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Stochastic homogenization of reinforced polymer with very small carbon inclusions

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ABSTRACT

In this study, we wish to determine a homogenized model of a material reinforced by spherical inclusion that is randomly distributed in space. The method used for the transition to the limit is Γ-convergence [1] in the stochastic case. In addition to the stochastic framework, the very small size compared to the characteristic size of the materials makes the homogenization procedure unconventional. In this study, we want to determine a homogenized model of a material reinforced by a spherical inclusion distributed randomly in space. The peculiarity here is that these particles are of very small size, this generating an energy due to the strong contrast of microstructure. The method used for the transition to the limit is Γ-convergence [1] in the stochastic case. The random distribution is taken into account during the transition of scales, so as to preserve the statistical information, and that in spite of the passage to the limit. In addition to the stochastic framework, the very small size compared to the characteristic size of the materials makes the homogenization procedure unconventional.

1. Introduction

We want here to obtain a homogenized model of a material reinforced by very rigid inclusions and distributed randomly. The model material considered as an example is a reinforced polymer with very small carbon inclusions; by the manufacturing method generating random distributions, as well as the strong contrast of property, this material seems to be a very good frame of application for this study. Indeed, the microstructural polymers have a certain characteristic length (denoted by ε), and to improve their resistance, it is possible to introduce very small carbon charges of radius $r << \varepsilon$. This charge is therefore distributed randomly in the material (see Fig. 1). We make the (realistic) hypothesis of an ergodic distribution (see [2–4]). In the case of high densities of inclusion, the ergodic side seems to be appropriate.

2. Notations and basic notions

We consider an open bounded cube $\mathcal{O} = \widehat{\mathcal{O}} \times (-L/2, L/2)$ of $\mathbb{R}^3$, where $\widehat{\mathcal{O}}$ is a bounded open interval of $\mathbb{R}^2$. We write ε to denote a sequence of positive real numbers intended to go to 0 and chosen in order to satisfy $|\widehat{\mathcal{O}} \setminus \bigcup_{z \in I_\varepsilon} \varepsilon(Y+z)| = 0$, where I_ε is a finite subset of $\mathbf{Z}^3$, and we define $Y \in \mathbb{R}^2$ the reference surface cell; this domain will be considered here as unitary.

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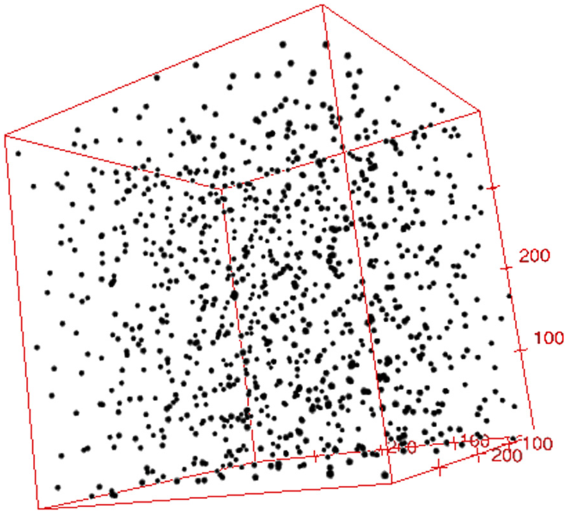

Fig. 1. Example of carbon distribution.

Given $r_\varepsilon << \varepsilon$, we consider the open ball

$$B_{r_\varepsilon}(\varepsilon\omega_0) := \left\{x \in \mathbb{R} : |x - \omega_{0,\varepsilon}| < r_\varepsilon\right\}$$

centered at the center $\omega_{0,\varepsilon}$ of εY and set, for each $\omega_{0,\varepsilon} + \varepsilon z$, $z \in I_\varepsilon$, as

$$D_{r_\varepsilon} := \bigcup_{z \in I_\varepsilon} B_{r_\varepsilon}(\omega_{z,\varepsilon})$$

The boundary of each ball $B_{r_\varepsilon}(\omega_{z,\varepsilon})$ is denoted by $S_{r_\varepsilon}(\omega_{z,\varepsilon})$. At the same time as the scale transition, we are going to move the Y domain so as to cover the entire structure under study. For stochastic homogenization to work, we therefore need this domain Y to always include at least one inclusion. This is why we need to use the ergodicty hypothesis. The union of the ball is B_{r_ε}.

