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Viscoelastic materials with a double porosity structure

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ABSTRACT

This paper in concerned with the linear theory of materials with memory that possess a double porosity structure. First, the formulation of the initial-boundary-value problem is presented. Then, a uniqueness result is established. The semigroup theory of linear operators is used to prove existence and continuous dependence of solutions. A minimum principle for the dynamical theory is also derived.

1. Introduction

The mechanics of solids with a double porosity structure is of interest in geophysics and in mechanics of bones. In the recent years the deformation of these materials has been a subject of intensive study (see, e.g., [1–7], and references therein).

In the first studies of the so-called double porosity model, the authors used Darcy's law and deduced the equations for the displacement vector and the pressures associated with the porous structure. In the equilibrium theory, the fluid pressures become independent of the displacement vector field. By using the theory of materials with voids, Ieşan and Quintanilla [6] have derived a theory of thermoelasticity for materials with a double porosity structure. In this theory, the porosities are coupled with the displacement field, even in the static case. The theory of elastic materials with voids has been established by Nunziato and Cowin [8] for the behavior of porous solids in which the skeletal or matrix materials are elastic and the interstices are voids of material.

Recently, some papers have been devoted to the rate theory of viscoelastic materials with double porosity (see, e.g., [9]). The present paper is concerned with the linear theory of materials with memory that possess a double porosity structure. The history of motion is important for rheological materials and, in the dissipation and relaxation phenomena, it plays a central role.

In the classical theory of viscoelastic materials, Day [10] proved that the work done in every closed strain path starting from zero is invariant under time reversal if and only if the stress relaxation function is symmetric. Gurtin [11,12] derived an extension of Day's result within the context of the thermodynamics of materials with memory. In the first part of this paper, we use the results established by Day [10] and Gurtin [11,12] to present the constitutive equations of a viscoelastic material with a double porosity structure. Then, by using the method given by Gurtin et al. [13], we derive a uniqueness theorem for the initial-boundary-value problem. In the second part of the paper, we consider the dynamic theory with Dirichlet boundary conditions. We use the semigroup theory of linear operators to obtain the existence and continuous dependence

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of solutions. With a view toward a presentation of a minimum principle, we first establish a reciprocity relation. Then we present a minimum principle of Reiss type.

2. Porous viscoelastic solids

Let us denote by B the reference configuration occupied by a body at time t_0. We refer to motion of the continuum to a fixed system of rectangular Cartesian axes Ox_i, $(i = 1, 2, 3)$. The conventions adopted with regard to tensor indices are as follows: Latin indices (unless otherwise specification) are understood to range over the integers $(1, 2, 3)$, whereas Greek indices are confined to the range $(1, 2)$. The usual summation convention applies to all indices. Moreover, subscripts preceded by a comma denote partial differentiation with respect to the corresponding material coordinate and a superposed dot denotes the material derivative with respect to the time t.

Following Gurtin [11], we write $\nabla^{(n)} f(\mathbf{x}, t)$ for the n-th gradient of f with respect to $\mathbf{x}$ holding t fixed and $f^{(n)}(\mathbf{x}, t)$ for the n-th derivative of f with respect to t holding $\mathbf{x}$ fixed. We say that f is of class $C^{M,N}$ on $B \times (0, t_0)$ if f is continuous on $B \times (0, t_0)$ and the functions $\nabla^{(m)} f^{(n)}$, $m \in \{0, 1, \ldots, M\}$, $n \in \{0, 1, \ldots, N\}$, $m + n \leq \max(M, N)$, exist and are continuous on $B \times (0, t_0)$.

In what follows, we consider the linear theory of viscoelastic materials with a double porosity structure. Let u_i be the components of the displacement field. The components of the infinitesimal strain field are given by

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{1}$$

We denote by ν_1 the volume fraction field corresponding to pores and by ν_2 the volume fraction field corresponding to fissures. Let us denote by ν_1^* and ν_2^* the volume fractions in the reference configuration. Let φ be the change in volume fraction ν_1 from the reference value ν_1^*, and let ψ be the change in volume fraction ν_2 from the reference value ν_2^*. The equations of motion can be expressed as [8]

$$t_{ji,j} + F_i = \rho \ddot{u}_i, \; \sigma_{j,j} + \xi + G = \kappa_1 \ddot{\varphi}, \; \tau_{j,j} + \zeta + L = \kappa_2 \ddot{\psi} \tag{2}$$

Here we have used the following notations: t_{ij} is the stress tensor, σ_j and τ_j are equilibrated stress vectors, ξ and ζ are the intrinsic equilibrated body forces, F_i is the body force, G is the extrinsic equilibrated body force associated with pores, L is the equilibrated body force associated with fissures, ρ is the reference mass density, and κ_1 and κ_2 are coefficients of inertia (cf. [8]). In the classical theory of viscoelastic materials, Day [10] proved that the work done in every closed strain path starting from zero is invariant under time reversal if and only if the stress relaxation function is symmetric. Gurtin [12] extended this result to the thermodynamics of materials with memory. By using the results of Day [10] and Nunziato and Cowin [8], we are led to the following constitutive equations of centrosymmetric viscoelastic materials with a double porosity structure:

$$\begin{aligned}
t_{ij}(\mathbf{x}, t) &= \int_{-\infty}^{t} [C_{ijkl}(\mathbf{x}, t-s)\dot{e}_{kl}(\mathbf{x}, s) + B_{ij}(\mathbf{x}, t-s)\dot{\varphi}(\mathbf{x}, s) \\
&\quad + D_{ij}(\mathbf{x}, t-s)\dot{\psi}(\mathbf{x}, s)]\mathrm{d}s \\
\sigma_i(\mathbf{x}, t) &= \int_{-\infty}^{t} [\alpha_{ij}(\mathbf{x}, t-s)\dot{\varphi}_{,j}(\mathbf{x}, s) + \beta_{ij}(\mathbf{x}, t-s)\dot{\psi}_{,j}(\mathbf{x}, s)]\mathrm{d}s \\
\tau_i(\mathbf{x}, t) &= \int_{-\infty}^{t} [\beta_{ji}(\mathbf{x}, t-s)\dot{\varphi}_{,j}(\mathbf{x}, s) + \gamma_{ij}(\mathbf{x}, t-s)\dot{\psi}_{,j}(\mathbf{x}, s)]\mathrm{d}s \\
\xi(\mathbf{x}, t) &= -\int_{-\infty}^{t} [B_{ij}(\mathbf{x}, t-s)\dot{e}_{ij}(\mathbf{x}, s) + \alpha_1(\mathbf{x}, t-s)\dot{\varphi}(\mathbf{x}, s) \\
&\quad + \alpha_3(\mathbf{x}, t-s)\dot{\psi}(\mathbf{x}, s)]\mathrm{d}s \\
\zeta(\mathbf{x}, t) &= -\int_{-\infty}^{t} [D_{ij}(\mathbf{x}, t-s)\dot{e}_{ij}(\mathbf{x}, s) + \alpha_3(\mathbf{x}, t-s)\dot{\varphi}(\mathbf{x}, s) \\
&\quad + \alpha_2(\mathbf{x}, t-s)\dot{\psi}(\mathbf{x}, s)]\mathrm{d}s
\end{aligned} \tag{3}$$

The relaxation functions are twice continuously differentiable and have the following symmetries

$$C_{ijrs} = C_{rsij} = C_{jirs},\ B_{ij} = B_{ji},\ D_{ij} = D_{ji},\ \alpha_{ij} = \alpha_{ji},\ \gamma_{ij} = \gamma_{ji} \tag{4}$$

By an admissible process, we mean and ordered array of functions $\pi = (u_i, \varphi, \psi, e_{ij}, t_{ij}, \sigma_i, \tau_i, \xi, \zeta)$, defined on $\overline{B} \times (-\infty, \infty)$ with the properties: (i) u_i, φ and ψ are of class C^2; (ii) $u_i, \dot{u}_i, \ddot{u}_i, \varphi, \dot{\varphi}, \ddot{\varphi}, \psi, \dot{\psi}, \ddot{\psi}, e_{ij}, \varphi_{,i}$ and $\psi_{,i}$ are continuous on $\overline{B} \times (-\infty, \infty)$; (iii) $e_{ij} = e_{ji}$, $t_{ij} = t_{ji}$; (iv) t_{ij}, σ_j and τ_k are functions of class $C^{1,0}$ on $\overline{B} \times (-\infty, \infty)$; (v) $t_{ij}, t_{ij,i}, \sigma_k, \sigma_{j,j}, \tau_j, \tau_{i,i}$, ξ and ζ are continuous on $\overline{B} \times (-\infty, \infty)$. To the above equations, we must adjoin the initial data and boundary conditions. The initial data consists of the functions $\pi^* = (u_i^*, \varphi^*, \psi^*, e_{ij}^*, t_{ij}^*, \sigma_i^*, \tau_i^*, \xi^*, \zeta^*)$ defined on $\overline{B} \times (-\infty, t^*)$. In what follows, we shall consider $t^* = 0$. We consider processes that correspond to initial data π^*,

$$\pi^{(r)} = \pi^* \tag{5}$$

where $\pi^{(r)} = (u_i, \varphi, \psi, e_{ij}, t_{ij}, \sigma_i, \tau_i, \xi, \zeta)$ is the restriction of the admissible process π to $\overline{B} \times (-\infty, 0)$. Clearly, if π is an admissible process that satisfies the condition (5), then u_i, φ and ψ satisfy the initial conditions

$$\begin{aligned}
&u_i(\mathbf{x}, 0) = \lim_{t\to 0} u_i^*(\mathbf{x}, t) \equiv u_i^0(\mathbf{x}), \dot{u}_i(\mathbf{x}, 0) = \lim_{t\to 0} \dot{u}_i^*(\mathbf{x}, t) \equiv v_i^0(\mathbf{x})\\
&\varphi(\mathbf{x}, 0) = \lim_{t\to 0} \varphi^*(\mathbf{x}, t) \equiv \varphi^0(\mathbf{x}), \dot{\varphi}(\mathbf{x}, 0) = \lim_{t\to 0} \dot{\varphi}^*(\mathbf{x}, t) \equiv \varphi_1^0(\mathbf{x})\\
&\psi(\mathbf{x}, 0) = \lim_{t\to 0} \psi^*(\mathbf{x}, t) \equiv \psi^0(\mathbf{x}), \dot{\psi}(\mathbf{x}, 0) = \lim_{t\to 0} \dot{\psi}^*(\mathbf{x}, t) \equiv \psi_1^0(\mathbf{x}), \mathbf{x} \in \overline{B}
\end{aligned} \tag{6}$$

