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A case of strong nonlinearity: Intermittency in highly turbulent flows

Un cas de forte non-linéarité : l'intermittence en milieu turbulent à grand nombre de Reynolds

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A B S T R A C T

It has long been suspected that flows of incompressible fluids at large or infinite Reynolds number (namely at small or zero viscosity) may present finite time singularities. We review briefly the theoretical situation on this point. We discuss the effect of a small viscosity on the self-similar solution to the Euler equations for inviscid fluids. Then we show that single-point records of velocity fluctuations in the Modane wind tunnel display correlations between large velocities and large accelerations in full agreement with scaling laws derived from Leray's equations (1934) for self-similar singular solutions to the fluid equations. Conversely, those experimental velocity–acceleration correlations are contradictory to the Kolmogorov scaling laws.

R É S U M É

On pense depuis longtemps que les écoulements fluides incompressibles à grand, sinon infini, nombre de Reynolds présentent des singularités localisées en temps et en espace. Nous étudions l'effet d'une petite viscosité sur les solutions auto-semblables des équations des fluides. Nous montrons ensuite que des enregistrements de fluctuations de vitesse dans la soufflerie de Modane présentent des corrélations entre grandes vitesses et grandes accélérations, en accord complet avec les lois d'échelle déduites des solutions auto-similaires des équations trouvées par Leray en 1934. En revanche, ces corrélations sont en contradiction avec les lois d'échelle déduites de la théorie de Kolmogorov.

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Foreword

Over the years, Pierre Coullet developed an outstanding research devoted to many aspects of nonlinearity in Science. With his very sure taste, he chose topics with a deep geometrical underpinning, nonlinearity being only one element in the structure of the scientific question. In fluid mechanics, nonlinearity and geometry concur to bring forward difficult and fascinating questions. We think first to the transition to turbulence by a cascade of period doubling, predicted almost simultaneously by Pierre Coullet and Charles Tresser [1] and by Mitch Feigenbaum [2]. This scenario of transition was observed slightly afterwards in fluid experiments at the "École normale" laboratory by Jean Maurer and Albert Libchaber [3]. The understanding of the transition to turbulence with a few degrees of freedom did not end research in fluid turbulence, a difficult field where real progress has always been very slow. The next step was the realization that the transition to turbulence in large systems with many degrees of freedom belongs to the class of directed percolation [4], but this cannot be seen as the end of the story. A big open question remaining in fluid turbulence was raised in the 1949 paper by Batchelor and Townsend [5], where the authors discuss observations of large velocity fluctuations, which they attribute in, we believe, a not fully convincing way to the large wavenumber limit of the Kolmogorov cascade [6]. This assumption (like many, if not most, works in turbulence theory) bypass any discussion of the time dependence of the fluctuations of the turbulent flow and their link to the basic fluid equations, whereas the observed intense and short bursts of turbulence have obviously something to do with the time dependence of solutions to the fluid equations, a point we expand on below. We hope that this contribution will show our admiration for Pierre and will be also of interest for fluid mechanics.

1. Introduction

One outstanding problems of turbulence in fluids was posed by Batchelor and Townsend in 1949 [5] and can be stated as follows. Kolmogorov's theory predicts a spectrum of velocity fluctuations decaying like $k^{-5/3}$ at large wave numbers, the Kolmogorov–Obukhov spectrum. Measurements made over the years agree well with this prediction [7,8]. Therefore, it was somewhat surprising to observe also that the largest velocity and acceleration fluctuations in a turbulent flow are short lived and are also associated with short distances. This looks contradictory with the Kolmogorov–Obukhov spectrum, which predicts that the intensity of the fluctuations of velocity decreases as the length scale (the inverse wave number) decreases, because the statistical weight of large velocity fluctuations (and acceleration) is not small, particularly in the Modane experiment, where about 4 per cent of the recorded data are for acceleration larger than 2.5 in units of its standard deviation. This phenomenon is called intermittency. A particular consequence of these very intense and quick bursts observed in the record is the wing widening of the probability distribution of the acceleration, a property that cannot be explained in a theory with a single scaling parameter, as the one in the original Kolmogorov theory of 1941 [6].

To describe fluid motion, besides Kolmogorov or Kolmogorov-inspired statistical theories, there are basic equations, the Navier–Stokes (NS) equations becoming the Euler equations in the inviscid limit. It makes sense to come back to those fundamental equations to see if the phenomenon of intermittency is explainable by them and, in particular, if predictions could be made concerning it. This is the purpose of this paper, which includes an analysis of velocity data recorded in the big wind tunnel of Modane, in southern France.

As no general solution to either the NS or Euler equations is known, it could seem hopeless to base a theory on solutions to those equations. However, the situation is not as bad as one could believe first, because of the idea of self-similar solutions for the fluid equations, an idea going back to Leray [9]. We explain in Sec. 2–3 what are those self-similar solutions. In Sec. 4, we apply this to predict the occurrence of quasi-singularities, namely singular solutions to the Euler equations becoming smooth under the effect of viscosity. This relies on two assumptions, first that the Euler equations have a finite time singularity, whereas the NS (Navier–Stokes) equations have not. Based on this, we predict a relation between the large fluctuations of the velocity and of the acceleration, which is amazingly well verified by hot-wire records made in Modane's wind tunnel, Sec. 5–6.

2. Self-similar fluid equations

In 1934, Leray [9] published a paper on the Navier–Stokes equations for an incompressible fluid in 3D,

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u} \qquad \nabla \cdot \mathbf{u} = 0 \tag{1}$$

where $\mathbf{u}$ is the velocity field, p is the pressure and ν is the kinematic viscosity of the fluid. For $\nu = 0$, one obtains the Euler equations (see next section). In his paper, Leray introduced many important ideas, among which the notion of weak solution and also the problem of the existence (or not) of a solution becoming singular after a finite time, when starting from smooth initial data. He looked for solutions of the self-similar type,

$$\mathbf{u}(\mathbf{r}, t) = (t^* - t)^{-\alpha} \mathbf{U}(\mathbf{r}(t^* - t)^{-\beta}) \qquad p(\mathbf{r}, t) = (t^* - t)^{-2\alpha} P(\mathbf{r}(t^* - t)^{-\beta}) \tag{2}$$

where t^* is the time of the singularity (set to zero later), α and β are real positive exponents to be found, and the pair of functions $(\mathbf{U}, P)$ with upper-case letters is to be derived from Euler, or NS equations, see below. That such a velocity field is a solution to Euler or NS equations implies to balance the two terms on the left-hand side of (1), which behave respectively as $t^{-(\alpha+1)}$ and $t^{-(2\alpha+\beta)}$. It yields a first relation between the two parameters,

$$\alpha + \beta = 1 \tag{3}$$

and the re-scaled equation for **U**

$$(\alpha \mathbf{U} + \beta \mathbf{R} \cdot \nabla \mathbf{U}) + \mathbf{U} \cdot \nabla \mathbf{U} = -\nabla P \qquad \nabla \cdot \mathbf{U} = 0 \tag{4}$$

In the case of the Navier–Stokes equation, the balance with the dissipative term $\nu \nabla^2 \mathbf{u}$, of order $t^{-(\alpha+2\beta)}$, imposes $\beta = 1/2$, which yields the exponents found by Leray,

$$\alpha = \beta = 1/2 \tag{5}$$

Let us give an outlook of the derivation of Leray's equation for $(\mathbf{U}, P)$ (see, for example, [10], but not done in this way by Leray). We consider the case $t - t^* < 0$ leading to what is sometimes called backward self-similar equation. If the NS equations admits self-similar solutions, the set $(\mathbf{u}, p)$ must be of the form

$$\mathbf{u}(\mathbf{r}, t) = \sqrt{\frac{\nu}{-t}} \mathbf{U}\left(\mathbf{r}(-\nu t)^{-\frac{1}{2}}\right) \qquad p(\mathbf{r}, t) = \left(\frac{\nu}{-t}\right) P\left(\mathbf{r}(-\nu t)^{-\frac{1}{2}}\right) \tag{6}$$

where t is for $t - t^*$ and ν is the kinematic viscosity.

Introducing the logarithmic time $\tau = -\log(-t)$, plus additional changes of variables $\mathbf{R} = \mathbf{r}(-\nu t)^{-1/2}$, $\mathbf{U}(\mathbf{R}, \tau) = \sqrt{\frac{-t}{\nu}} \mathbf{u}(\mathbf{r}, t)$, $P(\mathbf{R}, \tau) = \left(\frac{-t}{\nu}\right) p(\mathbf{r}, t)$, the equations for the pair of functions $(\mathbf{U}(\mathbf{R}, \tau), P(\mathbf{R}, \tau))$ become

$$\frac{\partial \mathbf{U}}{\partial \tau} + \frac{1}{2}(\mathbf{U} + \mathbf{R} \cdot \nabla \mathbf{U}) + \mathbf{U} \cdot \nabla \mathbf{U} = -\nabla P + \nabla^2 \mathbf{U} \qquad \nabla \cdot \mathbf{U} = 0 \tag{7}$$

The interest of using such a time dependence is that, for original differential equations of first order with respect to time t, the new differential equation is still first order and autonomous with respect to τ. The pair $(\mathbf{u}, p)$ is a self-similar solution to NS or Euler equations if and only if $\mathbf{U} = \mathbf{U}(\mathbf{R})$ and $P = P(\mathbf{R})$ are fixed points depending only of $\mathbf{R}$, that gives the Leray equation,

$$\frac{1}{2}(\mathbf{U} + \mathbf{R} \cdot \nabla \mathbf{U}) + \mathbf{U} \cdot \nabla \mathbf{U} = -\nabla P + \nabla^2 \mathbf{U} \qquad \nabla \cdot \mathbf{U} = 0 \tag{8}$$

In the following, equations of the form (8) are called NS–Leray equations if the viscosity is non-zero and Euler–Leray equations whenever the $\nabla^2 \mathbf{U}$ in (8) is absent and if, with respect to the scaled variables (6), the circulation Γ carried by the flow towards the singularity replaces the kinematic viscosity ν.