For R_ε satisfying $r_\varepsilon << R_\varepsilon << \varepsilon$, we also define in the same way the disks $B_{R_\varepsilon}(\omega_{z,\varepsilon})$, their boundary $S_{R_\varepsilon}(\omega_{z,\varepsilon})$, their union B_{R_ε}, and the union of the ball B_{R_ε}. We finally denote by $C_{r_\varepsilon R_\varepsilon}(\omega_{z,\varepsilon})$ the ring

$$B_{R_\varepsilon}(\omega_{z,\varepsilon}) \setminus B_{r_\varepsilon}(\omega_{z,\varepsilon})$$

Given $a_\varepsilon > 0$ satisfying $a_\varepsilon |B_\varepsilon| \to a$, i.e. $k_\varepsilon := a_\varepsilon \frac{r_\varepsilon^2}{\varepsilon^2} \to k$ with $k\pi|\widehat{\mathcal{O}}| = a$, we consider the energy functional defined in $W^{1,p}(\mathcal{O})$ by

$$F_\varepsilon(u) = \int_{\mathcal{O}\setminus B_{r_\varepsilon}} f(\nabla u)\,\mathrm{d}x + a_\varepsilon \int_{B_{r_\varepsilon}} f(\nabla u)\,\mathrm{d}x$$

that we sometimes write

$$F_\varepsilon(u) = \int_{\mathcal{O}\setminus B_{r_\varepsilon}} f(\nabla u)\,\mathrm{d}x + a_\varepsilon |B_\varepsilon| \fint_{B_{r_\varepsilon}} f(\nabla u)\,\mathrm{d}x \tag{1}$$

where $f : \mathbb{R}^3 \to \mathbb{R}$ is a convex function satisfying the standard growth condition of order $p > 1$

$$\alpha|\xi|^p \le f(\xi) \le \beta(1 + |\xi|^p) \tag{2}$$

for two given constants $\alpha > 0$ and $\beta > 0$. Let R_ε satisfying $r_\varepsilon << R_\varepsilon << \varepsilon$. Following [3,5], we split the functional F_ε into three terms:

$$F_\varepsilon(u) = \int_{\mathcal{O}\setminus B_{R_\varepsilon}} f(\nabla u)\,\mathrm{d}x + \int_{B_{R_\varepsilon}\setminus B_{r_\varepsilon}} f(\nabla u)\,\mathrm{d}x + a_\varepsilon |B_\varepsilon| \fint_{B_{r_\varepsilon}} f(\nabla u)\,\mathrm{d}x$$

We assume the following control between f and its p-recession $f^{\infty,p}$ defined by $f^{\infty,p}(\xi) = \lim\limits_{t\to+\infty} \dfrac{f(t\xi)}{|t|^p}$: there exists $\beta' > 0$, $0 < \gamma < p$ such that, for all $\xi \in \mathbb{R}^3$

$$\left|f(\xi) - f^{\infty,p}(\xi)\right| \le \beta'(1 + |\xi|^{p-\gamma}) \tag{3}$$

From (2), we infer that $f^{\infty,p}$ satisfies for all $\xi \in \mathbb{R}^3$

$$\alpha|\xi|^p \le f^{\infty,p}(\xi) \le \beta|\xi|^p \tag{4}$$

and, since clearly $f^{\infty,p}$ is a positively p-homogeneous convex function, its satisfies the local Lipschitz condition

$$|f^{\infty,p}(\xi) - f^{\infty,p}(\xi')| \le L|\xi - \xi'|(|\xi|^{p-1} + |\xi'|^{p-1}) \tag{5}$$

For more details see [6].