We consider the following boundary conditions

$$\begin{aligned}
&u_i = \widetilde{u}_i \text{ on } \overline{S}_1 \times I,\ t_{ji}n_j = \widetilde{t}_i \text{ on } S_2 \times I\\
&\varphi = \widetilde{\varphi} \text{ on } \overline{S}_3 \times I,\ \sigma_j n_j = \widetilde{\sigma} \text{ on } S_4 \times I\\
&\psi = \widetilde{\psi} \text{ on } \overline{S}_5 \times I,\ \tau_j n_j = \widetilde{\tau} \text{ on } S_6 \times I
\end{aligned} \tag{7}$$

where $I = (0, \infty)$, $S_k, (k = 1, 2, \ldots, 6)$, are subsets of the boundary ∂B so that $\overline{S}_1 \cup S_2 = \overline{S}_3 \cup S_4 = \overline{S}_5 \cup S_6$, $S_1 \cap S_2 = S_3 \cap S_4 = S_5 \cap S_6 = \emptyset$, and $\widetilde{u}_i, \widetilde{\varphi}, \widetilde{\psi}, \widetilde{t}_i, \widetilde{\sigma}$ and $\widetilde{\tau}$ are prescribed functions. Throughout this paper, we assume that: (i) F_i, G and L are continuous on $B \times I$; (ii) $\widetilde{u}_i, \widetilde{\varphi}$ and $\widetilde{\psi}$ are continuous on $S_1 \times I$, $S_3 \times I$ and $S_5 \times I$, respectively; (iii) $\widetilde{t}_i, \widetilde{\sigma}$ and $\widetilde{\tau}$ are continuous in time and piecewise regular on $S_2 \times I$, $S_4 \times I$ and $S_6 \times I$, respectively; (iv) ρ is continuous and strictly positive on $\overline{B}$. We say that $\pi = (u_i, \varphi, \psi, e_{ij}, t_{ij}, \sigma_i, \tau_i, \xi, \zeta)$ is a viscoelastic process corresponding to the body loads (F_i, G, L) if π is an admissible process that satisfies the equations (1)–(3). By a solution to the problem, we mean a viscoelastic process corresponding to the body loads (F_i, G, L) that satisfies the initial history condition (5) and the boundary conditions (7).

Let us present an alternative form of the constitutive equations. We introduce the convolution

$$(f * g)(\mathbf{x}, t) = \int_0^t f(\mathbf{x}, t - s)g(\mathbf{x}, s)\mathrm{d}s$$

where f and g are functions on $B \times I$ that are continuous in time. We denote

$$\begin{aligned}
s_{ij}(t) &= \int_0^\infty [\dot{C}_{ijmn}(t + s)e_{mn}(-s) + \dot{B}_{ij}(t + s)\varphi(-s)\\
&\quad + \dot{D}_{ij}(t + s)\psi(-s)]\mathrm{d}s\\
\pi_i(t) &= \int_0^\infty [\dot{\alpha}_{ij}(t + s)\varphi_{,j}(-s) + \dot{\beta}_{ij}(t + s)\psi_{,j}(-s)]\mathrm{d}s\\
\chi_i(t) &= \int_0^\infty [\dot{\beta}_{ji}(t + s)\varphi_{,j}(-s) + \dot{\gamma}_{ij}(t + s)\psi_{,j}(-s)]\mathrm{d}s\\
\nu(t) &= -\int_0^\infty [\dot{B}_{ij}(t + s)e_{ij}(-s) + \dot{\alpha}_1(t + s)\varphi(-s) + \dot{\alpha}_3(t + s)\psi(-s)]\mathrm{d}s\\
\vartheta(t) &= -\int_0^\infty [\dot{D}_{ij}(t + s)e_{ij}(-s) + \dot{\alpha}_3(t + s)\varphi(-s) + \dot{\alpha}_2(t + s)\psi(-s)]\mathrm{d}s
\end{aligned} \tag{8}$$

where, for convenience, we have expressed the argument $\mathbf{x}$. Since the process π^* is prescribed, it follows that $s_{ij}, \pi_i, \chi_i, \nu$ and ϑ are given functions. With the help of (8), we can express the constitutive equations (3) in the form

$$\begin{aligned}
t_{ij} &= s_{ij} + \frac{\mathrm{d}}{\mathrm{d}t}(C_{ijkl} * e_{kl} + B_{ij} * \varphi + D_{ij} * \psi) \\
\sigma_i &= \pi_i + \frac{\mathrm{d}}{\mathrm{d}t}(\alpha_{ij} * \varphi_{,j} + \beta_{ij} * \psi_{,j}) \\
\tau_i &= \chi_i + \frac{\mathrm{d}}{\mathrm{d}t}(\beta_{ji} * \varphi_{,j} + \gamma_{ij} * \psi_{,j}) \\
\xi &= \nu - \frac{\mathrm{d}}{\mathrm{d}t}(B_{ij} * e_{ij} + \alpha_1 * \varphi + \alpha_3 * \psi) \\
\zeta &= \vartheta - \frac{\mathrm{d}}{\mathrm{d}t}(D_{ij} * e_{ij} + \alpha_3 * \varphi + \alpha_2 * \psi)
\end{aligned} \tag{9}$$

3. Uniqueness

Uniqueness results in the classical viscoelasticity have been presented in various works (see, e.g., [14] and references therein). A uniqueness theorem in the case of viscoelastic materials with voids has been presented by Ciarletta and Scalia [15]. In this section, we use the results of Gurtin et al. [13] to derive a uniqueness result for the problem formulated in Section 2.

Let $\pi = (u_i, \varphi, \psi, e_{ij}, t_{ij}, \sigma_i, \tau_i, \xi, \zeta)$ be a viscoelastic process corresponding to the body loads (F_i, G, L). In view of the equations of motion (2), we find that

$$\begin{aligned}
&t_{ij}\dot{e}_{ij} + \sigma_i\dot{\varphi}_{,i} + \tau_i\dot{\psi}_{,i} - \xi\dot{\varphi} - \zeta\dot{\psi} = (t_{ji}\dot{u}_i + \sigma_j\dot{\varphi} + \tau_k\dot{\psi})_{,j} \\
&+ F_i\dot{u}_i + G\dot{\varphi} + L\dot{\psi} - \rho\dot{u}_i\ddot{u}_i - \kappa_1\dot{\varphi}\ddot{\varphi} - \kappa_2\dot{\psi}\ddot{\psi}
\end{aligned}$$

By using the divergence theorem, we get

$$\begin{aligned}
&\int_B (t_{ij}\dot{e}_{ij} + \sigma_i\dot{\varphi}_{,i} + \tau_i\dot{\psi}_{,i} - \xi\dot{\varphi} - \zeta\dot{\psi})\mathrm{d}v = \int_{\partial B} (t_{ji}\dot{u}_i + \sigma_j\dot{\varphi} + \tau_j\dot{\psi})n_j\mathrm{d}a \\
&+ \int_B (F_i\dot{u}_i + G\dot{\varphi} + L\dot{\psi})\mathrm{d}v - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_B (\rho\dot{u}_i\dot{u}_i + \kappa_1\dot{\varphi}^2 + \kappa_2\dot{\psi}^2)\mathrm{d}v
\end{aligned} \tag{10}$$

We define the functions f_{ij}, η and χ by

$$\begin{aligned}
&f_{ij}(\mathbf{x}, t_1, t_2) = e_{ij}(\mathbf{x}, t_1) - e_{ij}(\mathbf{x}, t_2), \eta(\mathbf{x}, t_1, t_2) = \varphi(\mathbf{x}, t_1) - \varphi(\mathbf{x}, t_2) \\
&\chi(\mathbf{x}, t_1, t_2) = \psi(\mathbf{x}, t_1) - \psi(\mathbf{x}, t_2), \ \mathbf{x} \in B, t_1, t_2 \in I
\end{aligned} \tag{11}$$

Let us introduce the notations

$$\begin{aligned}
&W(t_1, t_2; q) = \frac{1}{2}C_{ijmn}(q)f_{ij}(t_1, t_2)f_{mn}(t_1, t_2) + B_{ij}(q)f_{ij}(t_1, t_2)\eta(t_1, t_2) \\
&+ D_{ij}(q)f_{ij}(t_1, t_2)\chi(t_1, t_2) + \frac{1}{2}\alpha_{ij}(q)\eta_{,i}(t_1, t_2)\eta_{,j}(t_1, t_2) \\
&+ \beta_{ij}(q)\eta_{,i}(t_1, t_2)\chi_{,j}(t_1, t_2) + \frac{1}{2}\gamma_{ij}(q)\chi_{,i}(t_1, t_2)\chi_{,j}(t_1, t_2) \\
&+ \frac{1}{2}\alpha_1(q)\eta^2(t_1, t_2) + \alpha_3(q)\eta(t_1, t_2)\chi(t_1, t_2) + \frac{1}{2}\alpha_2(q)\chi^2(t_1, t_2) \\
&V(t_1, t_2; q) = \frac{1}{2}\dot{C}_{ijmn}(q)f_{ij}(t_1, t_2)f_{mn}(t_1, t_2) + \dot{B}_{ij}(q)f_{ij}(t_1, t_2)\eta(t_1, t_2) \\
&+ \dot{D}_{ij}(q)f_{ij}(t_1, t_2)\chi(t_1, t_2) + \frac{1}{2}\dot{\alpha}_{ij}(q)\eta_{,i}(t_1, t_2)\eta_{,j}(t_1, t_2) \\
&+ \dot{\beta}_{ij}(q)\eta_{,i}(t_1, t_2)\chi_{,j}(t_1, t_2) + \frac{1}{2}\dot{\gamma}_{ij}(q)\chi_{,i}(t_1, t_2)\chi_{,j}(t_1, t_2) \\
&+ \frac{1}{2}\dot{\alpha}_1(q)\eta^2(t_1, t_2) + \dot{\alpha}_3(q)\eta(t_1, t_2)\chi(t_1, t_2) + \frac{1}{2}\dot{\alpha}_2(q)\chi^2(t_1, t_2) \\
&U(t_1, t_2; q) = \frac{1}{2}\ddot{C}_{ijmn}(q)f_{ij}(t_1, t_2)f_{mn}(t_1, t_2) + \ddot{B}_{ij}(q)f_{ij}(t_1, t_2)\eta(t_1, t_2)
\end{aligned} \tag{12}$$

$$+ \ddot{D}_{ij}(q) f_{ij}(t_1,t_2)\chi(t_1,t_2) + \frac{1}{2}\ddot{\alpha}_{ij}(q)\eta_{,i}(t_1,t_2)\eta_{,j}(t_1,t_2) + \ddot{\beta}_{ij}(q)\eta_{,i}(t_1,t_2)\chi_{,j}(t_1,t_2)$$
$$+ \frac{1}{2}\ddot{\gamma}_{ij}(q)\chi_{,i}(t_1,t_2)\chi_{,j}(t_1,t_2) + \frac{1}{2}\ddot{\alpha}_1(q)\eta^2(t_1,t_2) + \ddot{\alpha}_3(q)\eta(t_1,t_2)\chi(t_1,t_2)$$
$$+ \frac{1}{2}\ddot{\alpha}_2(q)\chi^2(t_1,t_2), \quad t_1,t_2,q \in I$$

where we have suppressed the argument $\mathbf{x}$.