Over the years, the search for solutions to (8) motivated many works, mostly by mathematicians. The main effort was to try to prove (or disprove) the existence of such singularities assuming properties of the initial data [11]. Other attempts have been directed toward a direct numerical solution to NS and/or Euler equations, with the purpose of showing they have or not a finite time singularity [12].

3. Euler–Leray equations

In the case of Euler equations, the existence of a self-similar solution imposes (3), but the balance condition with the dissipative term $\nu \nabla^2 \mathbf{u}$ does not hold, allowing other sets of exponents different from (5). One exponent, β for instance, is seemingly free, namely it does not follow from simple algebraic manipulation of the Euler equations. There are several possibilities to get a second relation between the two exponents β and α. This relies on the existence of conservation laws, and the final result depends on what conservation law is considered.

Let consider first the conservation of circulation on closed curves. The circulation Γ along a closed curve carried by the flow toward the singularity, is of order $t^{\beta-\alpha}$. Therefore, the conservation of circulation implies $\alpha = \beta$, which gives (5), namely the same exponents as for the Navier–Stokes case. Moreover, velocity scales like $u(r, t) \sim \sqrt{\frac{\Gamma}{-t}}$ near the singularity. With such a choice, the total energy (L^2 norm) of solutions to the self-similar problem in R^3 evolves formally as $t^{1/2}$ times a diverging integral because $U \sim 1/R$ at large distance in the case of a solution independent of the time τ.

For reasons explained in [13], we shall consider the exponents (5) ensuring conservation of circulation. This yields self-similar solutions like (6), and the Euler–Leray first equation given by (8) without the diffusion term:

$$\frac{\partial \mathbf{U}}{\partial \tau} + \frac{1}{2}(\mathbf{U} + \mathbf{R} \cdot \nabla \mathbf{U}) + \mathbf{U} \cdot \nabla \mathbf{U} = -\nabla P \qquad \nabla \cdot \mathbf{U} = 0 \tag{9}$$

Now let us consider, more generally, a self-similar solution to Euler equations of the form (2) with arbitrary exponents α, β (see Eq. (4)). If one considers, instead of the conservation of circulation, the conservation of the L^2 norm, one must satisfy the constraint $-2\alpha + 3\beta = 0$, together with (3), which yields in the inviscid case

$$\alpha = 3/5 \qquad \beta = 2/5 \tag{10}$$

which are the Sedov–Taylor exponents [14]. No set of singularity exponents can satisfy both constraints of L^2 norm conservation *and* constant non zero circulation on carried closed curves [15]. In the following, we mainly focus on the case originally proposed by Leray, $\alpha = \beta = 1/2$, associated with the constraint of circulation conservation, although the case of Sedov–Taylor exponents (4) is potentially promising, as illustrated by Fig. 1b, and could be investigated by using the same approach as the one proposed below.

Of course, it is highly desirable to have a nontrivial solution to Eq. (9), either in an analytic form or resulting from numerical analysis. This may include an explicit dependence with respect to the "time" τ, and does not seem to be as hopeless as one might think first. Let us outline a possible solution to this problem. The idea is to consider a solution to Eq. (9) which is close to a steady solution to Euler's equation in the axisymmetric case. This could be valid for large-amplitude solutions, and amounts to solve, at leading order, the nonlinear part of Eq. (9),

$$\mathbf{U}^0 \cdot \nabla \mathbf{U}^0 = -\nabla P^0 \qquad \nabla \cdot \mathbf{U}^0 = 0 \tag{11}$$

In the limit of large U, the two other terms, namely $\frac{\partial \mathbf{U}}{\partial \tau} + \left(\frac{3}{5}\mathbf{U} + \frac{2}{5}\mathbf{R} \cdot \nabla \mathbf{U}\right)$ are relatively small perturbations. The expansion near the leading-order solution $\mathbf{U}^0$ leads to two solvability conditions, which are satisfied by tuning the amplitudes of two modes, $A_{1,2}\exp(\mathrm{i}\omega_{1,2}\tau)$ oscillating in time τ near the steady state solution. Such oscillations could manifest themselves in the time records, as observed in experimental signals of turbulent flows, see Fig. 7, which displays a decreasing oscillatory behavior in the decaying phase of huge fluctuations. In the case of the Euler equations, there is no direction of time because of the symmetry of the equations under time reversal, so that oscillations after the singularity may have the same explanation as oscillations before the singularity. But in presence of viscosity the amplitude of the oscillations before and after the time $t*$ of the singularity may be different because of the increased viscous dissipation expected near $t*$.

To conclude on Euler–Leray equations, they yield a well-defined schema for the existence of solutions to the Euler equations in 3D becoming singular in a finite time and at a single point. A by-product of this analysis is the set of exponents of the singularity, which may be compared to experimental data for the big fluctuations observed in the time records of the velocity in a turbulent flow, as done below.

One motivation for working on Euler–Leray singularities is their possible connection with the phenomenon of intermittency in high-Reynolds-number flows, a point we have not found in the literature, although numerous works are devoted to a direct investigation of Euler equations (see the impressive list in [12]). This possible relevance of Euler–Leray singularities for explaining observed features of turbulent flows raises several questions. Among them, one may quote the following ones.

(i) What is the difference between Euler–Leray and NS–Leray singularities?
(ii) What is specific to our interpretation in terms of Leray singularities compared to other schema for intermittency?
(iii) What would be specific of Euler–Leray singularities in a time record of large Reynolds number flow?

We comment about previous points below.

(i) Little is known about this difference, in particular do both have nontrivial solutions, or does none has nontrivial solutions or only one has nontrivial solution? Mathematicians have obtained over the years various constrains on the functional space where such solutions could exist. This point (i) is discussed below in Section 4, devoted to the understanding of the effect of adding a small but finite viscosity to the singularities of the Euler–Leray equations in order to agree with the real physical situation of viscous fluids.
(ii) If intermittency is linked to Leray-like singularities, they yield automatically a strong correlation between large values of the velocity and of the acceleration (see below). Compared to predictions derived from Kolmogorov's theory, this correlation is a strong indication of the occurrence of Leray-like singularities near large fluctuations. It is fair to say however that, as far as we know, Kolmogorov himself has never mentioned this question of finite time singularity of either NS or Euler equations. So it would be unfair to attribute to him any statement about those singularities.
(iii) In both Euler–Leray and NS–Leray equations ($\alpha = \beta = 1/2$), the velocity field at the time of blow-up scales like $1/r$, r distance to the singularity, so that knowledge of the flow structure near a singularity would not help to distinguish between the two kinds of singularities.

Our analysis of the experimental data relies on a relationship between velocity and acceleration at the same time (given by the time records of velocity). The theory for this relationship is fairly simple. First the velocity scales like $u(r,t) \sim (-t)^{-1/2}\Gamma^{1/2}$ near the singularity, as written above. In the absence of viscosity, the order of magnitude of the circulation Γ is constant in the collapsing domain, and the typical Reynolds number of the small length scales does not tend to zero, but stays constant because the velocity grows at the same pace as the space scale decreases. The circulation is the one along a closed curve that is wholly carried by the flow in the collapsing domain. A discussion of the existence of such closed curve in the case of a solution to the Euler–Leray problem close to a steady solution to Euler equations will be given soon [16]. These properties are opposite to what is expected from standard ideas on turbulence, according to which the role of viscosity increases and the Reynolds number decreases as one approaches small spatial scales.

From the scaling laws of velocity, one derives immediately the one for the acceleration $\gamma(r,t)$. This acceleration is not the one of a particle carried by the flow, sometimes called Lagrangian acceleration, but only the time derivative of the fluid velocity measured at a given point, the quantity we have access to from hot wire measurements, also named Eulerian acceleration. This acceleration scales like $\gamma(r,t) \sim (-t)^{-3/2}\Gamma^{1/2}$. Accordingly, one finds the time-independent relation,

$$u^3 \sim \gamma\Gamma \qquad \text{for (5)} \tag{12}$$

Let us make a step aside to see how the latter relation changes if one assumes that the energy E is conserved in the singular domain. Using the definition of the self-similar solution for the velocity field of type (2) with Sedov–Taylor exponents given by (10), we get $u(t) \sim (E/(-t)^3)^{1/5}$. In that case, the time-independent relation between u and $\gamma \sim u/(-t)$ becomes

$$u^{8/3} \sim \gamma E^{1/3} \qquad \text{for (10)} \tag{13}$$

The two relations (12) and (13) predict that large accelerations are associated with large velocity fluctuations, and should display a similar power-law dependence $\gamma \propto u^z$ with $z \simeq 3$. We shall return to this point in section 6.

Let us now turn to the relationship between velocity and acceleration derived from the Kolmogorov scaling. The starting point is the Kolmogorov relation $u_r \sim (\epsilon r)^{1/3}$ where u_r is the typical change of velocity over a distance r and ϵ the rate of dissipation of the kinetic energy density per unit mass of the turbulent fluid. With those scalings, the time derivative of the velocity is of order $\gamma \sim \epsilon^{2/3} r^{-1/3}$. Therefore, one has the following relationship, independent of r, between u_r and γ,

$$u_r \gamma \sim \epsilon \tag{14}$$

an expression that can be derived directly from the definition of ϵ.

