ and sufficiently small $\varepsilon_0 > 0$.

If $U\left(\dfrac{\mathrm{d}U}{\mathrm{d}x} + B^2\right) \ge \beta_0 > 0$, then in the domain G_ε, the inequality

$$w_\varepsilon(x,\psi) \ge w_\varepsilon(x,0) + f(\psi) \tag{12}$$

holds true.

Lemma 3.7. *Let $w_\varepsilon(x,\psi)$ be a solution to the problem (3), (9). Then there exist positive constants M_1, M_2, M_3, independent of ε, such that*

$$0 < w_\varepsilon(x,\psi) < M_1, \tag{13}$$

$$M_2 < \left.\frac{\partial w}{\partial \psi}\right|_{\psi=0} < M_3. \tag{14}$$

Lemma 3.8. *There exists a constant M_4 such that*

$$\left|\frac{\partial w}{\partial \psi}\right| \le M_4, \qquad (x,\psi) \in G_\varepsilon.$$

Lemma 3.9. *There exist positive constants C_1, C_2 independent of ε such that in G_ε the estimates*

$$\frac{\partial w_\varepsilon}{\partial x} \ge -C_1, \qquad \sqrt{w_\varepsilon}\,\frac{\partial^2 w_\varepsilon}{\partial \psi^2} \ge -C_2$$

hold.

Lemma 3.10. *There exist positive constants C_3, C_4, C_5 independent of ε such that in domain G_ε the estimates*

$$\sqrt{w_\varepsilon}\,\frac{\partial^2 w_\varepsilon}{\partial \psi^2} \le C_3, \qquad \frac{\partial w_\varepsilon}{\partial x} \le C_4; \tag{15}$$

$$\frac{\partial w_\varepsilon}{\partial \psi} \le C_5, \qquad 0 \le \psi \le \widetilde{\psi} \tag{16}$$

hold true for some $\widetilde{\psi} \ge 0$.

Lemma 3.11. *In domain G_ε, the estimate*

$$\left|w_\varepsilon^{\beta-1}\frac{\partial w_\varepsilon}{\partial x}\right| \le C_6 \tag{17}$$

takes place, where C_6 does not depend on ε, $0 < \beta < 1/2$.

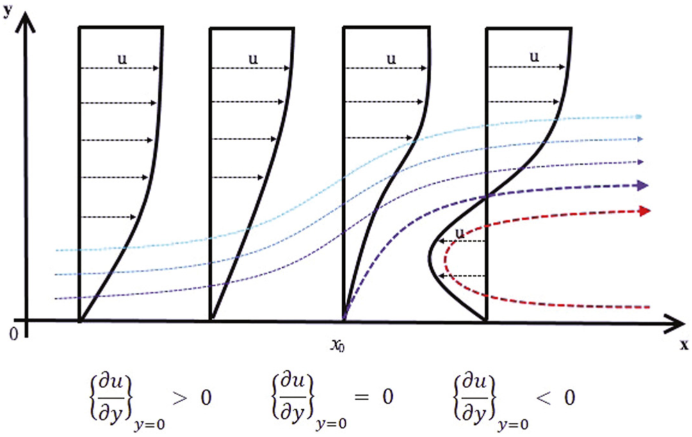

Fig. 2. Separation phenomenon.

4. Auxilliary propositions

Consider the domain

$$\omega = \{0 < x < X,\ \psi_1 < \psi < \psi_2\}$$

and denote by γ the part of its boundary

$$\gamma = \{x, \psi\ :\ 0 < x < X, \psi = \psi_1\} \cup \{x, \psi\ :\ 0 < x < X, \psi = \psi_2\} \cup \\ \cup \{x, \psi\ :\ x = 0, \psi_1 \le \psi \le \psi_2\}.$$

Let us consider a differential operator

$$A(w) \equiv a\Big(w, \frac{\partial w}{\partial \psi}\Big)\frac{\partial^2 w}{\partial \psi^2} - \frac{\partial w}{\partial x} + b(x,\psi)\frac{\partial w}{\partial \psi} + c(x,\psi)w, \tag{18}$$

where $a(w, p) \ge 0$.