Lemma 1. *Let $\pi = (u_i, \varphi, \psi, e_{ij}, t_{ij}, \sigma_i, \tau_i, \xi, \zeta)$ be an admissible process that corresponds to null initial history and satisfies the equations (3). Then*

$$\int_0^t (t_{ij}\dot{e}_{ij} + \sigma_i\dot{\varphi}_{,i} + \tau_i\dot{\psi}_{,i} - \xi\dot{\varphi} - \zeta\dot{\psi})\mathrm{d}v = W(t,0;t)$$
$$- \int_0^t V(s,0;s)\mathrm{d}s - \int_0^t V(t,s;t-s)\mathrm{d}s \tag{13}$$
$$+ \frac{1}{2}\int_0^t\int_0^t U(r,s;|r-s|)\mathrm{d}r\mathrm{d}s$$

Proof. Since π corresponds to null initial history, we have

$$u_i(\mathbf{x},t) = 0, \ \ \varphi(\mathbf{x},t) = 0, \ \ \psi(\mathbf{x},t) = 0, \ \ \mathbf{x} \in B, t \in (-\infty, 0] \tag{14}$$

In this case, from (11), we find that

$$f_{ij}(\mathbf{x},t,0) = e_{ij}(\mathbf{x},t), \ \ \eta(\mathbf{x},t,0) = \varphi(\mathbf{x},t), \ \chi(\mathbf{x},t,0) = \psi(\mathbf{x},t), \ \ t \in I \tag{15}$$

Moreover, the constitutive equations (3) become

$$\begin{aligned}
t_{ij} &= C_{ijmn}(0)e_{mn}(t) + B_{ij}(0)\varphi(t) + D_{ij}(0)\psi(t)\\
&\quad + \int_0^t [\dot{C}_{ijmn}(t-s)e_{mn}(s) + \dot{B}_{ij}(t-s)\varphi(s) + \dot{D}_{ij}(t-s)\psi(s)]\\
\sigma_i &= \alpha_{ij}(0)\varphi_{,j}(t) + \beta_{ij}(0)\psi_{,j}(t) + \int_0^t [\dot{\alpha}_{ij}(t-s)\varphi_{,j}(s) + \dot{\beta}_{ij}(t-s)\psi_{,j}(s)]\mathrm{d}s\\
\tau_i &= \beta_{ji}(0)\varphi_{,j}(t) + \gamma_{ij}(0)\psi_{,j}(t) + \int_0^t [\dot{\beta}_{ji}(t-s)\varphi_{,j}(s) + \dot{\gamma}_{ij}(t-s)\psi_{,j}(s)]\mathrm{d}s\\
\xi &= -B_{ij}(0)e_{ij}(t) - \alpha_1(0)\varphi(t) - \alpha_3(0)\psi(t)\\
&\quad - \int_0^t [\dot{B}_{ij}(t-s)e_{ij}(s) + \dot{\alpha}_1(t-s)\varphi(s) + \dot{\alpha}_3(t-s)\psi(s)]\mathrm{d}s\\
\zeta &= -D_{ij}(0)e_{ij}(t) - \alpha_3(0)\varphi(t) - \alpha_2(0)\psi(t) - \int_0^t [\dot{D}_{ij}(t-s)e_{ij}(s)\\
&\quad + \dot{\alpha}_3(t-s)\varphi(s) + \dot{\alpha}_2(t-s)\psi(s)]\mathrm{d}s
\end{aligned} \tag{16}$$

By using (14), we can write

$$\int_0^t (t_{ij}\dot{e}_{ij} + \sigma_i\dot{\varphi}_{,i} + \tau_i\dot{\psi}_{,i} - \xi\dot{\varphi} - \zeta\dot{\psi})\mathrm{d}s = t_{ij}e_{ij} + \sigma_i\varphi_{,i} + \tau_i\psi_{,i} - \xi\varphi - \zeta\psi$$
$$- \int_0^t (\dot{t}_{ij}e_{ij} + \dot{\sigma}_i\varphi_{,i} + \dot{\tau}_i\psi_{,i} - \dot{\xi}\varphi - \dot{\zeta}\psi)\mathrm{d}s \tag{17}$$

It follows from (4), (12), (15) and (16) that

$$t_{ij}e_{ij} + \sigma_i\varphi_{,i} + \tau_i\psi_{,i} - \xi\varphi - \zeta\psi = 2W(t, 0; 0) + J(t) \tag{18}$$

where

$$\begin{aligned} J(t) = \int_0^t \{&\dot{C}_{ijmn}(t-s)e_{ij}(t)e_{mn}(s) + \dot{B}_{ij}(t-s)[e_{ij}(t)\varphi(s) + e_{ij}(s)\varphi(t)] \\ &+ \dot{D}_{ij}(t-s)[e_{ij}(t)\psi(s) + e_{ij}(s)\psi(t)] + \dot{\alpha}_{ij}(t-s)\varphi_{,j}(s)\varphi_{,i}(t) \\ &+ \dot{\beta}_{ij}(t-s)[\psi_{,j}(s)\varphi_{,i}(t) + \psi_{,j}(t)\varphi_{,i}(s)] + \dot{\gamma}_{ij}(t-s)\psi_{,j}(s)\psi_{,i}(t) \\ &+ \dot{\alpha}_1(t-s)\varphi(s)\varphi(t) + \dot{\alpha}_3(t-s)[\psi(s)\varphi(t) + \psi(t)\varphi(s)] \\ &+ \dot{\alpha}_2(t-s)\psi(s)\psi(t)\}\mathrm{d}s \end{aligned} \tag{19}$$

In a similar way we obtain

$$\dot{t}_{ij}e_{ij} + \dot{\sigma}_i\varphi_{,i} + \dot{\tau}_i\psi_{,i} - \dot{\xi}\varphi - \dot{\zeta}\psi = \frac{\mathrm{d}}{\mathrm{d}t}W(t, 0; 0) + 2V(t, 0; 0) + P(t) \tag{20}$$

where

$$\begin{aligned} P(t) = \int_0^t \{&\ddot{C}_{ijmn}(t-s)e_{mn}(s)e_{ij}(t) \\ &+ \ddot{B}_{ij}(t-s)[\varphi(s)e_{ij}(t) + \varphi(t)e_{ij}(s)] + \ddot{D}_{ij}(t-s)[\psi(s)e_{ij}(t) \\ &+ \psi(t)e_{ij}(s)] + \ddot{\alpha}_{ij}(t-s)\varphi_{,j}(s)\varphi_{,i}(t) + \ddot{\beta}_{ij}(t-s)[\psi_{,j}(s)\varphi_{,i}(t) \\ &+ \psi_{,j}(t)\varphi_{,i}(s)] + \ddot{\gamma}_{ij}(t-s)\psi_{,j}(s)\psi_{,i}(t) + \ddot{\alpha}_1(t-s)\varphi(s)\varphi(t) \\ &+ \ddot{\alpha}_3(t-s)[\psi(s)\varphi(t) + \psi(t)\varphi(s)] + \ddot{\alpha}_2(t-s)\psi(s)\psi(t)\} \end{aligned} \tag{21}$$

By (17), (18) and (20), we get

$$\int_0^t (t_{ij}\dot{e}_{ij} + \alpha_i\dot{\varphi}_{,i} + \tau_i\dot{\psi}_{,i} - \xi\dot{\varphi} - \zeta\dot{\psi})\mathrm{d}s = W(t, 0; 0) - 2\int_0^t V(s, 0; 0)\mathrm{d}s + Q(t) \tag{22}$$

where

$$Q(t) = J(t) - \int_0^t P(\tau)\mathrm{d}\tau, \quad t \in I \tag{23}$$

We shall use the identities [13]

$$\int_0^t \dot{B}_{ij}(t-s)[e_{ij}(t)\varphi(s) + e_{ij}(s)\varphi(t)]\mathrm{d}s = [B_{ij}(t) - B_{ij}(0)]e_{ij}(t)\varphi(t)$$
$$+ \int_0^t \dot{B}_{ij}(t-s)e_{ij}(s)\varphi(s)\mathrm{d}s - \int_0^t \dot{B}_{ij}(t-s)[e_{ij}(t) - e_{ij}(s)][\varphi(t) - \varphi(s)]\mathrm{d}s$$
$$\int_0^t \ddot{\alpha}_{ij}(|s-\tau|)\mathrm{d}s = \dot{\alpha}_{ij}(\tau) + \dot{\alpha}_{ij}(t-\tau) - 2\dot{\alpha}_{ij}(0)$$

$$2\int_0^t\int_0^s \ddot{\alpha}_{ij}(s-\tau)\varphi_{,i}(s)\varphi_{,j}(\tau)\mathrm{d}s\,\mathrm{d}\tau$$

$$=\int_0^t\int_0^t \ddot{\alpha}_{ij}(|s-\tau|)\varphi_{,i}(\tau)\varphi_{,j}(\tau)\mathrm{d}\tau\,\mathrm{d}s \tag{24}$$

$$-\frac{1}{2}\int_0^t\int_0^t \ddot{\alpha}_{ij}(|s-\tau|)[\varphi_{,i}(s)-\varphi_{,i}(\tau)][\varphi_{,j}(s)-\varphi_{,j}(\tau)]\mathrm{d}\tau\,\mathrm{d}s$$

$$\int_0^t\int_0^s \ddot{B}_{ij}(t-s)[e_{ij}(t)\varphi(s)+e_{ij}(s)\varphi(t)]\mathrm{d}\tau\,\mathrm{d}s$$

$$=\int_0^t\int_0^t \ddot{B}_{ij}(|s-\tau|)e_{ij}(s)\varphi(\tau)\mathrm{d}\tau\,\mathrm{d}s$$

$$=\int_0^t\int_0^s \ddot{B}_{ij}(|s-\tau|)e_{ij}(\tau)\varphi(\tau)\mathrm{d}\tau\,\mathrm{d}s$$

$$-\frac{1}{2}\int_0^t\int_0^t \ddot{B}_{ij}(|s-\tau|)[e_{ij}(s)-e_{ij}(\tau)][\varphi(s)-\varphi(\tau)]\mathrm{d}\tau\,\mathrm{d}s$$

It follows from (19), (21), (23), and (24) that

$$Q(t)=W(t,0;t)-W(t,0;0)-\int_0^t [V(t,s;t-s)+V(s,0;s) \tag{25}$$

$$-2V(s,0;0)]\mathrm{d}s+\frac{1}{2}\int_0^t\int_0^t U(s,\tau;|s-\tau|)\mathrm{d}s\,\mathrm{d}\tau,\ t\in I$$

By (22), (23), and (25), we obtain (13). □

Lemma 1 forms the basis of the following uniqueness theorem.