Note that, if the Taylor hypothesis is used, the partial time derivative of the velocity should be equal to $v_0 \partial u/\partial x$, where v_0 is the advection velocity. In the case of a self-similar solution like (6), the Eulerian acceleration becomes $\gamma_{\text{Taylor}} \sim v_0 U'/t$ and (12) becomes

$$u_r^2 \sim \frac{\Gamma}{v_0} \gamma_{\text{Taylor}} \tag{15}$$

On the other hand, Kolmogorov scalings lead to $\gamma_{\text{Taylor}} \sim v_0 \epsilon u_r^{-2}$, and the relation (14) between u_r and γ_{Taylor} must be replaced by

$$u_r^2 \gamma_{\text{Taylor}} \sim v_0 \epsilon \tag{16}$$

The two relations (12)–(14) deduced without the Taylor hypothesis (and also the two relations (15)–(16) deduced for the case of frozen turbulence) are so sharply different that it makes sense to see whether some of them agree with experimental data, as done in section 6. We discuss in Sec. 6.2 the pertinence of using the Eulerian acceleration $\gamma(r,t) = \partial u/\partial t$ to test those scaling laws.

4. Effect of a small viscosity on singularities of the Euler–Leray equations

This section does not rely on proved results on solutions to NS–Leray or Euler–Leray equations. It attempts to show a possible scenario of what happens concerning the occurrence of singularities in the physically relevant situation of a large, but not infinite, Reynolds number. Our main assumption is that the Euler–Leray problem has a *bona fide* solution, whereas the NS–Leray one has none. This statement is unproved on either side as far as we can tell and requires some explanation.

Over the years, mathematicians studied rather intensively Leray's equation [11]. To our knowledge, a still incomplete understanding has been reached yet. In the case of the NS–Leray equation, various negative results have been presented, which exclude (non-zero) solutions belonging to certain functional space. We think that the expected slow decay like $1/r$ of the solution to the NS–Leray equation is a source of difficulties to reach a definite conclusion. Nevertheless, we shall make the hypothesis of the absence of solutions to the NS–Leray problem with this long-range dependence. Obviously, we exclude unbounded solutions at large distances. The case of the Euler–Leray problem seems to be more complex. At this point, as far as we call tell, the situation is very uncertain. Some numerical simulations point to a self-similar solution with measurable exponents, whereas mathematics exclude the existence of such solutions or give bounds, including lower bounds, for a norm of solutions depending on a free exponent introduced at the beginning. This seems not to exclude a non-trivial solution.

Below we shall assume that:

(i) there is no convenient non-zero solution to the NS–Leray equation. By "convenient", we mean that $\mathbf{U}(\mathbf{R})$ is a smooth solution (not growing at infinity); in other words, $\mathbf{u}(\mathbf{r}, t = 0)$ is non-singular;
(ii) the Euler–Leray equation has convenient non-zero solutions.

The next step in our analysis is to consider the NS–Leray singularity problem in the (realistic) limit of small but non-vanishing viscosity. It is legitimate to study the behavior of an initial condition that is exactly the solution to the Euler–Leray equations and to find what happens to this solution if viscosity is small but not zero. Within our assumption of lack of solution to the NS–Leray equation, the evolution of such an initial condition is changed dramatically by a small viscous term, as we are going to explain.

The Euler–Leray equations have an interesting structure, pointed out in [17]; they are invariant under dilation. This means that if (9) has a solution $\mathbf{U}(\mathbf{R})$, then the function $\mu\mathbf{U}(\mu\mathbf{R})$ is also a solution, with μ an arbitrary real number. This continuous symmetry in the set of solutions to the Euler–Leray equations will play, as usual in this type of situation, an important role in the perturbation brought by a small change in the equations. Such a change is the addition of a small viscosity, which breaks the dilation invariance because NS–Leray equations are not invariant under dilation at constant non-vanishing viscosity, unless $\mu = -1$, which does not correspond to a continuous symmetry. Note that besides this dilation symmetry, there is also a continuous symmetry under rotations, which is preserved by the viscosity term and so does not bring any dynamics of the parameter μ, contrary to the breaking of dilation invariance.

Because of the breaking of dilation invariance, it is not possible to find by regular expansion a solution to NS–Leray equations close to a solution to Euler–Leray equations in the limit of a small but non-zero viscosity. This is because, at first order with respect to the small viscosity, one finds a solvability condition that is impossible to satisfy in the framework of the equations of similarity as they stand. Let us sketch a more detailed explanation. To get a solution to (7) for small viscosity (more properly $\frac{1}{Re} = \nu/\Gamma \ll 1$), we start from a solution $\bar{\mathbf{U}}(\mathbf{R})$ to Eq. (9), associated with an arbitrary value $\mu = 1$ of the dilation parameter. The solution $\mathbf{U}_{\rm EL} = \mu\bar{\mathbf{U}}(\mu\mathbf{R})$ of (9) is the leading-order term of our unknown solution to (7), expanded in powers of the small parameter ν. The effect of a non-vanishing viscosity is to cause a drift of the solution to Euler–Leray equations in the space of the parameter μ, and also to introduce another time dependence of the singular solutions, via the logarithm $\tau = -\ln(-t)$, $t = 0$ being the instant of the blow-up. With respect to this new “time” variable, the blow-up time is sent to plus infinity.

At first order with respect to the small parameter ν, a small (unknown) perturbation $\mathbf{U}_{\rm c}(\mathbf{R},\mu)$ added to $\mathbf{U}_{\rm EL}(\mathbf{R},\mu)$ must satisfy a solvability condition deduced by introducing a slow dependence of μ with respect to τ, hence a slow variation with respect to τ of the solution. The first correction $\mathbf{U}_{\rm c}$ is of order ν/Γ. It has to satisfy a linear equation derived by putting $\mathbf{U}_{\rm EL} + \mathbf{U}_{\rm c}$ into Eq. (7) and keeping only terms linear with respect to $\mathbf{U}_{\rm c}$, ν and $\partial/\partial\tau$. We get

$$\frac{\partial \mathbf{U}_{\rm EL}}{\partial \tau} + \mathcal{L}[\mathbf{U}_{\rm EL}]\mathbf{U}_{\rm c} = \nu\nabla^2\mathbf{U}_{\rm EL} \tag{17}$$

where $\mathcal{L}$ is a linear operator acting on functions of $\mathbf{R}$, derived by linearization of (9) near the solution $\mathbf{U}_{\rm EL}$. This operator is such that $\mathcal{L}[\mathbf{U}_{\rm EL}]\mathbf{U}_{\rm d} = 0$, for $\mathbf{U}_{\rm d} = \frac{\partial \mathbf{U}}{\partial \mu}$ because of the dilation invariance of the Euler–Leray equations. In technical terms, the function $\frac{\partial \mathbf{U}}{\partial \mu}$ belongs to the non-empty kernel of the linear operator $\mathcal{L}[\mathbf{U}_{\rm EL}]$ (we set aside for the moment the question of the way the pressure enters into this). Define now an inner product, a real number in the space of functions of $\mathbf{R}$, namely a bilinear quantity $\langle \mathbf{U}_e(\mathbf{R})|\mathbf{U}_f(\mathbf{R})\rangle$ where e and f are arbitrary indices. This inner product must be defined by convergent integrals, which requires some care, because many functions under consideration decay slowly at large R.

The dynamical equation for $\mu(\tau)$ is derived as a solvability condition for Eq. (17), because once the equation is multiplied by the kernel of the operator adjoint of $\mathcal{L}$, the unknown function $\mathbf{U}_{\rm c}$ disappears completely out of Eq. (17) and the only freedom to cancel the result is to impose an equation of motion for $\mu(\tau)$. This is done by writing $\frac{\partial \mathbf{U}_{\rm EL}}{\partial \tau} = \frac{\mathrm{d}\mu}{\mathrm{d}\tau}\frac{\partial \mathbf{U}_{\rm EL}}{\partial \mu}$, with the final result,

$$\frac{\mathrm{d}\mu}{\mathrm{d}\tau}\left\langle U^{\dagger}(\mathbf{R})|\frac{\partial \mathbf{U}_{\rm EL}}{\partial \mu}\right\rangle = \nu\left\langle U^{\dagger}(\mathbf{R})|\nabla^2\mathbf{U}_{\rm EL}\right\rangle \tag{18}$$

In this equation, the function $U^{\dagger}(\mathbf{R})$ belongs to the kernel of the linear operator conjugate of the kernel of $\mathcal{L}[\mathbf{U}_{\rm EL}]$ with the inner product still to be chosen, that is $\mathcal{L}^{\dagger}U^{\dagger}(\mathbf{R}) = 0$. The end result of this is a dynamical equation for the dilation parameter μ.