Lemma 4.1. *Assume that the functions $a(w, p)$, $b(x,\psi)$, $c(x,\psi)$ are bounded as $(x,\psi) \in \overline{\omega}$ and $-\infty < p < \infty$; $w(x,\psi)$ is continuous in $\overline{\omega}$ and has in ω continuous derivatives appearing in (18). If $A(w) \le 0$ in $\overline{\omega} \setminus \gamma$ and $w \ge 0$ on γ, then $w \ge 0$ in $\overline{\omega}$. If $A(w) \ge 0$ in $\overline{\omega} \setminus \gamma$ and $w \le 0$ on γ, then $w \le 0$ in $\overline{\omega}$.*

Lemma 4.2. *Suppose that the conditions of Lemma 4.1 are fulfilled and moreover that $a(w, p)$ has bounded partial derivatives on w and p in ω. Suppose also that $\Phi(x,\psi)$ is a continuous function in $\overline{\omega}$, which has in $\overline{\omega} \setminus \gamma$ continuous partial derivatives appearing in (18). If $A(w) - \Phi(w) \le 0$ in $\overline{\omega} \setminus \gamma$ and $w \ge \Phi$ on γ, then $w \ge \Phi$ in $\overline{\omega}$. If $A(w) - \Phi(w) \ge 0$ in $\overline{\omega} \setminus \gamma$ and $w \le \Phi$ on γ, then $w \le \Phi$ in $\overline{\omega}$.*

The proof of these Lemmas can be found in [3, Ch. 2, §2.1].

5. On the separation of the boundary layer

The boundary layer with large Reynolds numbers exhibits a phenomenon called separation of boundary layer. The meaning of this phenomenon is that a backward flow appearing for some movement of viscous medium separates (detach) the boundary layer from the streamline surface and takes it in the outer flow (see Fig. 2).

Definition 5.1. The separation point of a boundary layer is a point x_0 such that $u_y(x, 0) > 0$ for $0 < x < x_0$, and $u_y(x_0, 0) = 0$. Then x_0 is the supremum of $x > 0$ such that, in domain D, the problem (1), (2) has a solution $u(x, y)$, $v(x, y)$, for which $u_y(x, 0) > 0$.

In the point x_0 as $y = 0$, the derivative of the velocity on y equals to zero. Hence an inflection point appears, and the stability of the flow is violated. Then, it appears a domain with negative derivative on y, i.e. the back flow inducing vorticity.

It is possible to avoid this phenomenon imposing a magnetic field perpendicular to the flow. Under this influence, the separation point moves to the right (see Fig. 3).

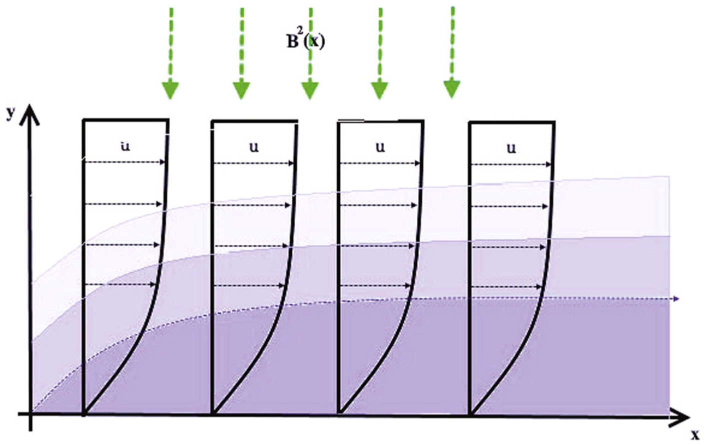

Fig. 3. Influence of a magnetic field on the separation of boundary layer.