Theorem 1. *Assume that*

(i) ρ,κ_1 *and* κ_2 *are strictly positive;*
(ii) $W\geq 0, V\leq 0, U\geq 0$ *on* $B\times I$, *for any* f_{ij},η *and* γ *with* $f_{ij}=f_{ji}$.

Then the boundary-initial-value problem of viscoelasticity has at most one solution.

Proof. We define the addition of two admissible processes and multiplication of an admissible process by a scalar in the usual manner [11]. Suppose that there are two solutions. Then their difference $(u_i,\varphi,\psi,e_{ij},t_{ij},\sigma_i,\tau_i,\xi,\zeta)$ corresponds to null history and to null boundary data. From (10) and (13), we obtain

$$\int_B\{\frac{1}{2}(\rho\dot{u}_i\dot{u}_i+\kappa_1\dot{\varphi}^2+\kappa_2\dot{\psi}^2)+W(t,0;t)-\int_0^t V(s,0;s)\mathrm{d}s$$

$$-\int_0^t V(t,s;t-s)\mathrm{d}s+\frac{1}{2}\int_0^t\int_0^t U(\tau,s;|\tau-s|)\mathrm{d}\tau\,\mathrm{d}s\}\mathrm{d}v=0 \tag{26}$$

In view of the hypothesis (i) and (ii), we find that (26) implies $\dot{u}_i=0$, $\dot{\varphi}=0$, and $\dot{\psi}=0$ on $B\times I$. With the help of the initial data, we conclude that u_i,φ and ψ vanish on $B\times I$. □

The results of Edelstein and Gurtin [16] can be used to derive another uniqueness result.

4. Existence of solutions

In this section, we present an existence result for the solutions to the dynamical problem of nonhomogeneous viscoelastic materials with double porous structure. Our results extend the ones obtained by Martinez and Quintanilla [17] for the classical theory of viscoelastic materials with voids.

First, we recall an alternative way to write the basic system of equations. We have

$$\begin{aligned}
\rho\ddot{u}_i &= F_i + \Big(C_{ijmn}(0)e_{mn}(t) + B_{ij}(0)\varphi(t) + D_{ij}(0)\psi(t) \\
&\quad + \int_0^\infty [\dot{C}_{ijmn}(s)e_{mn}(t-s) + \dot{B}_{ij}(s)\varphi(t-s) + \dot{D}_{ij}(s)\psi(t-s)]\mathrm{d}s\Big)_{,j} \\
\kappa_1\ddot{\varphi} &= G + \Big(\alpha_{ij}(0)\varphi_{,j}(t) + \beta_{ij}(0)\psi_{,j}(t) + \int_0^\infty [\dot{\alpha}_{ij}(s)\varphi_{,j}(t-s) + \dot{\beta}_{ij}(s)\psi_{,j}(t-s)]\mathrm{d}s\Big)_{,j} \\
&\quad - B_{ij}(0)e_{ij}(t) - \alpha_1(0)\varphi(t) - \alpha_3(0)\psi(t) \\
&\quad - \int_0^\infty [\dot{B}_{ij}(s)e_{ij}(t-s) + \dot{\alpha}_1(s)\varphi(t-s) + \dot{\alpha}_3(s)\psi(t-s)]\mathrm{d}s \\
\kappa_2\ddot{\psi} &= L + \Big(\beta_{ji}(0)\varphi_{,j}(t) + \gamma_{ij}(0)\psi_{,j}(t) + \int_0^\infty [\dot{\beta}_{ji}(s)\varphi_{,j}(t-s) + \dot{\gamma}_{ij}(s)\psi_{,j}(t-s)]\mathrm{d}s\Big)_{,j} \\
&\quad - D_{ij}(0)e_{ij}(t) - \alpha_3(0)\varphi(t) - \alpha_2(0)\psi(t) - \int_0^\infty [\dot{D}_{ij}(s)e_{ij}(t-s) \\
&\quad + \dot{\alpha}_3(s)\varphi(t-s) + \dot{\alpha}_2(s)\psi(t-s)]\mathrm{d}s
\end{aligned} \tag{27}$$

In this section, we assume homogenous Dirichlet boundary conditions. That is

$$u_i(\mathbf{x},t) = \varphi(\mathbf{x},t) = \psi(\mathbf{x},t) = 0 \quad (\mathbf{x},t) \in \partial B \times I \tag{28}$$

The initial history condition implies:

$$u_i(\mathbf{x},-s) = u_i^*(\mathbf{x},-s), \quad \varphi(\mathbf{x},-s) = \varphi^*(\mathbf{x},-s), \quad \psi(\mathbf{x},-s) = \psi^*(\mathbf{x},-s) \tag{29}$$

for $(\mathbf{x},s) \in B \times [0,\infty)$.

In this section we assume that

(a) the mass density and the equilibrated inertias are positive,

$$\rho(\mathbf{x}) \geq \rho_1 > 0, \quad \kappa_1(\mathbf{x}) \geq \kappa_1^* > 0, \quad \kappa_2(\mathbf{x}) \geq \kappa_2^* > 0 \tag{30}$$

(b) there exists a positive constant C_0 such that

$$\begin{aligned}
&C_{ijkl}(\infty)\xi_{ij}\xi_{kl} + 2B_{ij}(\infty)\xi_{ij}m + 2D_{ij}(\infty)\xi_{ij}n + \alpha_1(\infty)m^2 + \alpha_2(\infty)n^2 \\
&+ 2\alpha_3(\infty)mn + \alpha_{ij}(\infty)\xi_i\xi_j + \gamma_{ij}(\infty)\eta_i\eta_j + 2\beta_{ij}(\infty)\xi_i\eta_j \\
&\geq C_0(\xi_{ij}\xi_{ij} + m^2 + n^2 + \xi_i\xi_i + \eta_j\eta_j)
\end{aligned} \tag{31}$$

for any $\xi_{ij}, m, n, \xi_i, \eta_j$ such that $\xi_{ij} = \xi_{ji}$.

(c) there exists a positive function $\delta(s)$ for $s \geq 0$ such that

$$\begin{aligned}
&\ddot{C}_{ijkl}(s)\xi_{ij}\xi_{kl} + 2\ddot{B}_{ij}(s)\xi_{ij}m + 2\ddot{D}_{ij}(s)\xi_{ij}n + \ddot{\alpha}_1(s)m^2 + \ddot{\alpha}_2(s)n^2 \\
&+ 2\ddot{\alpha}_3(s)mn + \ddot{\alpha}_{ij}(s)\xi_i\xi_j + \ddot{\gamma}_{ij}(s)\eta_i\eta_j + 2\ddot{\beta}_{ij}(s)\xi_i\eta_j \\
&\geq \delta(s)(\xi_{ij}\xi_{ij} + m^2 + n^2 + \xi_i\xi_i + \eta_j\eta_j)
\end{aligned} \tag{32}$$

for any $\xi_{ij}, m, n, \xi_i, \eta_j$ such that $\xi_{ij} = \xi_{ji}$.

It is worth noting that, from these assumptions, we can obtain the existence of a positive function $\delta_1(s)$ such that

$$-\Big(\dot{C}_{ijkl}(s)\xi_{ij}\xi_{kl}+2\dot{B}_{ij}(s)\xi_{ij}m+2\dot{D}_{ij}(s)\xi_{ij}n+\dot{\alpha}_1(s)m^2+\dot{\alpha}_2(s)n^2$$
$$+2\dot{\alpha}_3(s)mn+\dot{\alpha}_{ij}(s)\xi_i\xi_j+\dot{\gamma}_{ij}(s)\eta_i\eta_j+2\dot{\beta}_{ij}(s)\xi_i\eta_j\Big)$$
$$\geq\delta_1(s)(\xi_{ij}\xi_{ij}+m^2+n^2+\xi_i\xi_i+\eta_j\eta_j) \tag{33}$$

for any $\xi_{ij}, m, n, \xi_i, \eta_j$ such that $\xi_{ij}=\xi_{ji}$.

We now consider the Hilbert space where we are going to study our problem. We will treat with the elements of the form

$$\mathcal{U}=(\mathbf{u},\mathbf{v},\varphi,\phi,\psi,\chi,\mathbf{z},l,k) \tag{34}$$

where $\mathbf{v}=\dot{\mathbf{u}}$, $\phi=\dot{\varphi}$, $\chi=\dot{\psi}$, $\mathbf{z}(s)=\mathbf{u}(t-s)$, $l(s)=\varphi(t-s)$ and $k(s)=\psi(t-s)$.

We consider the Hilbert space $\mathcal{Z}=\{(\mathbf{u},\mathbf{v},\varphi,\phi,\psi,\chi,\mathbf{z},l,k)\}$ such that $\mathbf{u}\in\mathbf{W}_0^{1,2}$, $\varphi,\psi\in W_0^{1,2}$, $\mathbf{v}\in\mathbf{L}^2$, $\phi,\chi\in L^2$ meanwhile $\mathbf{z},l,k$ are in the completion of the elements in $C_0^\infty([0,\infty),\mathbf{W}_0^{1,2}\cap\mathbf{W}^{2,2})$, $C_0^\infty([0,\infty),W_0^{1,2}\cap W^{2,2})$ with respect to the inner product