Let us turn now to the definition of the inner product $\langle \mathbf{U}_e(\mathbf{R})|\mathbf{U}_f(\mathbf{R})\rangle$. This is in principle arbitrary, except that a change in its definition leads to a change in the operator conjugate of $\mathcal{L}$ and then of the function $U^{\dagger}(\mathbf{R})$. The velocity field $\mathbf{U}_{\rm EL}$ decays like $1/R$ at large R. The same kind of argument used to derive the long-distance behavior of $\mathbf{U}_{\rm EL}$ shows also that $U^{\dagger}(\mathbf{R})$ decays like $1/R$ at large R. Let us introduce the usual inner product as the integral over space of the scalar product of two vector fields,

$$\langle \mathbf{U}_e(\mathbf{R})|\mathbf{U}_f(\mathbf{R})\rangle = \int \mathbf{U}_e(\mathbf{R})\cdot\mathbf{U}_f(\mathbf{R})\,\mathrm{d}\mathbf{R} \tag{19}$$

It is not hard to check that $\nabla^2\mathbf{U}_{\rm EL}$ decays like $1/R^3$ as R becomes large, whereas $U^{\dagger}(\mathbf{R})$ is of order $1/R$ in the same limit. Therefore, the integrand on the right-hand side of the solvability condition (18) decays like $1/R^4$ at R large, so that the integral converges at large distances. The left-hand side is less simple. The field $U^{\dagger}(\mathbf{R})$ decays like $1/R$ at large R. The derivative $\frac{\partial \mathbf{U}_{\rm EL}}{\partial \mu}$ does not include terms of order $1/R$ because the term $1/R$ of $\mu\mathbf{U}_{\rm EL}(\mu\mathbf{R})$ is independent of μ. Therefore, the first non-zero contribution at R large to this derivative is of order $1/R^3$, so that the inner product $\left\langle U^{\dagger}(\mathbf{R})|\frac{\partial \mathbf{U}_{\rm EL}}{\partial \mu}\right\rangle$ is given by a converging integral at R large.

There remains to settle the question of the pressure. This can be done, at least formally, by relating the pressure to the square of the velocity field by taking the divergence of (9). This yields a Poisson equation for the pressure, where the source term is the gradient of the Reynolds tensor. This Poisson equation can be solved formally for the pressure. The result can be inserted into the equation for **U**, which becomes an equation without the pressure but with an integral term. This allows us to use the formalism introduced above and yields an expression for $\frac{d\mu}{d\tau}$ that is explicit, but rather complicated. The integrals giving the inner products are still converging because, as was shown in [17], the pressure decays like $1/R^3$ at large R.

Because we have no explicit form, either analytical or even numerical, of the field $\mathbf{U}_{EL}$ solution to Euler–Leray equations, it is not possible to say anything precise concerning the solution to Eq. (18), namely concerning the ultimate fate of the self-similar solution to the Euler–Leray equations once the viscosity is turned on. This equation for $\mu(\tau)$ has a very simple mathematical structure, being first order with respect to τ and autonomous. Even without knowing explicitly $\mathbf{U}_{EL}$, one can say that there are two possibilities: either the solution $\mu(\tau)$ tends to zero as τ tends to infinity or tends to a non-zero fixed point. In the first case, the approximation made in deriving Eq. (18) breaks down at a certain time τ because, if μ tends to zero, the coefficient on the right-hand side, which should remain small by assumption, is of order $\nu/(\Gamma\mu^2)$, Γ being the initial value of the circulation. The quantity $\nu/(\Gamma\mu^2)$, which is initially small because $\nu \ll \Gamma$, grows indefinitely as μ tends to zero. Therefore, the initial assumption of a small viscosity breaks down as τ tends to infinity, namely as time gets close to the singularity time. This means that a new regime is reached where viscosity cannot be considered anymore as small. It is reasonable to guess that, in the absence of external forcing, the solution decays to zero then. Of course, in a real turbulent flow, there is always forcing by fluctuations of pressure, so that the time dependence does not stop at this time and continues. This could be represented mathematically by a random forcing term in the fluid equations.

Let us consider now the possibility that the equation for $\mu(\tau)$ tends to a non-zero fixed point. Such a fixed point would be a non-trivial solution to NS–Leray equations, the existence of which is still unsettled, and have been discarded here. Note that if a non-zero fixed point exists, it would be a way to find one solution to (8) by perturbation of solutions to (9). Besides that, it would be highly conjectural to say anything more, again because of the lack of known explicit non-trivial solution to (8) or (9).

As two side remarks, let us notice first that this breaking of the dilation invariance by the viscosity term could be also operative in the case of a direct numerical search of a singular solution to the Euler equations because the numerical method always represent imperfectly the original equations. The numerical noise could break the original dilation invariance, which could interrupt the blow-up by a drift of the dilation parameter, as it happens when a small viscosity is turned on. Another significant effect of adding viscosity effects to the self-similar solutions to Euler–Leray equations concerns the dissipation of the energy in the singular domain. Recall that this energy is given by a diverging integral, but if the dilation parameter μ tends to zero, the energy (which scales formally as μ) must tend to zero. This paradox could be explained by the spatial spreading of the perturbation $\mathbf{U}_c$, whose length scale increases as $1/\mu$, a situation irrelevant for a collapse in real flows, because large-distance coherence should be destroyed by the field of turbulent fluctuations.

5. On the possibility of observing a Leray-like singularity on hot wire records

The point developed in this paper is that the occurrence of Leray-like singularities in flows at high Reynolds number can be put in evidence by measuring the time-dependent velocity at a single point. This raises two questions: unless one is very lucky, there is little chance that the point of measurement is exactly the one where blow-up occurs (this neglecting that there could be no actual blow-up because of the viscosity, a point to which we will come back below). Furthermore, in the wind-tunnel measurements that we will report on in Sec. 6, a mean advection velocity carries along any time-dependent event. If the turbulence intensity is low (turbulent velocity fluctuations small compared to the mean velocity), Taylor's hypothesis of frozen turbulence can be used to convert temporal experimental measurements into a measurement of a quasi-instantaneous space dependence of velocity. We shall deal with those points now.

The self-similar solution of the "generic type" (6) considered above, which has a dilation invariance, is not the most general solution. Besides the dilation invariance, there are also a translation and a Galilean invariance of the solution to the fluid equation, which leads to a more general self-similar solution of the form

$$\mathbf{v}(\mathbf{r},t) = (t^* - t)^{-\frac{1}{2}} \mathbf{U}((\mathbf{r} - \mathbf{r}_0 - \mathbf{v}_0 t)(t^* - t)^{-\frac{1}{2}}) + \mathbf{v}_0 \tag{20}$$

where $\mathbf{r}_0$ is the position of the point of measurement, assuming that the singularity occurs at $r = 0$ and $t = t^*$. This expression represents a flow structure convected with the mean velocity v_0, which gets an infinite velocity at $r = r_0 + v_0 t^*$ at time t^*. A local Eulerian probe will record the velocity $\mathbf{v}(\mathbf{r},t)$ given in Eq. (20) at a given location $\mathbf{r}$, which may be taken as $\mathbf{r} = 0$. To simplify the expressions, let us take also $t^* = 0$ as the time when the singularity (located at r_0) occurs.

Consider first cases without advection velocity, namely with $v_0 = 0$. As time goes on, the velocity fluctuation recorded at $r = 0$ can be seen as follows. As a function of time t, the size of the singularity domain decreases, it is of order $(-\Gamma t)^{1/2}$, because $\Gamma \sim ur \sim r^2/(-t)$. Therefore, when t becomes much smaller than (r_0^2/Γ), the singular domain becomes much smaller than r_0 (its distance to the point of measurement), so that the velocity growth close to the singularity cannot reach the detector. In other words, the growth of the velocity and acceleration will be measured until time $t \sim -(r_0^2/\Gamma)$. As the time delay between t and the singular time gets smaller than (r_0^2/Γ), the velocity field due to the singularity localized in r_0

becomes time independent at large distance $r' \gg r_0$, and given by the $\Gamma/|r' - r_0|$ law of spatial decay, which gives Γ/r_0 at the detector's place.

When the mean velocity is taken into account, the law of decay in space like $1/|r' - r_0|$ becomes a law of decay in time, as recorded by the hot wire, like $\Gamma/|r_0 + v_0 t|$. By writing $|r_0 + v_0 t| = |r_0^2 + v_0^2 t^2 + 2 r_0 v_0 t|^{1/2}$ and shifting time as $t = t' - \frac{r_0}{v_0}$, one finds that the signal becomes $\Gamma/|v_0 t'|$, which is the same for $t' < 0$ and $t' > 0$. This assumes that at positive (unshifted) time, an Euler–Leray singularity "bounces" from negative to positive times, its dynamics for positive times (after the singularity) being the same as before the singularity, just because of the symmetry of Euler's equation under time reversal. However, as shown in section 4, even a little bit of viscosity should yield a strong asymmetry of the time signal because it makes vanish the singularity in the scale of the logarithmic time.

The standard view on measurements of velocity fluctuations in wind tunnels by hot wires is that the only thing one can observe is the space-dependent part of the velocity field because the typical time of evolution of the turbulent fluctuations is much longer than the typical time of advection of the structure, just because those two times are related to velocities by the simple formula r/v, so that the bigger velocity yields the shortest time for a given distance r. This is called Taylor assumption/hypothesis of frozen turbulence. In the Modane experiments described in Sec. 6, the standard deviation of v is smaller than $v_0/10$, but the maximum amplitude of the velocity fluctuations ($v_{\max} - v_{\min}$) is of the same order as or even bigger than the mean advection velocity v_0. Therefore, applying Taylor's assumption is obviously not permitted. Let us return to the scaling relation (15) and clarify the role of advection in the time derivative of the velocity, an important point to compare the scaling laws with experimental data (see next section). From (20), we get:

$$\frac{\mathrm{d}\mathbf{v}(\mathbf{r},t)}{\mathrm{d}t} = \frac{1}{(-t)^{3/2}}\left(U + R\frac{\partial U}{\partial R}\right) + \frac{v_0}{(-t)}\frac{\partial U}{\partial R} \tag{21}$$

On the r.h.s. of (21), the leading order term is the first one (as t tends to zero). The second term, coming from the advection, is like $1/(-t)$. To be more precise, let us compare the order of magnitude of these two contributions to the acceleration. The first one is of order $\Gamma^{1/2}/(-t)^{3/2}$, whereas the second one, which was used to derive (15), is of order $v_0/(-t)$. When v_0 gets very big, the advection effect can dominate over the self-similar dynamics, but it is not always dominant, since the first term leading to the scaling law (12) becomes dominant as t gets closer and closer to 0.