Theorem 3.1 says that, for $B^2 + \dfrac{\mathrm{d}U}{\mathrm{d}x} \geq \alpha$, where α is a positive constant, there is no separation. The following statement is valid.

Theorem 5.1. *If there exists a solution to the problem* (1), (2) *in domain* $D = \{0 < x < X,\ 0 < y < \infty\}$*, then* $X < x_0$*, where* x_0 *is the separation point that can be found from the conditions*

$$\max_y u_0^2(y) - \int_0^{x_0} \left(-2U(x)\frac{\mathrm{d}U}{\mathrm{d}x} - 2B^2(x)U(x) \right) dx = 0 \quad \textit{and} \quad \frac{\mathrm{d}U(x_0)}{\mathrm{d}x} < 0.$$

Proof. Consider the problem (3), (4) under the condition $v_0(x) \equiv 0$. Suppose that $\dfrac{\mathrm{d}U}{\mathrm{d}x} < 0, \quad B^2 < \left|\dfrac{\mathrm{d}U}{\mathrm{d}x}\right|$.

Let us denote $p_0 := \max\limits_\psi w_0(\psi)$ and define

$$\widetilde{w}(x) = p_0 - \int_0^x \left(-2U(x)\frac{\mathrm{d}U}{\mathrm{d}x} - 2B^2(x)U(x) \right) \mathrm{d}x.$$

Define the operator

$$L(V) := \nu\sqrt{w}\left(1 + \frac{3}{4}k\left(\frac{\partial w}{\partial \psi}\right)^2\right)\frac{\partial^2 V}{\partial \psi^2} - \frac{\partial V}{\partial x} - 2B^2\sqrt{V}.$$

Assume that, in the domain $G = \{0 < x < X,\ 0 < \psi < \infty\}$, there exists a solution $w(x,\psi)$ to the problem (3), (4) and there exists x_0 such that $\widetilde{w}(x_0,\psi) = 0$, $\widetilde{w}(x,\psi) > 0$ as $x < x_0$, $\dfrac{\mathrm{d}U}{\mathrm{d}x} < 0$.

Consider the difference $L(\widetilde{w}) \equiv -2U\dfrac{\mathrm{d}U}{\mathrm{d}x} - 2B^2U - 2B^2\sqrt{\widetilde{w}}$ and $L(w) \equiv -2U\dfrac{\mathrm{d}U}{\mathrm{d}x} - 2B^2U$. We have

$$L(\widetilde{w}) - L(w) = -2B^2\sqrt{\widetilde{w}} \leq 0 \quad \text{as} \quad \widetilde{w} \geq 0, \quad 0 < x \leq x_0.$$

Since

$$\max_\psi w_0(\psi) + 2\int_0^x \left(U\frac{\mathrm{d}U}{\mathrm{d}x} + B^2U\right)\mathrm{d}x > 0 = w(x,0)$$

and $w(0,\psi) \leq \max\limits_\psi w_0(\psi) \leq \widetilde{w}(0,\psi)$, then by the maximum principle (see Lemma 4.2), we get the inequality $w(x,\psi) \leq \widetilde{w}(x,\psi)$ for $x \leqslant x_0$.

Hence, if $\widetilde{w}(x_0) = 0$ and $\dfrac{\partial w(x_0,\psi)}{\partial x} = \displaystyle\int_0^x 2\left(U\frac{\mathrm{d}U}{\mathrm{d}x} + B^2U\right)\mathrm{d}x < 0$, then there is no positive solution $w(x,\psi)$ as $x > x_0$. It means that $X < x_0$, i.e. for $0 < x < X$ there is no separation of the boundary layer. Theorem 5.1 is proved. □

Remark 1. Due to Theorem 5.1 there is no separation of the boundary layer provided that:

$$B^2(x) > \left|\frac{\mathrm{d}U}{\mathrm{d}x}\right|.$$

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