Definition 2.1. Given two concentric disks $B_r(\omega) \subset B_R(\omega)$ of $\mathbb{R}^3$ with arbitrary center ω, we define the capacity of $B_r(\omega)$ in $B_R(\omega)$ associated with the function f by

$$\mathrm{Ener}^f_{r,R} = \inf\left\{ \int\limits_{B_R(\omega)} f^{\infty,p}(\nabla w)\,\mathrm{d}x : w \in W^{1,p}_0(B_R(\omega)) : w = 1 \text{ in } B_r(\omega) \right\}$$

Note that $\mathrm{Ener}^f_{r,R}$ does not depend on the choice of the center ω of the disks. We assume the following limit behavior of $\mathrm{Ener}^f_{r_\varepsilon,\mathbb{R}_\varepsilon}$

$$\lim_{\varepsilon\to0} \frac{\mathrm{Ener}^f_{r_\varepsilon,R_\varepsilon}}{\varepsilon^2} = \gamma \tag{6}$$

with $\gamma \in [0, +\infty[$.

3. Technical lemmas

With each sequence $(u_\varepsilon)_{\varepsilon>0}$ of functions in $W^{1,p}(\mathcal{O})$ we associate the sequence $(v_\varepsilon)_{\varepsilon>0}$ of the functions in $L^p(\mathcal{O})$ defined by $v_\varepsilon = \frac{|\mathcal{O}|}{|B_{r_\varepsilon}|} \mathbb{1}_{B_{r_\varepsilon}} u_\varepsilon$. Recall the following compactness property (see [3,5,7,8]).

Lemma 3.1. *Let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence of finite energy, i.e. satisfying $\sup_{\varepsilon>0} F_\varepsilon(u_\varepsilon) < +\infty$. Then there exist (u, v) in $W^{1,p}(\mathcal{O}) \times L^p(\widehat{\mathcal{O}}, W^{1,p}(0,\pi))$ and a subsequence not relabeled such that*

$$u_\varepsilon \to u \; W^{1,p}(\mathcal{O})$$
$$v_\varepsilon \overset{*}{\rightharpoonup} v \text{ in } L^p(\widehat{\mathcal{O}}, W^{1,p}(0,\pi))$$
$$+ \text{ boundary conditions}$$

We denote by Dom the subset of $W^{1,p}(\mathcal{O}) \times L^p(\widehat{\mathcal{O}}, W^{1,p}(0,\pi))$ made up of functions satisfying the boundary conditions.

We are going to establish the Γ-convergence of the sequence $(F_\varepsilon)_{\varepsilon>0}$ in $L^p(\mathcal{O})$ equipped with its strong topology to the functional F_0 defined in $W^{1,p}(\mathcal{O})$ by $F_0(u) = \inf_{v\in L^p(\mathcal{O})} G(u, v)$, where the bifunctional $G : L^p(\mathcal{O}) \times L^p(\mathcal{O}) \to \mathbb{R}^+ \cup \{+\infty\}$ is given by

$$G(u,v) := \begin{cases} \displaystyle\int\limits_{\mathcal{O}} f(\nabla u)\,\mathrm{d}x + \gamma \int\limits_{\mathcal{O}} |u - v|^p\,\mathrm{d}x + a \int\limits_{\mathcal{O}} f^{\infty,p}\Big(\frac{\partial v}{\partial \psi}\Big)\,\mathrm{d}x & \text{if } (u,v) \in \mathrm{Dom} \\ +\infty & \text{otherwise} \end{cases}$$

The second term corresponds to a non-local energy (see [6,9]). This induced energy of influencing to long range inclusions. Therefore, the constant γ would be a characteristic length. This size depends here on the material, which is not always the case in the literature. In fact, in some works, this length corresponds either to a regularization parameter ([10] and [11]) or it depends on a macroscopic energy [12–14] not related to the microstructure. In our case, it also corresponds to an energy (see (6)), but this energy depends on the microstructure. Another important quantity here, v, is a virtual field corresponding to the displacement of the inclusions; this can be considered as a memory effect of the inclusions. It is thanks to this field that we can maintain information from the micro scale to the macro scale.