$$< (\mathbf{u},\mathbf{v},\varphi,\phi,\psi,\chi,\mathbf{z},l,k),(\mathbf{u}',\mathbf{v}',\varphi',\phi',\psi',\chi',\mathbf{z}',l',k') >_{\mathcal{Z}}=$$
$$=\int_B(\rho v_i v_i'+\kappa_1\phi\phi'+\kappa_2\chi\chi')\mathrm{d}v$$
$$+\int_B\Big(C_{ijkl}(\infty)u_{i,j}u'_{k,l}+B_{ij}(\infty)(u_{i,j}\varphi'+u'_{i,j}\varphi))+D_{ij}(\infty)(u_{i,j}\psi'+u'_{i,j}\psi))$$
$$+\alpha_1(\infty)\varphi\varphi'+\alpha_2(\infty)\psi\psi'+\alpha_3(\infty)(\varphi\psi'+\varphi'\psi)$$
$$+\alpha_{ij}(\infty)\varphi_{,i}\varphi'_{,j}+\gamma_{ij}(\infty)\psi_{,i}\psi'_{,j}+\beta_{ij}(\infty)(\varphi_{,i}\psi'_{,j}+\varphi'_{,i}\psi_{,j})\Big)\mathrm{d}v$$
$$-\int_B\int_0^\infty\Big(\dot{C}_{ijkl}(s)(u_{i,j}-z_{i,j}(s))(u'_{k,l}-z'_{k,l}(s))$$
$$+\dot{B}_{ij}(s)[(u_{i,j}-z_{i,j}(s))(\varphi'-l'(s))+(u'_{i,j}-z'_{i,j}(s))(\varphi-l(s))]$$
$$+\dot{D}_{ij}(s)[(u_{i,j}-z_{i,j}(s))(\psi'-k'(s))+(u'_{i,j}-z'_{i,j}(s))(\psi-k(s))]$$
$$+\dot{\alpha}_1(s)(\varphi-l(s))(\varphi'-l'(s))+\dot{\alpha}_2(s)(\psi-k(s))(\psi'-k'(s))$$
$$+\dot{\alpha}_3(s)[(\varphi-l(s))(\psi'-k'(s))+(\varphi'-l'(s))(\psi-k(s))] \tag{35}$$
$$+\dot{\alpha}_{ij}(s)(\varphi_{,i}-l_{,i}(s))(\varphi'_{,j}-l'_{,j}(s))+\dot{\gamma}_{ij}(s)(\psi_{,i}-k_{,i}(s))(\psi'_{,j}-k'_{,j}(s))$$
$$+\dot{\beta}_{ij}(s)[(\varphi_{,i}-l_{,i}(s))(\psi'_{,j}-k'_{,j}(s))+(\varphi'_{,i}-l'_{,i}(s))(\psi_{,j}-l_{,j}(s))]\Big)\mathrm{d}s\,\mathrm{d}v$$

In view of the assumptions (30)–(32), the norm defined by this inner product is equivalent to the usual one in $\mathcal{Z}$.

We define the following operators:

$$B_i(\mathbf{u})=\rho^{-1}(C_{ijkl}(0)u_{k,l})_{,j},\quad \mathbf{B}=(B_i)$$
$$C_i(\varphi)=\rho^{-1}(B_{ij}(0)\varphi)_{,j},\quad \mathbf{C}=(C_i)$$
$$D_i(\psi)=\rho^{-1}(D_{ij}(0)\psi)_{,j},\quad \mathbf{D}=(D_i)$$
$$P_i(\mathbf{z})=\rho^{-1}\left(\int_0^\infty\dot{C}_{ijkl}(s)z_{k,l}(s)\mathrm{d}s\right)_{,j},\quad \mathbf{P}=(P_i)$$
$$E_i(l)=\rho^{-1}\left(\int_0^\infty\dot{B}_{ij}(s)l(s)\mathrm{d}s\right)_{,j},\quad \mathbf{E}=(E_i)$$
$$F_i^*(k)=\rho^{-1}\left(\int_0^\infty\dot{D}_{ij}(s)k(s)\mathrm{d}s\right)_{,j},\quad \mathbf{F}^*=(F_i)$$

$$G^*(\mathbf{u}) = -\kappa_1^{-1} B_{ij}(0)u_{i,j}, \quad H(\varphi) = \kappa_1^{-1}[(\alpha_{ij}(0)\varphi_{,i})_{,j} - \alpha_1(0)\varphi]$$

$$J(\psi) = \kappa_1^{-1}[(\beta_{ij}(0)\psi_{,i})_{,j} - \alpha_3\psi], \quad K(\mathbf{z}) = \kappa_1^{-1}[\int_0^\infty \dot{B}_{ij}(s)z_{i,j}(s)\mathrm{d}s]$$

$$L^*(l) = \kappa_1^{-1}[(\int_0^\infty \dot{\alpha}_{ij}l_{,i}(s)\mathrm{d}s)_{,j} - \int_0^\infty \dot{\alpha}_1 l(s)\mathrm{d}s]$$

$$M(k) = \kappa_1^{-1}[(\int_0^\infty \dot{\beta}_{ij}k_{,i}(s)\mathrm{d}s)_{,j} - \int_0^\infty \dot{\alpha}_3 k(s)\mathrm{d}s]$$

$$O(\mathbf{u}) = -\kappa_2^{-1} D_{ij}(0)u_{i,j}, \quad P(\varphi) = \kappa_2^{-1}[(\beta_{ij}(0)\varphi_{,i})_{,j} - \alpha_3(0)\varphi]$$

$$Q(\psi) = \kappa_2^{-1}[(\gamma_{ij}(0)\psi_{,i})_{,j} - \alpha_2(0)\psi]$$

$$R(\mathbf{z}) = -\kappa_2^{-1}[\int_0^\infty \dot{D}_{ij}(s)z_{i,j}(s)\mathrm{d}s]$$

$$S(l) = \kappa_2^{-1}[(\int_0^\infty \dot{\beta}_{ij}l_{,i}(s)\mathrm{d}s)_{,j} - \int_0^\infty \dot{\alpha}_3 l(s)\mathrm{d}s]$$

$$T(k) = \kappa_2^{-1}[(\int_0^\infty \dot{\gamma}_{ij}k_{,i}(s)\mathrm{d}s)_{,j} - \int_0^\infty \dot{\alpha}_2 k(s)\mathrm{d}s]$$

$$U_i(\mathbf{z}) = \frac{\mathrm{d}}{\mathrm{d}s}z_i(s), \quad \mathbf{U} = (U_i)$$

$$V(l) = \frac{\mathrm{d}}{\mathrm{d}s}l(s), \quad W(k) = \frac{\mathrm{d}}{\mathrm{d}s}k(s)$$

We consider the matrix operator $\mathcal{A}$:

$$\mathcal{A} = \begin{pmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{B} & \mathbf{0} & \mathbf{C} & \mathbf{0} & \mathbf{D} & \mathbf{0} & \mathbf{P} & \mathbf{E} & \mathbf{F} \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ G^* & 0 & H & 0 & J & 0 & K & L^* & M \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ O & 0 & P & 0 & Q & 0 & R & S & T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{U} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & V & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & W \end{pmatrix} \tag{36}$$

It is worth noting that the domain of the operator is the subspace $(\mathbf{u}, \mathbf{v}, \varphi, \phi, \psi, \chi, \mathbf{z}, l, k)$ such that $\mathbf{u} \in \mathbf{W}_0^{1,2} \cap \mathbf{W}^{2,2}$, $\mathbf{v} \in \mathbf{W}_0^{1,2}$, $\varphi, \psi \in W_0^{1,2} \cap W^{2,2}$, $\phi, \chi \in W_0^{1,2}$ and such that $\mathbf{z}(0) = \mathbf{u}$, $l(0) = \varphi$, $k(0) = \psi$.

It is clear that the domain of the operator is a dense subspace of $\mathcal{Z}$. We can write our problem in the form

$$\frac{\mathrm{d}\mathcal{U}}{\mathrm{d}t} = \mathcal{A}\mathcal{U} + \mathcal{F}(t), \ \mathcal{U}(0) = \mathcal{U}_0 \tag{37}$$

where

$$\mathcal{U} = (\mathbf{u}, \mathbf{v}, \varphi, \phi, \psi, \chi, \mathbf{z}, l, k) \in \mathcal{D}$$

and

$$\mathcal{F} = (\mathbf{0}, \rho^{-1}\mathbf{F}, 0, \kappa_1^{-1}G, 0, \kappa_2^{-1}L, \mathbf{0}, 0, 0)$$

Lemma 2. *The operator $\mathcal{A}$ defined previously satisfies that*

$$< \mathcal{A}\mathcal{U}, \mathcal{U} > \leq 0$$

for every $\mathcal{U} \in \mathcal{D}$.

Proof. If we make use of the boundary conditions and of the divergence theorem, we see that

$$
\begin{aligned}
< \mathcal{AU},\mathcal{U} >= -\frac{1}{2}\int_B\int_0^\infty [&\ddot{C}_{ijkl}(s)(u_{i,j}-z_{i,j}(s))(u_{i,j}-z_{i,j}(s)) \\
&+2\ddot{B}_{ij}(s)(u_{i,j}-z_{i,j}(s))(\varphi-l(s)) \\
&+2\ddot{D}_{ij}(s)(u_{i,j}-z_{i,j}(s))(\psi-k(s)) \\
&+\ddot{\alpha}_1(\varphi-l(s))^2+\ddot{\alpha}_2(\psi-k(s))^2 \\
&+2\ddot{\alpha}_3(\varphi-l(s))(\psi-k(s)) \\
&+\ddot{\alpha}_{ij}(\varphi_{,i}-l_{,i}(s))(\varphi_{,j}-l_{,j}(s)) \\
&+\ddot{\gamma}_{ij}(\psi_{,i}-k_{,i}(s))(\psi_{,j}-k_{,j}(s)) \\
&+\ddot{\beta}_{ij}(\varphi_{,i}-l_{,i}(s))(\psi_{,j}-k_{,j}(s))]\mathrm{d}s\,\mathrm{d}v
\end{aligned} \tag{38}
$$

In view of the assumption (32), the lemma is proved. □

Lemma 3. *The operator $\mathcal{A}$ defined previously satisfies the condition:*

$$\text{Range}\,(\mathcal{I}-\mathcal{A})=\mathcal{Z}$$

Proof. Let $\mathcal{U}'=(\mathbf{v}',\mathbf{v}',\varphi',\phi',\psi',\chi',\mathbf{z}',l',k')\in\mathcal{Z}$. We need to solve the system:

$$
\begin{aligned}
\mathbf{u}-\mathbf{v}&=\mathbf{u}' \\
\mathbf{v}-\mathbf{Bu}-\mathbf{C}\varphi-\mathbf{D}\psi-\mathbf{Pz}-\mathbf{E}l-\mathbf{F}k&=\mathbf{v}' \\
\varphi-\phi&=\varphi' \\
\phi-G^*\mathbf{u}-H\varphi-J\psi-K\mathbf{z}-L^*l-Mk&=\phi' \\
\psi-\chi&=\psi' \\
\chi-O\mathbf{u}-P\varphi-Q\psi-R\mathbf{z}-Sl-Tk&=\chi' \\
\mathbf{z}-\mathbf{Uz}&=\mathbf{z}' \\
l-Vl&=l' \\
k-Wk&=k'
\end{aligned} \tag{39}
$$