We show below that the experimental observations agree well with the relationship (12) between velocity and acceleration where the advection effect on the measurements is neglected, although the experiment does not fit the relation (15) deduced with the Taylor hypothesis. In summary, the assumption of Taylor frozen turbulence has to be used with caution when looking at extreme events.

6. Analysis of wind-tunnel records

We tested the two relations (12)–(14) and also (15)–(16) against experimental results by comparing the values of the velocity fluctuations $u = v - v_0$, and of the acceleration recorded at the same place and same time, $\gamma_i = (v_{i+1} - v_i) f$, with f the sampling frequency. We looked at the data obtained in the S1MA wind-tunnel from ONERA in Modane, where turbulent velocity was recorded by hot wires. The first subsection below is relative to data taken in the return vein of the tunnel in the 1990s [7,8]. Subsection 6.2 uses recent measurements made in 2014 in the framework of a ESWIRP European project, also in the wind-tunnel of Modane [18].

Our aim was to use experimental data in order to conclude about the presence of self-similar solutions in the turbulent flow, and more precisely if self-similar solutions of type (6) do show-up. If they do, even as rare events, they should be seeable at least for large γ and u values, where one expects a relation of type (12), or (13), between acceleration and velocity fluctuations. On the contrary, if Kolmogorov scaling rules the dynamics, large accelerations (resp. velocity) should occur when velocity fluctuations (resp. acceleration) are small.

6.1. Data taken in the return vein in the 1990s

We first present our study of a 10-min record of the wind velocity, taken at sampling frequency $f = 25$ kHz ($\mathcal{N} =$ 13.7 millions of points in time) by a single hot wire located in the return vein of the tunnel. The mean wind velocity was 20.55 m/s, with 1.7 m/s as the standard deviation. The Reynolds number $Re_\lambda = \sqrt{15 Re}$ is about 2500 (one of the largest values in this kind of experiment). The "acceleration" in these data (as defined above) has standard deviation $\sigma_\gamma =$ 1803 m/s^2, and the maximum acceleration was about 30 times this value, reaching about 5000 times the gravitational constant. In order to test if the scalings associated with self-similar solutions can be extracted from the experimental data, we have studied the behavior of two sets of conditional moments, the first one given γ, which is presented just below, the second one given the velocity u, see subsection 6.1.2.

6.1.1. Moments conditioned on acceleration

From the set of paired values (v_i, γ_i), $i = 1 \ldots \mathcal{N}$, taken at the same time and same place, we first look at conditional moments of the velocity fluctuations given the acceleration, noted $\langle u^n \rangle_\gamma$, with $n = 1, 2, 3$.[1] Let us compare the scalings for Euler–Leray and Kolmogorov predictions in terms of the size of the structures. For the Leray case, self-similar solutions close to t^* are obviously associated with short-lived and small-sized spatial structures, as seen from (2), which gives

$$\left\langle u^3 \right\rangle_\gamma \sim \Gamma\gamma \sim 1/r^3 \quad \text{or} \quad \left\langle u^2 \right\rangle_\gamma \sim \frac{\Gamma}{v_0}\gamma_{\text{Taylor}} \sim 1/r^2 \qquad \text{(Leray)} \tag{22}$$

The two possibilities in (22) correspond to the relations (12) and (15); the former is deduced with $\gamma = \partial u/\partial t$, the latter is based on Taylor's hypothesis, $\gamma \sim v_0 \partial u/\partial x$. In addition, we note that for singularities associated with self-similar solutions with Sedov–Taylor exponents, the above relations becomes $\langle u^{8/3} \rangle_\gamma \sim 1/r^4$. Such singular events associated with small r values may be identified by large values of acceleration *and* of velocity fluctuations. On the other hand, Kolmogorov scalings (14)–(16) predict that small scales are also connected to large acceleration, but they are linked to small velocity fluctuation according to the rule

$$\langle u \rangle_\gamma \sim r^{1/3} \;\; ; \;\; \gamma \sim r^{-1/3} \;\text{or}\; \gamma_{\text{Taylor}} \sim r^{-2/3} \qquad (K) \tag{23}$$

As above, the two possibilities depend on the validity of Taylor's hypothesis; they correspond to (14) or (16), respectively.

If Leray-like solutions are formed in the flow, they have to co-exist with Kolmogorov fluctuations, then large acceleration events can appear either with a large velocity (due to Leray's condition), or with a small velocity (due to Kolmogorov scalings).

Therefore, if one observes a linear relation between γ and $\langle u^3 \rangle_\gamma$ (or $\langle u^{8/3} \rangle_\gamma$) one could expect that the ratio $\langle u^3 \rangle_\gamma / \gamma$ (or $\langle u^{8/3} \rangle_\gamma / \gamma$) would be smaller than the true value of the circulation Γ (or energy E) around the singular point, because this ratio should result from a kind of competition between the two processes of building small-scale fluctuations.

The conditional moments deduced from the experimental data are defined formally as

$$\langle u^n \rangle_\gamma = \int u^n P_\gamma(u) \mathrm{d}u \tag{24}$$

where $P_\gamma(u)$ is the conditional probability of the velocity fluctuation for a given value γ of the acceleration, which is deduced from the joint probability $P(u_i, \gamma_j)\mathrm{d}u\mathrm{d}\gamma$ for the pair of variables (u, γ) to be inside the domain $(u_i, u_i + \mathrm{d}u) \times (\gamma_j, \gamma_j + \mathrm{d}\gamma)$. From the raw data, it is given by the number of points recorded in this domain divided by the total number of recorded points,

$$P(u_i, \gamma_j)\mathrm{d}u\mathrm{d}\gamma = N_{i,j}/\mathcal{N} \tag{25}$$

The conditional probability $P_{\gamma_j}(u_i)\mathrm{d}u$ for the velocity to be inside the interval $[u_i, u_i + \mathrm{d}u]$ given the event that acceleration is inside $[\gamma_j, \gamma_j + \mathrm{d}\gamma]$, is given by $N_{i,j}/N_j$, where $N_j = \sum_i N_{i,j}$. Using Eqs. (24)–(25), we get the following expression for $\langle u^n \rangle_\gamma$ in terms of the number of points recorded in the elementary domains,

$$\langle u^n \rangle_\gamma = \sum_i u_i^n \frac{N_{i,j}}{N_j} \tag{26}$$

To see which one, if any, of the relations (12)–(13) or (15), on the one hand, and (14) or (16), on the other hand, agrees with the experimental data, we plot in Figs. 1 and 2, respectively, the observed values of $\langle u^3 \rangle_\gamma$, and $\langle u^2 \rangle_\gamma$, and we show in Fig. 3a that Kolmogorov relations do not fit the data.

Fig. 1a shows that, in average, the power 3 of the velocity fluctuations increases quasi-linearly with the acceleration, in agreement with the relation (12). This statement is completed by Fig. 1b, which displays the ratio $\langle u^3 \rangle_\gamma / \gamma$ (red curve). Because this ratio slightly increases with γ at large values of γ, we plot on the same curve the ratio corresponding to Sedov–Taylor scalings, $\langle u^{8/3} \rangle_\gamma / \gamma$. Although the exponents 3 and 8/3 are very close, we have to remark that the blue curve displays a clearer flat behavior than the red one. In both cases, the constant (or quasi-constant) behavior extends on a wide range of order $|\gamma| \gtrsim 1.5\sigma_\gamma$. Differently, Fig. 2 shows that Leray's relation (15) with Taylor's hypothesis does not fit so well the data, because the domain where $u^2 \propto \gamma$ is very short or non-existent, see captions. The latter poor fit illustrates that the events associated with large acceleration and large velocity fluctuations are beyond the validity of Taylor's hypothesis.

In summary, the experimental data agree well with our hypothesis of the existence of Leray-type singular events in turbulent flow. We observed a good-enough fit between Leray scalings (12) and the experimental conditional moment, which behaves as $\langle u^3 \rangle_\gamma \sim \gamma$. Surprisingly, we note an even better fit when comparing the data with the relation (13)

[1] We use the notation $\langle u^n \rangle_\gamma$ for a conditional moment given γ, instead of the standard notation $\langle u^n | \gamma \rangle$, to avoid confusion with the ratio $\langle u^n / \gamma \rangle$ also used below.

