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A short historical account of period doublings in the pre-renormalization era

Une courte histoire du doublage de période dans l'ère de la pré-normalisation

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ABSTRACT

I will shortly review the history of experimental and theoretical findings on period doubling until the discovery of the quantitative universal properties of the infinite period-doubling cascade.

RÉSUMÉ

Cet article décrit brièvement l'histoire des expériences et des développements théoriques du doublage de période jusqu'à la découverte des propriétés quantitatives universelles de la cascade infinie.

Version française abrégée

Avant d'être renormalisé à Nice et à Los Alamos et de devenir une route robuste (ouverte) vers le chaos, le phénomène de doublage de période a vécu une longue histoire. Cet article présente quelques étapes de ces développements aussi bien théoriques qu'expérimentaux. Les notions de familles à un paramètre et de bifurcation (bien que cette dernière fut introduite par Poincaré) ont mis un certain temps avant d'émerger comme les concepts naturels. La lecture des articles « anciens » frappe par la capacité de calcul (manuelle) et l'inventivité des dispositifs expérimentaux. De nombreuses observations et développements théoriques ont suivi la découverte des propriétés d'universalité dans la suite d'accumulations de doublage de période, et ce flux de résultats n'est certainement pas près de s'arrêter.

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1. Introduction

More than one hundred years separate the first observations of the doubling period phenomenon and the discovery of the quantitative universal properties of accumulation of period doublings. As we will see, there were many experimental and theoretical works on the subject, from which the contemporary concepts emerged. It is interesting to try to understand in the light of our present knowledge what was observed or computed, and how some concepts emerged quite early (like perturbation techniques), while some remained without use for a long time (like the Poincaré map). Of course, we should keep in mind that Science is in constant evolution and we are somewhat biased by our actual knowledge. The major concepts and ideas of tomorrow may largely encompass those of today and shed a different light on the subject.

It is sometimes difficult to ascertain what was really observed in experiments (notwithstanding the fact that some experimental reports seem to be lost). Very clever experimental settings had to compensate for the lack of sophisticated equipments and measurements techniques. For example, in the famous frequency demultiplication experiment [1], the system was observed through a telephone receiver and a human ear was used as a frequency analyzer. The amazing precision and details of observations had probably a lot to do with the fact that Balthazar Van der Pol had absolute pitch [2].

Quite probably, there were many more manifestations of period doubling, which were missed or discarded. It is of course much easier to search experimentally for a phenomenon that has been already predicted theoretically or previously observed in other situations. This probably explains why many experiments concentrated on the frequency demultiplication and in particular on period 3. Note also that frequency demultiplication found early applications [2] for example in building a television system [3]. Quite probably, higher period-doubling bifurcations were observed without being reported.

This paper is organized as follows, based more or less on successive epochs of more and more sophisticated theory and experiments. We will first discuss the history of the period-doubling phenomenon in itself and the development of the theoretical analysis until the occurrence of the notion of period-doubling bifurcation. We then describe some early observations of secondary period-doubling bifurcations and of the beginning of the cascade. This is followed by the qualitative observation of the infinite cascade and some mathematics-related results. Then came the quantitative results on the infinite cascade including universality and openness. The bibliography does not pretend to be exhaustive, but tries to mention the earliest articles about experimental and theoretical discoveries related to the phenomenon of period doubling.

In the sequel, we will say period n to mean period nT when the reference period T is obvious from the context.

2. Double period, subharmonics, undertone, frequency demultiplication, fractional harmonics

One of the first observations of a double period is due to Faraday [4] on surfaces of vibrating liquid layers. A few years later, Savart [5] observed a “son rauque” in cane vibrations, see also [6] and [7]. Von Melde observed a similar phenomenon in the vibration of strings, see Raman [8]. The related phenomenon of parametric resonance had been observed many times even much earlier (the swing was known in very ancient times, like, for example, in the Aiora festival of ancient Greece or in a statue of Minoean time). See also [9] for the history of electrical circuits.

In all the previously mentioned experiments, there is a periodic forcing at a frequency that is roughly twice the “natural” frequency of the system. Note that this is very different from the harmonic generation by nonlinearities. For a nice review about these early experiments and theory, see Von Kármán’s Gibbs lecture [10].

Surprisingly enough, some scientists have raised doubts on the existence of subharmonic solutions; we refer the reader to [11] section 2 for a discussion and historical references. The landmark paper of Van der Pol and Van der Mark [1] on frequency demultiplication also played a major role in the further developments. It is also one of the first papers analyzing the evolution of the dynamics along a one-parameter family.

The first theoretical approaches were made by Lord Rayleigh [12] and [13] (§65b) and Stephenson [14] using perturbation theory. A more mathematical approach was described by H. Poincaré [15], chapter XXVIII “Solutions du deuxième genre” using the time T map.

In order to briefly describe the findings at this stage of history, we will use the example of an electrical system discussed by Pedersen [11] (formula 3.31), namely (with slightly different notations and some rescalings),

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \gamma\,\frac{\mathrm{d}x}{\mathrm{d}t} + \left(\frac{1}{4} + \xi + g\,x\right) x = \sin(t) \tag{1}$$

where $\gamma \geq 0$ is a friction coefficient, ξ measures the detuning with respect to the period 4π, and g measures the intensity of the nonlinearity. Considering the equation for the function $-x(t+\pi)$, one can see that one can assume $g \geq 0$, and it is enough to consider the parameter space $\mathcal{P} = \{(\gamma, \xi, g)\}$ (with $\gamma \geq 0$ and $g \geq 0$).