From (39) we see that

$$
\begin{aligned}
\mathbf{z}(s)&=\mathrm{e}^{-s}(\mathbf{u}+\int_0^s \mathrm{e}^{\tau}\mathbf{z}'(\tau)\mathrm{d}\tau) \\
l(s)&=\mathrm{e}^{-s}(\varphi+\int_0^s \mathrm{e}^{\tau}l'(\tau)\mathrm{d}\tau) \\
k(s)&=\mathrm{e}^{-s}(\psi+\int_0^s \mathrm{e}^{\tau}k'(\tau)\mathrm{d}\tau)
\end{aligned} \tag{40}
$$

Therefore, we can obtain the system

$$
\begin{pmatrix} \mathbf{B}^* & \mathbf{C}^* & \mathbf{D}^* \\ E^* & F^* & H^* \\ J^* & K^* & M^* \end{pmatrix}\begin{pmatrix} \mathbf{u} \\ \varphi \\ \psi \end{pmatrix}=\begin{pmatrix} \mathbf{m} \\ n_1 \\ n_2 \end{pmatrix} \tag{41}
$$

where

$$B_i^*(\mathbf{u})=u_i-\rho^{-1}[(C_{ijkl}(0)+\int_0^\infty \dot{C}_{ijkl}(s)\mathrm{e}^{-s}\mathrm{d}s)u_{k,l}]_{,j},\quad \mathbf{B}^*=(B_i^*)$$

$$C_i^*(\varphi) = -\rho^{-1}[(B_{ij}(0) + \int_0^\infty \dot{B}_{ij}(s)\mathrm{e}^{-s}\mathrm{d}s)\varphi]_{,j}, \quad \mathbf{C}^* = (C_i^*)$$

$$D_i^*(\psi) = -\rho^{-1}[(D_{ij}(0) + \int_0^\infty \dot{D}_{ij}(s)\mathrm{e}^{-s}\mathrm{d}s)\psi]_{,j}, \quad \mathbf{D}^* = (D_i^*)$$

$$E^*(\mathbf{u}) = -\kappa_1^{-1}[(B_{ij}(0) + \int_0^\infty \dot{B}_{ij}(s)\mathrm{e}^{-s}\mathrm{d}s)u_{i,j}]$$

$$F^*\varphi = \varphi - \kappa_1^{-1}[(\alpha_{ij}(0) + \int_0^\infty \dot{\alpha}_{ij}(s)\mathrm{e}^{-s}\mathrm{d}s)\varphi_{,i}]_{,j} + \kappa_1^{-1}[(\alpha_1(0) + \int_0^\infty \dot{\alpha}_1(s)\mathrm{e}^{-s}\mathrm{d}s)\varphi]$$

$$H^*\psi = -\kappa_1^{-1}[(\beta_{ij}(0) + \int_0^\infty \dot{\beta}_{ij}(s)\mathrm{e}^{-s}\mathrm{d}s)\psi_{,j}]_{,i} + \kappa_1^{-1}[(\alpha_3(0) + \int_0^\infty \dot{\alpha}_3(s)\mathrm{e}^{-s}\mathrm{d}s)\psi]$$

$$J^*(\mathbf{u}) = -\kappa_2^{-1}[(D_{ij}(0) + \int_0^\infty \dot{D}_{ij}(s)\mathrm{e}^{-s}\mathrm{d}s)u_{i,j}]$$

$$K^*\varphi = -\kappa_2^{-1}[(\beta_{ij}(0) + \int_0^\infty \dot{\beta}_{ij}(s)\mathrm{e}^{-s}\mathrm{d}s)\varphi_{,i}]_{,j} + \kappa_2^{-1}[(\alpha_3(0) + \int_0^\infty \dot{\alpha}_3(s)\mathrm{e}^{-s}\mathrm{d}s)\varphi]$$

$$M^*\psi = \psi - \kappa_2^{-1}[(\gamma_{ij}(0) + \int_0^\infty \dot{\gamma}_{ij}(s)\mathrm{e}^{-s}\mathrm{d}s)\psi_{,j}]_{,i} + \kappa_2^{-1}[(\alpha_2(0) + \int_0^\infty \dot{\alpha}_2(s)\mathrm{e}^{-s}\mathrm{d}s)\psi]$$

$$m_i = u_i' + v_i' + \rho^{-1}\int_0^\infty \int_0^s \mathrm{e}^{\tau-s}\big(\dot{C}_{ijkl}(s)z'_{k,l}(\tau) + \dot{B}_{ij}(s)k'(\tau) + \dot{D}_{ij}(s)l'(\tau)\big)_{,j}\mathrm{d}\tau\,\mathrm{d}s$$

$$n_1 = \varphi' + \phi' + \kappa_1^{-1}\int_0^\infty \int_0^s \mathrm{e}^{\tau-s}\big(\dot{\alpha}_{ij}(s)l'_{,i}(\tau) + \dot{\beta}_{ij}(s)k'_{,i}(\tau)\big)_{,j}\mathrm{d}\tau$$

$$-\kappa_1^{-1}\int_0^\infty \int_0^s \mathrm{e}^{\tau-s}\big(\dot{B}_{ij}(s)z'_{i,j}(\tau) + \dot{\alpha}_1(s)l'(\tau) + \dot{\alpha}_3(s)k'(\tau)\big)\mathrm{d}\tau\,\mathrm{d}s$$

$$n_2 = \psi' + \chi' + \kappa_2^{-1}\int_0^\infty \int_0^s \mathrm{e}^{\tau-s}\big(\dot{\beta}_{ij}(s)l'_{,i}(\tau) + \dot{\gamma}_{ij}(s)k'_{,i}(\tau)\big)_{,j}\mathrm{d}\tau\,\mathrm{d}s$$

$$-\kappa_1^{-1}\int_0^\infty \int_0^s \mathrm{e}^{\tau-s}\big(\dot{D}_{ij}(s)z'_{i,j}(\tau) + \dot{\alpha}_3(s)l'(\tau) + \dot{\alpha}_2(s)k'(\tau)\big)\mathrm{d}\tau$$

In order to study the system (41), we consider the bilinear form defined on $\mathbf{W}_0^{1,2} \times W_0^{1,2} \times W_0^{1,2}$:

$$\mathcal{R}[(\mathbf{u},\varphi,\psi),\hat{\mathbf{u}},\hat{\varphi},\hat{\psi})] = \tag{42}$$

$$= < (\mathbf{B}^*\mathbf{u} + \mathbf{C}^*\varphi + \mathbf{D}^*\psi, \mathbf{E}^*\mathbf{u} + \mathbf{F}^*\varphi + \mathbf{H}^*\psi, \mathbf{J}^*\mathbf{u} + \mathbf{K}^*\varphi + \mathbf{M}^*\psi), (\rho\hat{\mathbf{u}}, \kappa_1\hat{\varphi}, \kappa_2\hat{\psi} >$$

where this product is taken in $\mathbf{L}^2 \times L^2 \times L^2$. It is clear that this product is bounded. On the other side, we see that

$$\mathcal{R}[(\mathbf{u},\varphi,\psi),(\mathbf{u},\varphi,\psi)] = \tag{43}$$

$$\int_B (\rho u_i u_i + \kappa_1\varphi^2 + \kappa_2\psi^2)\mathrm{d}v$$

$$+\int_B\int_0^\infty e^{-s}[C_{ijkl}(s)u_{i,j}u_{k,l}+2B_{ij}(s)u_{i,j}\varphi+2D_{ij}(s)\psi+\alpha_1(s)\varphi^2+\alpha_2(s)\psi^2+2\alpha_3(s)\varphi\psi]\mathrm{d}s\,\mathrm{d}v$$
$$+\int_B\int_0^\infty e^{-s}[\alpha_{ij}(s)\varphi_{,i}\varphi_{,j}+2\beta_{ij}(s)\varphi_{,i}\psi_{,j}+\gamma(s)\phi_{,i}\psi_{,j}]\mathrm{d}s\,\mathrm{d}v$$

Therefore, in view of (30) and (31), $\mathcal{R}$ is a coercive bilinear form. On the other side, it is clear that $(\mathbf{m},n_1,n_2)\in\mathbf{W}^{-1,2}\times W^{-1,2}\times W^{-1,2}$. The Lax–Milgram theorem implies the existence of a solution to our problem. We also see the existence of $(\mathbf{v},\phi,\chi)$ in $\mathbf{W}_0^{1,2}\times W_0^{1,2}\times W_0^{1,2}$ and we can prove that $\mathbf{z}(s),l(s),k(s)$ belongs in the corresponding space. □

We have proved Lemma 3.

Theorem 2. *The operator $\mathcal{A}$ defined previously generates a contractive semigroup in $\mathcal{Z}$.*

As a consequence, we obtain Theorem 3.

Theorem 3. *Let us assume that*

$$F_i,G,L\in C^1([0,\infty,L^2])\cap C^0([0,\infty,W_0^{1,2}]),\ \textit{and}\ \mathcal{U}_0\in\mathcal{D}$$

Then, there exists a unique solution to the problem determined by the system (27) and the conditions (28), (29), such that $\mathcal{U}(t)\in C^1([0,\infty,\mathcal{Z}])$.

5. Minimum principle

In this section, we use the results of Reiss [18] and Reiss and Haug [19] to derive a minimum principle for the porous viscoelastic materials. First, we give an alternative characterization of the problem formulated in Section 2 and derive a reciprocity relation. We introduce the functions j and l on $[0,\infty)$ by

$$l(t)=1,\ j=(l*l)(t)=t,\ t\in[0,\infty) \tag{44}$$

Let H_i, S and T be functions on $\overline{B}\times[0,\infty)$ defined by

$$\begin{aligned}&H_i=j*F_i+\rho(tv_i^0+u_i^0),\ s=j*G+\kappa_1(t\varphi_1^0+\varphi^0)\\&T=j*L+\kappa_2(t\psi_1^0+\psi^0)\end{aligned} \tag{45}$$

Following Gurtin [11], we have Lemma 4.

Lemma 4. *Let $u_i,\varphi,\psi\in C^{0,2}$, $t_{ij},\sigma_i,\tau_i\in C^{1,0}$, and $\xi,\zeta\in C^0$. Then $u_i,\varphi,\psi,t_{ij},\sigma_i,\tau_i,\xi$ and ζ satisfy the equations (2) and the initial conditions (6) if and only if*

$$j*t_{ki,k}+H_i=\rho u_i,\ j*(\sigma_{i,i}+\xi)+S=\kappa_1\varphi,\ j*(\tau_{i,i}+\zeta)+T=\kappa_2\psi \tag{46}$$

on $B\times[0,\infty)$.

The next proposition is an immediate consequence of Lemma 4.