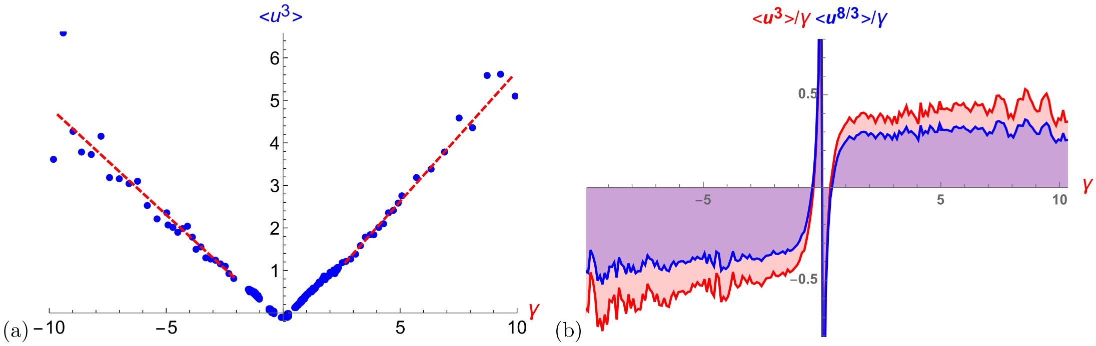

Fig. 1. Experimental test to investigate the validity of the scalings associated with self-similar solutions to the Euler equations. In (a), the conditional average $\langle u^3\rangle_\gamma$ (for a given value of γ) versus γ agrees with (12). Curves (b) (drawn with filling to axis) compare the data with predictions of relations (12) and (13), red and blue curves, respectively. The ratio $\langle u^3\rangle_\gamma/\gamma$ in red increases slightly with γ with a quasi-plateau for $|\gamma| \gtrsim 1.5\sigma_\gamma$, the physical value of the constant Γ is given in the text. The blue curve shows that the ratio $\langle u^{8/3}\rangle_\gamma/\gamma$, which displays a cleaner plateau agrees well with the Sedov–Taylor relation (13). In the figures, u and γ are in units of their respective standard deviation or rms. The bin width for the velocity is $\delta u = 0.5$; for the acceleration, $\delta\gamma$ increases from the origin to the edges (from 0.01 to 0.5), as indicated by the interval between the points in (a).

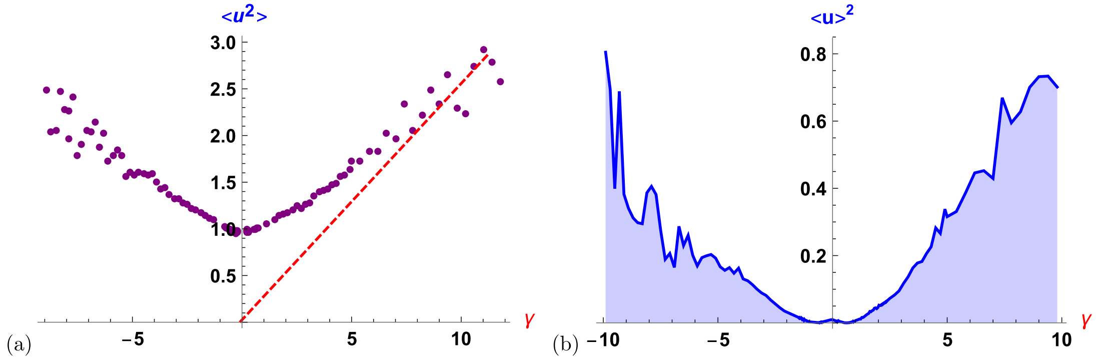

Fig. 2. Experimental test, which shows that the scaling (15) derived from Euler–Leray solutions within Taylor's hypothesis is not good. (a) displays the conditional moment $\langle u^2\rangle_\gamma$ versus γ. The dashed line indicates a short range where eventually a linear relation exists between $\langle u^2\rangle_\gamma$ and γ, which corresponds to less than 0.5 point per thousand, see the inset in Fig. 3b. Curve (b) filled to the axis, displays $(\langle u\rangle_\gamma)^2$ versus γ, which is also non-linear. In the figures, u and γ are in units of their respective standard deviation. Same bin widths as in previous figures.

associated with Sedov–Taylor exponents, as illustrated in Fig. 1b. The large domain spanned by these promising fits is a striking result, which is even a bit unexpected. It implies that the prefactor Γ in (12) or $E^{1/3}$ in (13) is not changing much from one singular event to the other, and that eddies of different size do not change this relation on average.

In the following, we compare Leray scalings with the experiment in order to support our theory, making the hypothesis that the circulation or the energy is conserved in the singular domain.

Fig. 3a shows that Kolmogorov scalings leading to relations (14)–(16) do not agree with the data, because no plateau shows up in the dependence of the products $\gamma\langle u\rangle_\gamma$ and $\gamma\langle u^2\rangle_\gamma$ with respect to γ. In particular, those quantities strongly increase for large accelerations. This result is in favor of the occurrence of Leray singularities in the flow. Furthermore, it shows that the singular structures have a stronger effect on the moments than any other kind of fluctuations (called "normal" later on). In the large acceleration domain, the effect of normal fluctuations would be to lower the circulation or energy value around singular points, since small eddies are associated with small u values for Kolmogorov scalings, as written in (23). Therefore, one expects that the value of the slope in Fig. 1a is smaller than the real value of the circulation close to a singular point. In physical variables, the circulation is $\Gamma = s\sigma_v^3/\sigma_\gamma$ where s is the slope of curve (a), or the height of the plateau in (b). From the observed value $\Gamma_\gamma = 1.6\ 10^{-3}$ m^2/s of the circulation, one may find the local Reynolds number $Re_\gamma = \Gamma_\gamma/\nu$, which is about 160 (taking the kinematic viscosity of air about $\nu \simeq 10^{-5}$ m^2/s at room temperature). It is a large but not very large Reynolds number; see the discussion in the next subsection.

Finally, let us emphasize that there is about 5 to 10 per cent of points in the whole record that agree with Leray's scaling (12) or (13), namely which correspond to acceleration values (scaled to σ_γ) in the domain $|\gamma| \gtrsim 2$, see Fig. 3b, where Leray's or Sedov's scaling is observed. If such events are really associated with singular solutions, this should indicate that the formation of Leray-type self-similar solutions is not so rare.

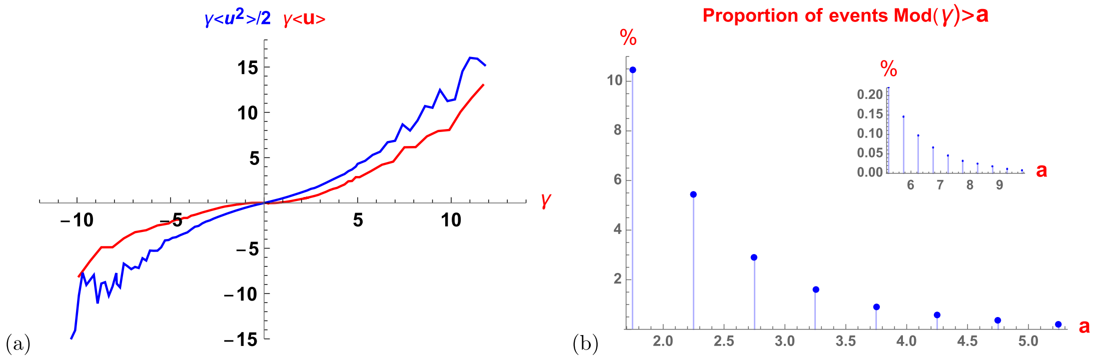

Fig. 3. (a) Experimental test for Kolmogorov scalings (14) and (16). Conditional moments $\gamma \langle u \rangle_\gamma$ (red curve) and $1/2\gamma \langle u^2 \rangle_\gamma$ (blue curve) given γ. The factor 1/2 in front of $\gamma \langle u^2 \rangle_\gamma$ is set to make easier the comparison between the behavior of the two curves. (b) Percentage of rare events with acceleration larger than a (in units of standard deviation). Same bin widths as in previous figures.

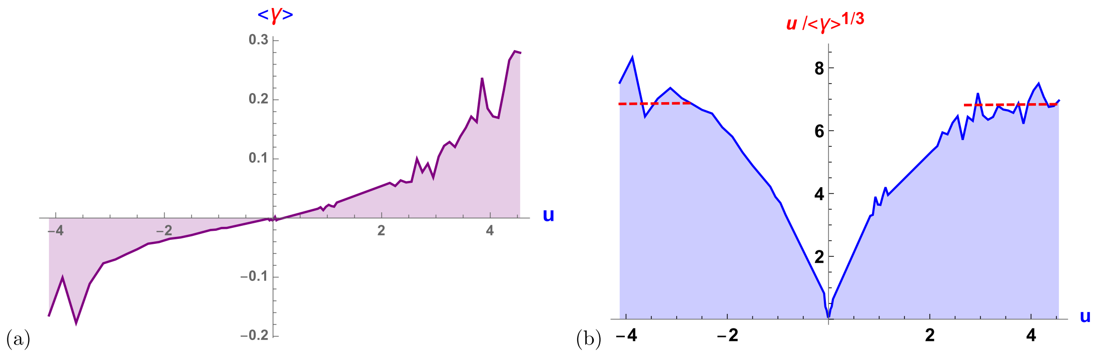

Fig. 4. (a) Experimental plot of the conditional moment given u, $\langle \gamma \rangle_u$, versus u, both in units of their rms. In (b) the ratio $u/\langle \gamma \rangle_u^{1/3}$ versus u displays a plateau for $u \gtrsim 2.5\sigma_u$.

6.1.2. Moments conditioned on velocity

Symmetrically we have also investigated the behavior of $\langle \gamma \rangle_u$, the conditional moment (average value of γ) given a velocity fluctuation u. A large velocity u is expected for small spatial scales with Leray's scaling (22), and for large scales with Kolmogorov's scalings (23). Therefore, peaks of u can be due either to the presence of singular events of small size, if they exist, or associated with large "normal" structures (K relation). But, for such large eddies, large velocities should occur with small acceleration, contrary to singular events, which are related to large acceleration (as explained in section 6.1.1). The result of this study is shown in Fig. 4 (curves drawn with filling to axis).