To establish the gamma convergence, we must increase and minimize the preceding energy by the inferior and superior limits of (1). For the upper bound, it is enough to follow the same strategy as in [1] and [6] for the first term. For the second one, it is obtained by using the following lemma.

Lemma 3.2. *Let* $(u, v) \in Dom$. *There exist* $(u_\varepsilon, v_\varepsilon = \frac{|\mathcal{O}|}{|B_{r_\varepsilon}|} \mathbb{1}_{B_{r_\varepsilon}} u_\varepsilon)$ *in* $W^{1,p}(\mathcal{O}) \times L^p(\widehat{\mathcal{O}}, W^{1,p}(0,\pi))$ *such that* $u_\varepsilon \to u$ $L^p(\mathcal{O})$, $u \in W^{1,p}(\mathcal{O})$, $v_\varepsilon \overset{*}{\rightharpoonup} v$ *in* $L^p(\widehat{\mathcal{O}}, W^{1,p}(0,\pi))$ *and*

$$\lim_{\varepsilon\to 0} \int_{B_{R_\varepsilon}\setminus B_{r_\varepsilon}} f(\nabla u_\varepsilon)\,\mathrm{d}x = \gamma \int_{\mathcal{O}} |u-v|^p\,\mathrm{d}x \tag{7}$$

Proof. *Step 1.* We establish (7) when $(u, v) \in \mathcal{C}^1(\overline{\mathcal{O}})$. Let $\theta_\varepsilon \in W_0^{1,p}(D_{R_\varepsilon}(\omega_{0,\varepsilon}))$ be the solution to the capacitary problem

$$\int_{B_{R_\varepsilon}(\omega_{0,\varepsilon})} f^{\infty,p}(\nabla\theta_\varepsilon)\,\mathrm{d}\hat{x} = \mathrm{Ener}^f_{r_\varepsilon,R_\varepsilon}$$

We extend by εY-periodicity in $\mathcal{O} = \bigcup_{z\in I_\varepsilon} \varepsilon(Y+z)$, so as to cover the entire domain $\mathcal{O}$.

On the other hand, consider

$$w_\varepsilon(\hat{x},\psi) := \sum_{z\in I_\varepsilon} \Big(\fint_{B_{r_\varepsilon}(\omega_{z,\varepsilon})+z} v(\hat{y},\psi)\,\mathrm{d}\hat{y} \Big) \mathbb{1}_{\varepsilon(Y+z)}(x)$$

and set $u_\varepsilon = (1-\theta_\varepsilon)u + \theta_\varepsilon w_\varepsilon$. One has

$$\lim_{\varepsilon\to 0} \int_{B_{R_\varepsilon}\setminus B_{r_\varepsilon}} f(\nabla u_\varepsilon)\,\mathrm{d}x = \lim_{\varepsilon\to 0} \int_{B_{R_\varepsilon}\setminus B_{r_\varepsilon}} f^{\infty,p}(\nabla u_\varepsilon)\,\mathrm{d}x$$

Indeed, from (3) and Hölder's inequality,

$$\begin{aligned}\int_{B_{R_\varepsilon}\setminus B_{r_\varepsilon}} \left| f(\nabla u_\varepsilon)\,\mathrm{d}x - f^{\infty,p}(\nabla u_\varepsilon)\right| \mathrm{d}x &\le \beta'\Big(|B_{r_\varepsilon R_\varepsilon}| + \int_{B_{R_\varepsilon}\setminus B_{r_\varepsilon}} |\nabla u_\varepsilon|^{p-\delta}\,\mathrm{d}x \Big) \\ &\le \beta'\Big(|B_{r_\varepsilon R_\varepsilon}| + |B_{r_\varepsilon R_\varepsilon}|^{\frac{\delta}{p}} \Big(\int_{B_{R_\varepsilon}\setminus B_{r_\varepsilon}} |\nabla u_\varepsilon|^p\,\mathrm{d}x \Big)^{\frac{p-\delta}{p}} \Big)\end{aligned}$$

and since clearly $\sup_{\varepsilon>0} \int_{B_{R_\varepsilon}\setminus B_{r_\varepsilon}} |\nabla u_\varepsilon|^p\,\mathrm{d}x < +\infty$, the claim follows from the fact that, by the choice $R_\varepsilon << \varepsilon$, one has $\lim_{\varepsilon\to 0} |B_{r_\varepsilon R_\varepsilon}| = 0$.