Theorem 4. *Let π be an admissible process. Then π is a solution to the problem if and only if π satisfies the equations (1), (9), (46) the initial history condition (5), and the boundary conditions (7).*

We consider two external data systems $\mathcal{L}^{(\alpha)}=\{F_i^{(\alpha)},G^{(\alpha)},L^{(\alpha)},\widetilde{u}_i^{(\alpha)},\widetilde{\varphi}^{(\alpha)},\ \widetilde{\psi}^{(\alpha)},\widetilde{t}_i^{(\alpha)},\widetilde{\sigma}^{(\alpha)},\widetilde{\tau}^{(\alpha)},\pi^{*(\alpha)}\}$, $(\alpha=1,2)$, and denote by $\pi^{(\alpha)}=(u_i^{(\alpha)},\varphi^{(\alpha)},\psi^{(\alpha)},e_{ij}^{(\alpha)},t_{ij}^{(\alpha)},\sigma_i^{(\alpha)},\tau_i^{(\alpha)},\xi^{(\alpha)},\zeta^{(\alpha)})$ a solution corresponding to $\mathcal{L}^{(\alpha)}$. We introduce the notations

$$\begin{aligned}&t_i^{(\alpha)}=t_{ji}^{(\alpha)}n_j,\sigma^{(\alpha)}=\sigma_j^{(\alpha)}n_j,\tau^{(\alpha)}=\tau_j^{(\alpha)}n_j\\&H_i^{(\alpha)}=j*F_i^{(\alpha)}+\rho(tv_i^{0(\alpha)}+u_i^{0(\alpha)})\\&S^{(\alpha)}=j*G^{(\alpha)}+\kappa_1(t\varphi_1^{0(\alpha)}+\varphi^{0(\alpha)}),T^{(\alpha)}=j*H^{(\alpha)}+\kappa_2(t\psi_1^{0(\alpha)}+\psi^{0(\alpha)})\end{aligned}$$

$$s_i^{(\alpha)} = s_{ji}^{(\alpha)} n_j, \pi^{(\alpha)} = \pi_k^{(\alpha)} n_k, \chi^{(\alpha)} = \chi_k^{(\alpha)} n_k \tag{47}$$
$$\widetilde{H}_i^{(\alpha)} = j * s_{ki,k}^{(\alpha)}, \widetilde{S}^{(\alpha)} = j * (\pi_{k,k}^{(\alpha)} + \nu^{(\alpha)})$$
$$\widetilde{T}^{(\alpha)} = j * (\chi_{k,k}^{(\alpha)} + \vartheta^{(\alpha)})$$
$$\Gamma_{\alpha\beta} = \int_B [(H_i^{(\alpha)} + \widetilde{H}_i^{(\alpha)}) * u_i^{(\beta)} + (S^{(\alpha)} + \widetilde{S}^{(\alpha)}) * \varphi^{(\beta)}$$
$$+ (T^{(\alpha)} + \widetilde{T}^{(\alpha)}) * \psi^{(\beta)}]\mathrm{d}v + \int_{\partial B} j * [(t_i^{(\alpha)} - s_i^{(\alpha)}) * u_i^{(\beta)}$$
$$+ (\sigma^{(\alpha)} - \pi^{(\alpha)}) * \varphi^{(\beta)} + (\tau^{(\alpha)} - \chi^{(\alpha)}) * \psi^{(\beta)}]\mathrm{d}a$$

Lemma 5. *If the body is subjected to two external data systems, then the corresponding solutions $\pi^{(\alpha)}$ ($\alpha = 1, 2$) satisfy the reciprocity relation*

$$\Gamma_{12} = \Gamma_{21} \tag{48}$$

Proof. If we denote

$$J_{\alpha\beta} = l * [(t_{ij}^{(\alpha)} - s_{ij}^{(\alpha)}) * e_{ij}^{(\beta)} + (\sigma_i^{(\alpha)} - \pi_i^{(\alpha)}) * \varphi_{,i}^{(\beta)} + (\tau_i^{(\alpha)} - \chi_i^{(\alpha)}) * \psi_{,i}^{(\beta)}$$
$$- (\xi^{(\alpha)} - \nu^{(\alpha)}) * \varphi^{(\beta)} - (\zeta^{(\alpha)} - \vartheta^{(\alpha)}) * \psi^{(\beta)}] \tag{49}$$

then from (9) we find that

$$J_{\alpha\beta} = C_{ijkl} * e_{kl}^{(\alpha)} * e_{ij}^{(\beta)} + B_{ij} * (\varphi^{(\alpha)} * e_{ij}^{(\beta)} + \varphi^{(\beta)} * e_{ij}^{(\alpha)})$$
$$+ D_{ij} * (\psi^{(\alpha)} * e_{ij}^{(\beta)} + \psi^{(\beta)} * e_{ij}^{(\alpha)}) + \alpha_{ij} * \varphi_{,j}^{(\alpha)} * \varphi_{,i}^{(\beta)}$$
$$+ \beta_{ij} * (\psi_{,j}^{(\alpha)} * \varphi_{,i}^{(\beta)} + \psi_{,j}^{(\beta)} * \varphi_{,i}^{(\alpha)}) + \gamma_{ij} * \psi_{,j}^{(\alpha)} * \psi_{,i}^{(\beta)} \tag{50}$$
$$+ \alpha_1 * \varphi^{(\alpha)} * \varphi^{(\beta)} + \alpha_3 * (\psi^{(\alpha)} * \varphi^{(\beta)} + \psi^{(\beta)} * \varphi^{(\alpha)}) + \alpha_2 * \psi^{(\alpha)} * \psi^{(\beta)}$$

In view of (4) and (50), we obtain

$$J_{12} = J_{21} \tag{51}$$

On the other hand, by (49), (1), and (46), we obtain

$$l * J_{\alpha\beta} = j * [(t_{ki}^{(\alpha)} - s_{ki}^{(\alpha)}) * u_i^{(\beta)} + (\sigma_k^{(\alpha)} - \pi_k^{(\alpha)}) * \varphi^{(\beta)} + (\tau_k^{(\alpha)} - \chi_k^{(\alpha)}) * \psi^{(\beta)}]_{,k}$$
$$- \rho u_i^{(\alpha)} * u_i^{(\beta)} - \kappa_1 \varphi^{(\alpha)} * \varphi^{(\beta)} - \kappa_2 \psi^{(\alpha)} * \psi^{(\beta)} + H_i^{(\alpha)} * u_i^{(\beta)} + S^{(\alpha)} * \varphi^{(\beta)} \tag{52}$$
$$+ T^{(\alpha)} * \psi^{(\beta)} + j * [s_{ki,k}^{(\alpha)} * u_i^{(\beta)} + (\pi_{k,k}^{(\alpha)} + \nu^{(\alpha)}) * \varphi^{(\beta)} + (\chi_{k,k}^{(\alpha)} + \vartheta^{(\alpha)}) * \psi^{(\beta)}]$$

If we integrate this relation over B, and use (47), then we get

$$\int_B l * J_{\alpha\beta}\, \mathrm{d}v = \int_{\partial B} j * [(t_i^{(\alpha)} - s_i^{(\alpha)}) * u_i^{(\beta)} + (\sigma^{(\alpha)} - \pi^{(\alpha)}) * \varphi^{(\beta)}$$
$$+ (\tau^{(\alpha)} - \chi^{(\alpha)}) * \psi^{(\beta)}]\mathrm{d}a + \int_B [(H_i^{(\alpha)} + \widetilde{H}_i^{(\alpha)}) * u_i^{(\beta)} \tag{53}$$
$$+ (S^{(\alpha)} + \widetilde{S}^{(\alpha)}) * \varphi^{(\beta)} + (T^{(\alpha)} + \widetilde{T}^{(\alpha)}) * \psi^{(\beta)} - \rho u_i^{(\alpha)} * u_i^{(\beta)}$$
$$- \kappa_1 \varphi^{(\alpha)} * \varphi^{(\beta)} - \kappa_2 \psi^{(\alpha)} * \psi^{(\beta)}]\mathrm{d}v$$

From (51) and (53), we obtain (48). □

We say that f has a Laplace transform $\overline{f}$ (or $\mathcal{L}f$) if there exists a real number $p_0 \geq 0$ such that, for every $p \in [p_0, \infty)$, the integral

$$\overline{f}(\mathbf{x},p)=\int_0^\infty \mathrm{e}^{-pt} f(\mathbf{x},t)\mathrm{d}t$$

converges uniformly on B.

We assume that

(A_1) $F_i, G, L, \widetilde{u}_i, \widetilde{\varphi}, \widetilde{\psi}, \widetilde{t}_i, \widetilde{\sigma}, \widetilde{\tau}$ and the constitutive functions possess Laplace transforms;
(A_2) $\overline{C}_{ijmn} f_{ij} f_{mn} + 2\overline{B}_{ij} f_{ij} v + 2\overline{D}_{ij} f_{ij} w + \overline{\alpha}_{ij} g_i g_j + 2\overline{\beta}_{ij} h_j g_i + \overline{\gamma}_{ij} h_j h_i + \overline{\alpha}_1 v^2 + 2\overline{\alpha}_3 v w + \overline{\alpha}_2 w^2 \geq 0$

for any f_{ij}, g_i, h_i, v and w with $f_{ij} = f_{ji}$.

The assumption (A_2) is similar to that used by Edelstein and Gurtin [16] and Reiss and Haug [19] in classical viscoelasticity. We shall write $F^{[n]}(\mathbf{x},t)$ for the nth derivative of F with respect to t holding $\mathbf{x}$ fixed. Following Reiss [18], we introduce the set $\mathcal{M}$ of admissible weight functions. We say that $g \in \mathcal{M}$ if g is a function on $[0,\infty)$ with the following properties

$$\int_0^\infty \int_0^\infty g^{[k]}(t+s)\mathrm{d}t\,\mathrm{d}s \;\text{ exists }\; \text{for } k>0 \tag{54}$$

$$g(t)=\int_0^t \Gamma(p)\mathrm{e}^{-pt}\mathrm{d}p,\quad t\in[0,\infty) \tag{55}$$

where Γ is a continuous and positive function on $[0,\infty)$ and has a finite limit at infinity. For example, $g(t)=(t+a)^{-n}$, $n>2$, $a>0$ with $\Gamma(t)=[t^{n-1}\exp(-at)]/(n-1)!$ is a weight function [18]. We denote

$$l * f = \widehat{f}$$

where l is defined by (44). We say that h is bounded at infinity if $\lim_{t\to\infty} h(\mathbf{x},t)$ exists for each $\mathbf{x}\in B$. We shall assume that the functions used to formulate the problem are bounded at infinity. We say that $\pi=(u_i,\varphi,\psi,e_{ij},t_{ij},\sigma_i,\tau_i,\xi,\zeta)$ is a kinematically admissible process if π is an admissible process that satisfies the equations (1), (3), the initial history conditions (5) and the boundary conditions

$$u_i=\widetilde{u}_i,\quad \varphi=\widetilde{\varphi},\quad \psi=\widetilde{\psi} \text{ on } \partial B\times I \tag{56}$$

We denote by Ω the set of all kinematically admissible processes π such that π and grad π possess Laplace transforms.