In Fig. 4a, the curve shows that large values of $|u|$ are associated with maxima of $\langle \gamma \rangle_u$. This result is in agreement with Leray's predictions and in contradiction with Kolmogorov's ones. But we have to notice that the values of $\langle \gamma \rangle_u$ fitting Leray's relation are small (in physical units, they are smaller than the standard deviation σ_γ). This is explainable by the competition between singular events and large "normal" eddies, both contributing to large velocities, as summarized in Eqs. (22)–(23). This drastic reduction of the observed $\langle \gamma \rangle_u$ with respect to what would be expected if singular events were not in competition with large eddies, points out that "normal" fluctuations contribute noticeably to what happens at large velocity. To clarify in what domain Leray's scalings prevail over Kolmogorov's ones, we show in Fig. 4b the ratio $u/\langle \gamma \rangle_u^{1/3}$, which is approximately constant for a velocity u larger than about $2.5\sigma_u$. In this domain, the data are in agreement with Leray's scalings (12); however we note that the plateau is narrower than the one of Fig. 1b for the previous study of conditional momenta given acceleration. Comparing with the previous subsection, this result could yield that the contribution of large normal eddies is more active in reducing $\langle \gamma \rangle_u$ than the contribution of small normal eddies to reduce the moment $\langle u^3 \rangle_\gamma$ calculated in the previous subsection.

Because of the small value of $\langle \gamma \rangle_u$, the apparent circulation Γ_u and the local Reynolds number Re_u are greatly enhanced with respect to the values of the corresponding quantities in Sec. 6.1.1. Here we get $Re_u \sim 10^5$, which is of order of the Reynolds number in the Modane experiment (where $Re = 4.2 \cdot 10^5$). Due to the huge discrepancy between the Reynolds number deduced by the two methods described in Sec. 6.1.1 and in Sec. 6.1.2, the local Reynolds number at the singularity cannot be fairly estimated; nevertheless we can assert that it is much larger than unity, which is the typical value around small eddies in the dissipative range.

Note that we have used the Eulerian definition of the acceleration, $\gamma_E = \frac{\partial \mathbf{u}}{\partial t}$, to compare the Leray and Kolmogorov scaling laws with the experiments. This is correct for Leray scalings, because the self-similar solution is derived with the hypothesis

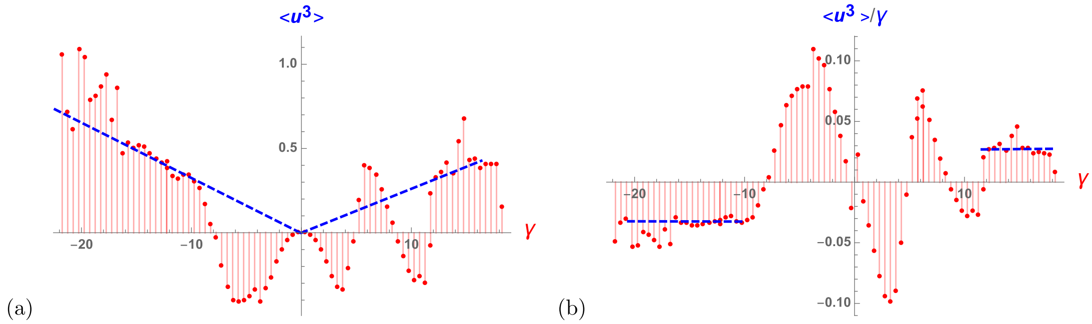

Fig. 5. Conditional moment calculated from data recorded far behind the grid, (a) $\langle u^3\rangle_\gamma$, (b) $\langle u^3\rangle_\gamma/\gamma$ versus γ, in units of their rms. The mean velocity and standard deviations are $\langle v_0\rangle = 40.6$ m/s, $\sigma_v = 0.9$ m/s and $\sigma_\gamma = 11000$ m/s^2. The bin widths are $\delta u = \delta\gamma = 0.5$ in units of their rms.

that the two terms of the Lagrangian acceleration $\gamma_L = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}$, are of same order. Actually, Kolmogorov scalings laws should be written as $u\gamma_L \sim \epsilon$, or $u^2\gamma_L \sim v_0\epsilon$ (if Taylor's hypothesis is assumed). Because the experimental data display a strong increase of $\langle u^2\rangle_{\gamma_E}$ and $\langle u\rangle_{\gamma_E}$ as γ_E increases, it is very unlikely that both quantities become functions decreasing like $1/\gamma$, if Lagrangian acceleration data were used in place of the Eulerian acceleration used here. We conjecture that scaling laws could not make a so large difference between two quantities that represent the same thing.

In summary, the record taken in the return vein of the wind tunnel agrees with the predictions of our analysis, based on the existence of self-similar solutions in the turbulent flow. We observed that in average the events with large acceleration are associated with large velocity fluctuations. The statistical study using probability distributions conditionally to a given value of acceleration, and of velocity fluctuation, shows that, on average, there is a linear relation between γ and u^z with $z \simeq 3$, for large γ and u. This proves that singular events (if they exist) are not rubbed out by the contributions of small eddies in the former case (Sec. 6.1.1), and by large structures in the latter (this subsection). On the contrary, singularities show up as winning in the competition with normal eddies (small and large ones), since the linear relation between u^z and γ is verified experimentally in average, for a large range of acceleration and velocity values.

6.2. 2014 data: grid turbulence

Recent experimental data have been taken behind a grid put in the test section of the wind tunnel in Modane, see Fig. 1 of Ref. [18]. We got some of them in order to see if the location of the hot wires has an effect on the detection of possible singularities in the flow. Among the several files of velocity recorded by hot wires that we have investigated, we present here two of them recorded at two different locations. In the first record the hot-wire was placed far from the grid (at 23 m behind it), where the turbulence is supposed to have relaxed to an isotropic and homogeneous state. The second record was taped closer to the grid (at 8 m behind it). In both cases, the mean velocity is twice larger than in the previous study, the Reynolds number is about five times smaller, $Re_\lambda \simeq 500$, and the sampling frequency is 250 kHz (ten times larger than in the ancient data).[2] The record duration is 10 and 13 min, respectively, which gives files with 150 and 200 millions of points. The results of our analysis of conditional moments given the acceleration are presented in Figs. 5 and 6.

In the first case, far from the grid, the curves $\langle u^3\rangle_\gamma$ versus γ in Fig. 5a and $\langle u^3\rangle_\gamma/\gamma$ in Fig. 5b display oscillations that are not in agreement with self-similar scalings. One may possibly observe that a linear relation $\langle u^3\rangle_\gamma \propto \gamma$ occurs for very large acceleration values, but in this domain the number of points is very small, the total number points corresponding to the linear domain being less than 0.1 per thousand. The slope of the curve is very small; it would correspond to a small circulation and to a local Reynolds number in the order of $Re_{\text{local}} \approx 0.16$. This value smaller than unity points towards events occurring in the dissipative domain only and to rare singular events where inertia is dominant.

Our interpretation of the small circulation deduced from this record is that the hot wire is located in a zone of decaying turbulence, where few organized structures with a large circulation have survived, structures that could become singularities. To assert this, we have investigated other data files recorded closer to the grid (at 8 m from the grid). These data display some events with huge acceleration, the largest one being of order 300 times the rms (see Fig. 7 below)! The result of the statistical study is shown in Figs. 6. It displays a significant domain where $\langle u^3\rangle_\gamma$ is proportional to γ; moreover, the slope is about 30 times larger than far from the grid, that gives a local Reynolds number larger than unity, $Re_\gamma \simeq 5$. Therefore, the proximity of the grid clearly helps the formation of singular structures (if any). For comparison with the ancient

[2] When studying the recent data with a sampling frequency equal to 250 kHz, we have observed white areas in the joint probability $P(u,\gamma)\delta u\delta\gamma$ when the bin widths were smaller than a certain value. For that reason, we chose bin widths equal to 0.5 in units of rms to calculate the conditional momenta. The white areas (without any points) show up as quasi-parallel rows, regularly arranged in the plane (u,γ), even in the domain where the number of points is maximum. We attribute this effect to the fact that the sampling time is probably too small, perhaps 3 or 5 times shorter than the response time of the hot wire. A way to suppress these white zones is to filter the raw data. We used also this technique, which reduces the amplitude of the fluctuations and smoothen the signals, and checked that it gives results (not shown here) in agreement with those presented in Sec. 6.2.