For each $z \in I_\varepsilon$, choose $\omega_{z,\varepsilon}$ the center of sphere. From (5), the regularity assumption on (u, v) and since $f^{\infty,p}$ is p-positively homogeneous, we infer

$$\begin{aligned}\lim_{\varepsilon\to 0} \int_{B_{R_\varepsilon}\setminus B_{r_\varepsilon}} f(\nabla u_\varepsilon)\,\mathrm{d}x &= \lim_{\varepsilon\to 0} \int_{B_{R_\varepsilon}\setminus B_{r_\varepsilon}} f^{\infty,p}(\nabla u_\varepsilon)\,\mathrm{d}x \\ &= \lim_{\varepsilon\to 0} \int_{B_{R_\varepsilon}\setminus B_{r_\varepsilon}} f^{\infty,p}((u-v)\nabla\theta_\varepsilon)\,\mathrm{d}x \\ &= \lim_{\varepsilon\to 0} \int_{B_{R_\varepsilon}\setminus B_{r_\varepsilon}} |u-v|^p f^{\infty,p}(\nabla\theta_\varepsilon)\,\mathrm{d}x \\ &= \lim_{\varepsilon\to 0} \int_{B_{R_\varepsilon}} f^{\infty,p}(\nabla\theta_\varepsilon)\,\mathrm{d}\hat{x} \sum_{z\in I_\varepsilon} \int_0^\pi |(u-v)(\omega_{z,\varepsilon},\psi)|^p\,\mathrm{d}\psi \\ &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2} \int_{B_{R_\varepsilon}} f^{\infty,p}(\widehat{\nabla}\theta_\varepsilon, 0)\,\mathrm{d}\hat{x} \sum_{z\in I_\varepsilon} \varepsilon^2 \int_0^\pi |(u-v)(\omega_{z,\varepsilon},\psi)|^p\,\mathrm{d}\psi\end{aligned}$$

$$= \lim_{\varepsilon\to 0} \frac{\mathrm{Ener}^{f}_{r_\varepsilon,R_\varepsilon}}{\varepsilon^2} \sum_{z\in I_\varepsilon} \varepsilon^2 \int_0^{\pi} |(u-v)(\omega_{z,\varepsilon},\psi)|^p \,\mathrm{d}\psi$$

$$= \gamma \int_{\mathcal{O}} |u-v|^p \,\mathrm{d}x$$

where we have used assumption (6) and the fact that $\sum_{z\in I_\varepsilon} \varepsilon^2 \int_0^{\pi} |(u-v)(\omega_{z,\varepsilon},\psi)|^p \,\mathrm{d}\psi$ is a Riemann sum in the last equality.

Step 2. We establish (7) in the general case by a standard approximation and diagonalization argument. □

Lemma 3.3. *For all* $(u_\varepsilon, v_\varepsilon)$, $v_\varepsilon = \frac{|\mathcal{O}|}{|T_{r_\varepsilon}|} \mathbb{1}_{B_{r_\varepsilon}} u_\varepsilon)$ *such that* $u_\varepsilon \to u$ *in* $L^p(\mathcal{O})$ *and* $v_\varepsilon \overset{*}{\rightharpoonup} v$ *in* $L^p(\widehat{\mathcal{O}}, W^{1,p}(0,\pi))$ *one has*

$$\liminf_{\varepsilon\to 0} \int_{B_{R_\varepsilon}\setminus B_{r_\varepsilon}} f(\nabla u_\varepsilon)\,\mathrm{d}x = \liminf_{\varepsilon\to 0} \int_{B_{R_\varepsilon}\setminus B_{r_\varepsilon}} f^{\infty,p}(\nabla u_\varepsilon)\,\mathrm{d}x \geq \gamma \int_{\mathcal{O}} |u-v|^p \mathrm{d}x \tag{8}$$