Theorem 5. *Assume that hypotheses (A_1) and (A_2) hold. Let $\Lambda_g\{\cdot\}$ be the functional on Ω defined by*

$$\begin{aligned}
\Lambda_g\{\pi\} = \int_B \int_0^\infty \int_0^\infty & g(t+s)\{(\widehat{C}_{ijmn} * e_{mn})(\mathbf{x},t)e_{ij}(\mathbf{x},s)\\
&+2(\widehat{B}_{ij}*\varphi)(\mathbf{x},t)e_{ij}(\mathbf{x},s)+2(\widehat{D}_{ij}*\psi)(\mathbf{x},t)e_{ij}(\mathbf{x},s)\\
&+(\widehat{\alpha}_{ij}*\varphi_{,j})(\mathbf{x},t)\varphi_{,i}(\mathbf{x},s)+2(\widehat{\beta}_{ij}*\psi_{,j})(\mathbf{x},t)\varphi_{,i}(\mathbf{x},s)\\
&+(\widehat{\gamma}_{ij}*\psi_{,j})(\mathbf{x},t)\psi_{,i}(\mathbf{x},s)+(\widehat{\alpha}_1*\varphi)(\mathbf{x},t)\varphi(\mathbf{x},s)\\
&+2(\widehat{\alpha}_3*\psi)(\mathbf{x},t)\varphi(\mathbf{x},s)+(\widehat{\alpha}_2*\psi)(\mathbf{x},t)\psi(\mathbf{x},s)\\
&+\rho u_i(\mathbf{x},t)u_i(\mathbf{x},s)+\kappa_1\varphi(\mathbf{x},t)\varphi(\mathbf{x},s)+\kappa_2\psi(\mathbf{x},t)\psi(\mathbf{x},s)\\
&-2(H_i+j*s_{ki,k})(\mathbf{x},t)u_i(\mathbf{x},s)-2[S+j*(\pi_{k,k}+\nu)](\mathbf{x},t)\varphi(\mathbf{x},s)\\
&-2[T+j*(\chi_{k,k}+\vartheta)](\mathbf{x},t)\psi(\mathbf{x},s)\}\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}v_x
\end{aligned} \tag{57}$$

for every $\pi\in\Omega$. If π is a solution to the Dirichlet problem, then

$$\Lambda_g\{\pi\}\leq\Lambda_g\{\widetilde{\pi}\} \tag{58}$$

for every $\widetilde{\pi}\in\Omega$.

Proof. We consider $\pi,\widetilde{\pi}\in\Omega$ and define $\pi'=\widetilde{\pi}-\pi$. Clearly, $\pi'=\{u_i',\varphi',\psi',e_{ij}',t_{ij}',\sigma_i',\tau_i',\xi',\zeta'\}$ is an admissible process that satisfies the equations (1), (3), null initial history and the boundary conditions

$$u_i' = 0, \varphi' = 0, \psi' = 0 \text{ on } \partial B \times I \tag{59}$$

With the help of (55) we find that

$$\int_B \int_0^\infty \int_0^\infty g(t+s)\Phi(\mathbf{x},t)\Psi(\mathbf{x},s)\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}v_x = \int_B \int_0^\infty \Gamma(p)\overline{\Phi}(\mathbf{x},p)\overline{\Psi}(\mathbf{x},p)\mathrm{d}p\,\mathrm{d}v_x \tag{60}$$

Clearly, we have

$$(\mathcal{L}l)(p) = p^{-1}, (\mathcal{L}j)(p) = p^{-2}, \mathcal{L}(\Phi * \Psi) = \overline{\Phi}\,\overline{\Psi} \tag{61}$$

Let us calculate $\Lambda_g\{\pi + \pi'\}$. In view of (60) and (61), we get

$$\begin{aligned}
&\int_B \int_0^\infty \int_0^\infty g(t+s)\{(\widehat{C}_{ijmn} * e_{mn} + \widehat{B}_{ij} * \varphi + \widehat{D}_{ij} * \psi)(\mathbf{x},t)e_{ij}'(\mathbf{x},s)\\
&+ (\widehat{C}_{ijmn} * e_{mn}' + \widehat{B}_{ij} * \varphi' + \widehat{D}_{ij} * \psi')(\mathbf{x},t)e_{ij}(\mathbf{x},s)\\
&+ (\widehat{\alpha}_{ij} * \varphi_{,j} + \widehat{\beta}_{ij} * \psi_{,j})(\mathbf{x},t)\varphi_{,i}'(\mathbf{x},s) + (\widehat{\alpha}_{ij} * \varphi_{,j}' + \widehat{\beta}_{ij} * \psi_{,j}')(\mathbf{x},t)\varphi_{,i}(\mathbf{x},s)\\
&+ (\widehat{\beta}_{ji} * \varphi_{,j} + \widehat{\gamma}_{ij} * \psi_{,j})(\mathbf{x},t)\psi_{,i}'(\mathbf{x},s) + (\widehat{\beta}_{ji} * \varphi_{,j}' + \widehat{\gamma}_{ij} * \psi_{,j}')(\mathbf{x},t)\psi_{,i}(\mathbf{x},s)\\
&+ (\widehat{B}_{ij} * e_{ij} + \widehat{\alpha}_1 * \varphi + \widehat{\alpha}_3 * \psi)(\mathbf{x},t)\varphi'(\mathbf{x},s)\\
&+ (\widehat{B}_{ij} * e_{ij}' + \widehat{\alpha}_1 * \varphi' + \widehat{\alpha}_3 * \psi')(\mathbf{x},t)\varphi(\mathbf{x},s) + (\widehat{D}_{ij} * e_{ij} + \widehat{\alpha}_3 * \varphi\\
&+ \widehat{\alpha}_2 * \psi)(\mathbf{x},t)\psi'(\mathbf{x},s) + (\widehat{D}_{ij} * e_{ij}' + \widehat{\alpha}_3 * \varphi' + \widehat{\alpha}_2 * \psi')(\mathbf{x},t)\psi(\mathbf{x},s)\}\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}v_x\\
&= 2\int_0^B \int_0^\infty p^{-1}\Gamma(p)\Pi(p)\mathrm{d}p\,\mathrm{d}v_x
\end{aligned} \tag{62}$$

where

$$\begin{aligned}
\Pi(p) = {}& \overline{C}_{ijmn}(p)\overline{e}_{mn}(p)\overline{e}_{ij}'(p) + \overline{B}_{ij}(p)(\overline{\varphi}\overline{e}_{ij}' + \overline{\varphi}'\overline{e}_{ij})(p)\\
&+ \overline{D}_{ij}(p)(\overline{\psi}\overline{e}_{ij}' + \overline{\psi}'\overline{e}_{ij})(p) + \overline{\alpha}_{ij}(p)\overline{\varphi}_{,j}'(p)\overline{\varphi}_{,i}(p)\\
&+ \overline{\beta}_{ij}(p)(\overline{\psi}_{,j}\overline{\varphi}_{,i}' + \overline{\psi}_{,j}'\overline{\varphi}_{,i})(p) + \overline{\gamma}_{ij}(p)\overline{\psi}_{,j}(p)\overline{\psi}_{,i}'(p)\\
&+ \overline{\alpha}_1(p)\overline{\varphi}(p)\overline{\varphi}'(p) + \overline{\alpha}_3(\overline{\psi}\overline{\varphi}' + \overline{\psi}'\overline{\varphi})(p) + \overline{\alpha}_2(p)\overline{\psi}(p)\overline{\psi}'(p)
\end{aligned} \tag{63}$$

With the help of (49) and (50), we obtain

$$\begin{aligned}
\Pi(p) = {}& p^{-1}[(\bar{t}_{ij} - \bar{s}_{ij})\overline{e}_{ij}' + (\overline{\sigma}_i - \overline{\pi}_i)\overline{\varphi}_{,i}' + (\overline{\tau}_i - \overline{\chi}_i)\overline{\psi}_{,i}'\\
&- (\overline{\xi} - \overline{\nu})\overline{\varphi}' - (\overline{\zeta} - \overline{\vartheta})\overline{\psi}']
\end{aligned} \tag{64}$$

Suppose that π is a solution. Then, in view of (1) and (46), we get

$$\begin{aligned}
&p^{-1}\Pi(p) = p^{-2}[(\bar{t}_{ki} - \bar{s}_{ki})\overline{u}_i' + (\overline{\sigma}_k - \overline{\pi}_k)\overline{\varphi}' + (\overline{\tau}_k - \overline{\chi}_k)\overline{\psi}']_{,k}\\
&- \rho\overline{u}_i\overline{u}_i' - \kappa_1\overline{\varphi}\overline{\varphi}' - \kappa_2\overline{\psi}\overline{\psi}' + (\overline{H}_i + p^2\overline{s}_{ki,k})\overline{u}_i'\\
&+ [\overline{S} + p^2(\overline{\pi}_{k,k} + \overline{\nu})]\overline{\varphi}' + [\overline{T} + p^2(\overline{\chi}_{k,k} + \overline{\vartheta})]\overline{\psi}'
\end{aligned} \tag{65}$$

From (57), (59), (61), (62), and (65), we find that

$$\begin{aligned}
\Lambda\{\widetilde{\pi}\} = {}& \Lambda\{\pi\} + \int_B \int_0^\infty \int_0^\infty g(t+s)\{(\widehat{C}_{ijmn} * e_{mn}')(\mathbf{x},t)e_{ij}'(\mathbf{x},s)\\
&+ 2(\widehat{B}_{ij} * \varphi_{,j}')(\mathbf{x},t)\varphi_{,i}'(\mathbf{x},s) + 2(\widehat{\beta}_{ij} * \psi_{,j}')(\mathbf{x},t)\varphi_{,i}'(\mathbf{x},s)\\
&+ (\widehat{\gamma}_{ij} * \psi_{,j}')(\mathbf{x},t)\psi_{,i}'(\mathbf{x},s) + (\widehat{\alpha}_1 * \varphi')(\mathbf{x},t)\varphi'(\mathbf{x},s)\\
&+ 2(\widehat{\alpha}_3 * \psi')(\mathbf{x},t)\varphi'(\mathbf{x},s) + (\widehat{\alpha}_2 * \psi')(\mathbf{x},t)\psi'(\mathbf{x},s)\}
\end{aligned} \tag{66}$$

It follows from (62), (66), and the hypothesis (A_2) that (58) holds. □

Minimum principles for other boundary conditions can be also derived (see [19]). Variational principles for linear theories of viscoelastic materials have been investigated in various papers (see, e.g., [14,20]).

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