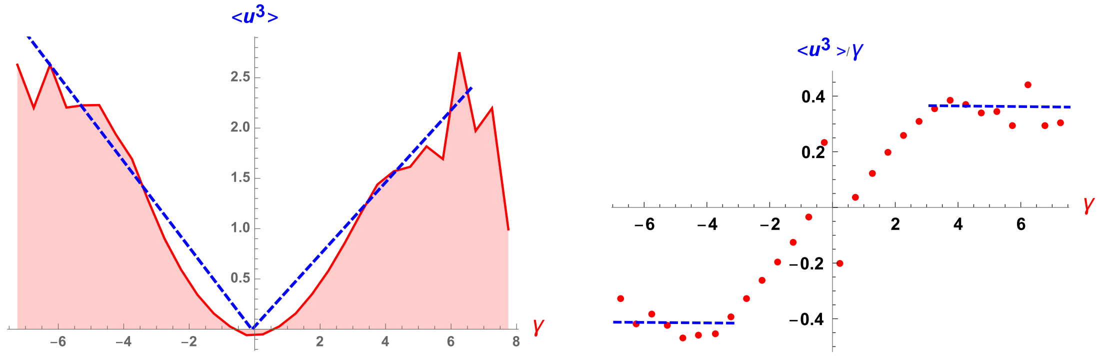

Fig. 6. Conditional moment for data recorded at 8 m behind the grid, (a) $\langle u^3\rangle_\gamma$, (b) $\langle u^3\rangle_\gamma/\gamma$ versus γ, in units of their standard deviations. The mean velocity and standard deviations are $v_0 = 43$ m/s, $\sigma_v = 2$ m/s, and $\sigma_\gamma = 38250$ m/s^2. The bin widths are $\delta u = \delta\gamma = 0.5$ in units of their rms.

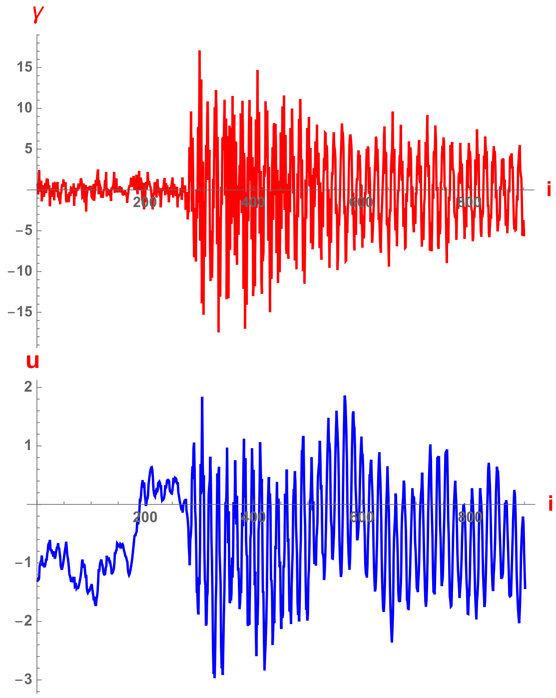

Fig. 7. Asymmetric burst of acceleration and velocity.

data taken in the return vein of the wind tunnel, Y. Gagne informed us that the hot wire was placed on the axis in a zone of "chunk" turbulence. In this place, the large-scale flow results from several contributions, a strongly diverging main flow (the diameter of the vein being 2.4 the one of the test domain), plus additional incoming cold wakes from far entrance. Moreover, a large protective grid is placed in the return vein, against possible flying objects. The good fit of these data with Leray's scalings allows us to conjecture that those additive contributions could help to the formation of singularities.

6.3. *Asymmetry of large fluctuations*

This section is to point out that the records of very large bursts are strongly asymmetric, meaning that the observed rise of the amplitude of velocity and acceleration is very quick and barely observable at our time resolution, whereas the decay of large fluctuations is quite long and involves many oscillations. The maximum amplitude of acceleration observed in Modane comes from the recent records of $v(t)$ made at high sampling frequency close to the grid, where the data show a strongly asymmetric burst, with acceleration values as large as 10^7 m/s^2 ($280\sigma_\gamma$). Similar bursts are also observed for lower amplitude peaks, in the order of 10^5–10^6 m/s^2, as shown in Fig. 7, where the maximum is about $15\sigma_\gamma$.

Such an asymmetry is expected, as explained on general grounds (irreversibility with respect to time) [13], but this general property does not help much to explain such spectacular recording. We believe now that the asymmetry of the time signal $u(t)$ close to a peak could provide one argument in favor of the existence of finite-time singularities (in Euler

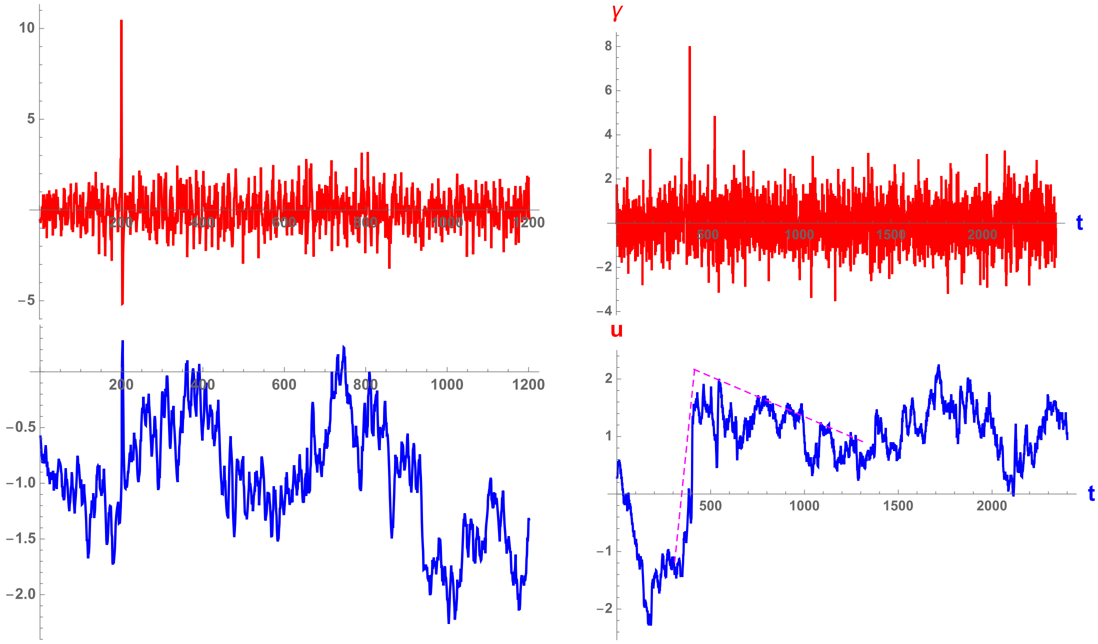

Fig. 8. Left: isolated peak of acceleration and velocity. Right: successive peaks of acceleration, the asymmetry of rising and decay time is suggested by the dashed line on the velocity plot. The acceleration (red curves) and velocity fluctuation (blue curves) are plotted in units of their respective standard deviation. Time is in units of the sampling time.

equations) to explain intermittency in high-Reynolds-number flows. The idea for explaining the striking asymmetry of Fig. 7 is based on the remark that the growth of the bursts is described as an incipient singularity of a solution to the Euler inviscid equation. Turning on the viscosity, as we show in Sec. 4 of this paper, makes drift the singular solution toward lower and lower amplitudes as it gets closer and closer to the time of blow-up. At the end of this smoothing, the fluid motion becomes ruled by the NS equation at a finite Reynolds number. We suggest that, when this happens, the fluctuation decays far more slowly than it has grown because of the decay of the nonlinear part of the dynamics. Therefore, the typical time scale becomes much longer, as observed, as well as the magnitude of the acceleration. The rather complex pattern of time dependence should be the result of oscillations linked to the fact that, even though the Reynolds number is not infinite, the relaxation is still oscillatory because of the effects of the finite nonlinearity of the fluid equations.

We stress that some large peaks show up as isolated ones, as in the left-hand part of Fig. 8, or else as successive peaks of acceleration associated with an asymmetric velocity signal, as shown in the right-hand part of the figures. It is important to notice that the two regimes (quick growth and slow decay) could well be even more different from each other than in the figures, due to the finite-time resolution of the measuring device.

7. Summary and conclusion

We discussed the existence of singular solutions to the inviscid and incompressible fluid equations and how this is related to experimental data. Assuming that the Navier–Stokes equations have no truly singular relevant solution, whereas the Euler equations have such singular solutions, we derive an equation for the decay of the singular solution to the Euler–Leray singularity under the effect of viscosity. Beyond the theoretical analysis, we compare a prediction of the self-similar dynamics with experimental records. It has been known since Batchelor and Townsend that turbulent flows generate large and short-lived derivatives of the velocity fluctuations. The relationship we uncover between large accelerations and large velocities agrees with our explanation of this observed intermittency as due to singularities of solutions to Euler equations.

On a wider point of view, this also shows that, perhaps, more is to be expected in the understanding of turbulence from solutions to the time-dependent fluid equations, including possible effects of a small but non-vanishing viscosity, something which is not so surprising after all!

Using scalings deduced from Leray singular solutions, we have shown that experimental data recorded in the wind tunnel of Modane are compatible with such sparse solutions that could well be not so rare in the case of high Reynolds numbers and for sufficient injected circulation. Because intermittency can be seen as a strong deviation from the K41 scaling law, it is not new to find experimental data that deviate from K41 scalings. We point out that the well-known relation $u_r = (\epsilon r)^{1/3}$ for the velocity fluctuations between two points separated by a distance r results from the hypothesis that the dissipation per unit mass, ϵ, is uniform in space and time. If the exponent of r is less than $1/3$, and if the power dissipated par unit mass is still assumed to be uniform in space, the dissipation should diverge. In Modane, we have found a negative exponent, namely a relation fitting the scaling $u_r \sim r^{-x}$ with x of the order of unity as predicted by (22). Therefore, finite dissipation (on average) and an exponent of r less than 1/3, as found in Modane, can be explained by a sparse (with zero measure) support in the space-time of dissipation events. Such a scenario is well explained by the random occurrence in space-time of singularities of the Leray type. Somehow this connects well the statistical properties of a turbulent flow with the solution to the fluid equations.

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