Proof. *Step 1.* Equality in (8) is clear. For $\eta > 0$ intended to go to 0 we decompose $(0,\pi)$ into a finite family of intervals $(K_j)_{j\in J_\eta}$ of size η: $\left|(0,\pi)\setminus \bigcup_{j\in J_\eta} K_j\right| = 0$. In this step, we modify the function u_ε into a Sobolev function $\tilde{u}_\varepsilon$ (depending on η) in each ring $\bigcup_{i\in I_\varepsilon} S_{r_\varepsilon R_\varepsilon}(\omega_{z,\varepsilon}) \times K_j$, satisfying:

$$\tilde{u}_\varepsilon = 0 \text{ on each surface } \bigcup_{i\in I_\varepsilon} S_{r_\varepsilon R_\varepsilon}(\omega_{z,\varepsilon}) \times K_j$$

$$\liminf_{\varepsilon\to 0} \int_{B_{R_\varepsilon}\setminus B_{r_\varepsilon}} f^{\infty,p}(\nabla u_\varepsilon)\,\mathrm{d}x \geq \liminf_{\varepsilon\to 0} \sum_{z\in I_\varepsilon} \sum_{j\in J_\eta} \int_{S_{r_\varepsilon R_\varepsilon}(\omega_{z,\varepsilon})\times K_j} f^{\infty,p}(\nabla \tilde{u}_\varepsilon)\,\mathrm{d}x$$

We make use of a standard truncation method in each slice of a neighborhood of each one of the two bases: we set $u_{\varepsilon,i} := \varphi_i u_\varepsilon$ where $\psi \mapsto \varphi_i(\psi)$ satisfies $\varphi_i = 1$ on the neighborhood $(K_j)_{\delta_i}$ and belongs to $\mathcal{C}_c((K_j)_{\delta_{i+1}})$, $i = 1,\dots,\nu$ where δ_i are small parameters. To perform the slicing method, the important points to note are $\lim_{\varepsilon\to 0} \int_{B_{R_\varepsilon}\setminus B_{r_\varepsilon}} |u_\varepsilon|^p \,\mathrm{d}x = 0$ and

$$\lim_{\nu\to+\infty} \frac{1}{\nu} \sup_{\varepsilon>0} \int_{B_{R_\varepsilon}\setminus B_{r_\varepsilon}} |\nabla u_\varepsilon|^p \,\mathrm{d}x = 0$$

Step 2. In what follows, we still denote by u_ε the function $\tilde{u}_\varepsilon$. By using Jensen's inequality, we infer

$$\liminf_{\varepsilon\to 0} \sum_{z\in I_\varepsilon} \sum_{j\in J_\eta} \int_{S_{r_\varepsilon R_\varepsilon}(\omega_{z,\varepsilon})\times K_j} f^{\infty,p}(\nabla \tilde{u}_\varepsilon)\,\mathrm{d}x \geq \liminf_{\varepsilon\to 0} \sum_{z\in I_\varepsilon} \sum_{j\in J_\eta} \eta \int_{S_{r_\varepsilon R_\varepsilon}(\omega_{z,\varepsilon})} f^{\infty,p}(\fint_{K_j} \widehat{\nabla} u_\varepsilon(\hat{x},s)\,\mathrm{d}s, 0)\,\mathrm{d}\hat{x} \tag{9}$$

In the last step we set $\widehat{u}_{\varepsilon,j}(\hat{x}) := \fint_{K_j} u_\varepsilon(\hat{x},s)\,\mathrm{d}s$ for all $j \in J_\eta$. Note that $\widehat{u}_{\varepsilon,j}$ depends on η.

Step 3. (End of the proof) We approximate the functions u and v as follows:

$$u_\eta(x) = \sum_{j\in J_\eta} \mathbb{1}_{K_j}(x_3) \fint_{K_j} u(\hat{x},s)\,\mathrm{d}s$$

$$v_\eta(x) = \sum_{j\in J_\eta} \mathbb{1}_{K_j}(x_3) \fint_{K_j} v(\hat{x},s)\,\mathrm{d}s$$

For all fixed $j \in J_\eta$, our strategy consists first in modifying in each ring $S_{r_\varepsilon R_\varepsilon}(\omega_{z,\varepsilon})$. The function $\widehat{u}_{\varepsilon,j}$ is such that the new function $\tilde{u}_{\varepsilon,j}$ agrees with $\fint_{S_{R_\varepsilon}(\omega_{z,\varepsilon})} \widehat{u}_{\varepsilon,j}(\hat{x})\, d\mathcal{H}^1$ on $C_{R_\varepsilon}(\omega_{z,\varepsilon})$, with $\fint_{S_{r_\varepsilon}(\omega_{z,\varepsilon})} \widehat{u}_{\varepsilon,j}(\hat{x})\,\mathrm{d}\mathcal{H}^1$ on $S_{r_\varepsilon}(\omega_{z,\varepsilon})$, and satisfies

$$\liminf_{\varepsilon\to 0} \sum_{z\in I_\varepsilon} \eta \int_{S_{r_\varepsilon R_\varepsilon}(\omega_{z,\varepsilon})} f^{\infty,p}(\nabla \widehat{u}_{\varepsilon,j}, 0)\,\mathrm{d}\hat{x} \geq \liminf_{\varepsilon\to 0} \sum_{z\in I_\varepsilon} \eta \int_{S_{r_\varepsilon R_\varepsilon}(\omega_{z,\varepsilon})} f^{\infty,p}(\nabla \tilde{u}_{\varepsilon,j}, 0)\,\mathrm{d}\hat{x} \tag{10}$$

This is done by using a standard De Giorgi slicing argument (see for instance [15], proof of Proposition 11.2.3).

To shorten notation, we still denote by $\widehat{u}_{\varepsilon,j}$ the function $\tilde{u}_{\varepsilon,j}$. For all fixed $j \in J_\eta$, we define the two step functions in $L^p(\mathcal{O} \times K_j)$

$$\overline{\widehat{u}_{\varepsilon,j}}(\hat{x}) = \sum_{z\in I_\varepsilon} \Big(\fint_{S_{R_\varepsilon}(\omega_{z,\varepsilon})} \widehat{u}_{\varepsilon,j}(\hat{x})\, \mathrm{d}\mathcal{H}^1(\hat{x})\Big) \mathbb{1}_{\varepsilon(Y+z)}$$

$$\overline{\widehat{v}_{\varepsilon,j}}(\hat{x}) = \sum_{z\in I_\varepsilon} \Big(\fint_{S_{r_\varepsilon}(\omega_{z,\varepsilon})} \widehat{u}_{\varepsilon,j}(\hat{x})\, \mathrm{d}\mathcal{H}^1(\hat{x})\Big) \mathbb{1}_{\varepsilon(Y+z)}$$

It is worth noticing that $\overline{\widehat{u}_{\varepsilon,j}} - \overline{\widehat{v}_{\varepsilon,j}} = \fint_{K_j} \left(\overline{u}_{\varepsilon,j} - \overline{v}_{\varepsilon,j}\right) \mathrm{d}x_3$ where $\overline{u}_{\varepsilon,j}$ and $\overline{v}_{\varepsilon,j}$ are two functions defined like $\overline{u}_\varepsilon$ and $\overline{v}_\varepsilon$ in [5] (replace $(0,\pi)$ by K_j). Proceeding like in [5,6], it is not difficult to establish that $\overline{u}_{\varepsilon,j} - \overline{v}_{\varepsilon,j}$ weakly converges in $L^p(\mathcal{O} \times K_j)$ to $u_\eta - v_\eta$.

By using the p-homogeneity of $f^{\infty,p}$ and a l.s.c. argument, we deduce

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} \sum_{z\in I_\varepsilon} \sum_{j\in J_\eta} \eta \int_{S_{r_\varepsilon R_\varepsilon}(\omega_{z,\varepsilon})} f^{\infty,p}(\nabla \tilde{u}_{\varepsilon,j}, 0)\, \mathrm{d}\hat{x} \\
&= \liminf_{\varepsilon\to 0} \sum_{z\in I_\varepsilon} \sum_{j\in J_\eta} \eta\varepsilon^2 \left| \fint_{S_{R_\varepsilon}(\omega_{z,\varepsilon})} \widehat{u}_{\varepsilon,j}(\hat{x})\, \mathrm{d}\mathcal{H}^1(\hat{x}) - \fint_{S_{r_\varepsilon}(\omega_{z,\varepsilon})} \widehat{u}_{\varepsilon,j}(\hat{x})\, \mathrm{d}\mathcal{H}^1(\hat{x}) \right|^p \frac{\mathrm{Ener}^f_{r_\varepsilon R_\varepsilon}}{\varepsilon^2} \\
&= \liminf_{\varepsilon\to 0} \frac{\mathrm{Ener}^f_{r_\varepsilon R_\varepsilon}}{\varepsilon^2} \sum_{j\in J_\eta} \eta \int_{\mathcal{O}} |\overline{\widehat{u}_{\varepsilon,j}} - \overline{\widehat{v}_{\varepsilon,j}}|^p\, \mathrm{d}x \geq \gamma \int_{\mathcal{O}} |u_\eta - v_\eta|^p\, \mathrm{d}x
\end{aligned} \tag{11}$$

The claim follows by combining the first step, (9), (10), (11), and letting $\eta \to 0$ □

4. Summary

In summary, in addition to the stochastic homogenization method, the important result of this work is the form of limit energy $F_0(u) = \inf_{v\in L^p(\mathcal{O})} G(u, v)$ with $G : L^p(\mathcal{O}) \times L^p(\mathcal{O}) \to \mathbb{R}^+ \cup \{+\infty\}$ given by

$$G(u,v) := \begin{cases} \displaystyle\int_{\mathcal{O}} f(\nabla u)\, \mathrm{d}x + \gamma \int_{\mathcal{O}} |u - v|^p\, \mathrm{d}x + a \int_{\mathcal{O}} f^{\infty,p}(\frac{\partial v}{\partial \psi})\, \mathrm{d}x & \text{if } (u,v) \in \mathrm{Dom} \\ +\infty & \text{otherwise} \end{cases}$$

This energy can be compared to non-local models of the type [14], but it has the advantage of not having a second gradient. Indeed, here the non-locality is represented, on the one hand, by the constant γ and, on the other hand, by the virtual field v. This model can be extended to a model of damage. In the latter case, the variable v will be the variable of damage and γ a term of regularization. This regulation constant depends on the distribution of inclusions, and can be computed using covariance methods [16].

5. Conclusion

In this study, we determined a homogeneous deterministic model of a randomly enhanced environment with spherical inclusion and very small relative to the size of a REV (Representative Elementary Volume). The method proposed here allows us to obtain a homogenized model of statistical data preserved in spite of the scale transition process. The same strategy can be proposed for other materials, such as randomly reinforced materials such as long fiber composites [17], but in the case of fibers with a very small radius. This result is a good start to understand the damage mechanisms of this type of material. Indeed, the second energy obtained at the limit corresponds to a stored energy, which can be at the origin of the damage to this type of composite. Following this study, we wish to proceed to a numerical resolution of this model so as to compare this result with mechanical experiments, for example.

In addition, from the development of a numerical resolution scheme, we wish to extend this model to a nonlinear behavior, more precisely to a damage model [12,14,18]. In fact, the energy γ corresponds to a stored energy that can create compensation. This damage is localized at the inclusion-matrix interface. This generates a decohesion very difficult to model; it is also by this aspect that this work is original and will help better understand these phenomena.

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