Note that, if $\gamma = \xi = g = 0$, there is a unique periodic solution of period 2π given by

$$x(t) = -\frac{4}{3}\,\sin(t)$$

One can prove that if γ, ξ and g are small enough, equation (1) still has a unique solution of period 2π denoted by $x_0(t)$. It is natural to ask for the stability of this solution. One first transforms the equation by the translation $x(t) = x_0(t) + q(t)$, which leads to the equation

$$\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} + \gamma \, \frac{\mathrm{d}q}{\mathrm{d}t} + \left(\frac{1}{4} + \xi + 2\, g\, x_0(t) + g\, q \right) q = 0 \tag{2}$$

or in the more convenient system form

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} q \\ v \end{pmatrix} = \begin{pmatrix} v \\ -\gamma\, v - \left(\frac{1}{4} + \xi + 2\, g\, x_0(t) + g\, q \right) q \end{pmatrix} \tag{3}$$

and we are interested in the stability of the trivial solution $q(t) = v(t) = 0$. The Poincaré time 2π map denoted by $\mathscr{P}$ is particularly convenient for this purpose. The 2×2 matrix $D\mathscr{P}(0,0)$ is obtained as the value at $t = 2\pi$ of the solution to the matrix differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} W(t) = \begin{pmatrix} 0 & 1 \\ -\left(\frac{1}{4} + \xi + 2\, g\, x_0(t) \right) & -\gamma \end{pmatrix} W(t)$$

with initial condition $W(0) = \mathrm{Id}$. It follows from Theorem 7.3 chapter **1** in [16] that

$$\det \big(D\mathscr{P}(0,0) \big) = \mathrm{e}^{-2\pi\gamma}$$

Therefore, if $\gamma > 0$, a change of stability in the matrix $D\mathscr{P}(0,0)$ (as a function of (γ, ξ, g)) can only occur when an eigenvalue is equal to ± 1 (the other eigenvalue being of modulus strictly smaller than one). In other words, a Neimark–Sacker bifurcation is impossible.

One can prove that, in a neighborhood of the origin in $\mathcal{P}$, there is a surface (essentially a cone) defined as the zero set of a regular function $H(\gamma, \xi, g)$ such that

$$H = \frac{16\, g^2}{9} - \xi^2 - \frac{\gamma^2}{4} + \mathcal{O}\big(|\gamma|^3 + |\xi|^3 + g^3 \big)$$

The equation $H = 0$ can also be solved in g and we get

$$g = F(\gamma, \xi) = \frac{3}{8} \sqrt{\gamma^2 + 4\xi^2} + \mathcal{O}\big(\gamma^2 + \xi^2 \big) \tag{4}$$

where $F \geq 0$ is a regular function, except at the origin, such that, for $0 \leq g < F(\gamma, \xi)$ $(H < 0)$, Eq. (1) has a stable periodic solution of period 2π. For g larger but near $F(\gamma, \xi)$ $(H > 0)$, this solution is unstable and the instability develops a solution of period 4π. In other words, if one considers a one-parameter family $(\gamma(s), \xi(s), g(s))$ in $\mathcal{P}$, an instability occurs when crossing upward the surface $g = F$ $(H = 0)$.

In [11], one has to combine equations 3.36 and the equation following 3.39 to get the above function F (H) to the leading order (with different notations).

Equation (2) is a particular case of the equation

$$\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} + \gamma \left(q, \frac{\mathrm{d}q}{\mathrm{d}t}, t \right) \frac{\mathrm{d}q}{\mathrm{d}t} + \left[\frac{1}{4} + A \left(q, \frac{\mathrm{d}q}{\mathrm{d}t}, t \right) \right] q = 0 \tag{5}$$

where the functions $\gamma(u, v, t)$ and $A(u, v, t)$ are regular and periodic of period 2π in t. This equation includes also many models of parametric resonance, and we are interested in the stability of the trivial solution $q = \mathrm{d}q/\mathrm{d}t = 0$ which can be analyzed using Floquet theory (see for example [17]). Let

$$\gamma(0,0,t) = \gamma_0 + \gamma_1 \cos(t + \varphi) + \tilde{\gamma}(t)$$

and

$$A(0,0,t) = A_0 + A_1 \cos(t + \psi) + \tilde{A}(t)$$

where γ_0, γ_1, A_0, A_1, φ and ψ are constants and the Fourier spectrum of $\tilde{\gamma}$ and $\tilde{A}$ do not contain 0 neither ± 1. Then, if $\gamma_0 < 0$, an instability occurs only at an eigenvalue ± 1 of the differential of the Poincaré map at the origin (the determinant of this matrix is $\exp(-2\pi\, \gamma_0)$), and there is a real valued functional $\mathscr{F}(\gamma, A)$ such that

$$\mathscr{F}(\gamma, A) = 16 A_0^2 + 4\gamma_0^2 - \gamma_1^2 - 4A_1^2 + 4A_1\gamma_1 \sin(\varphi - \psi) + h.o.t.$$

where the higher-order terms are at least cubic (also in $\tilde{\gamma}$ and $\tilde{A}$), and such that if $\mathscr{F} > 0$ the periodic solution is stable, while if $\mathscr{F} < 0$ and small enough, it is unstable and the instability develops a solution of period 4π.

In the previously mentioned theoretical papers (see also [11]), the function F was computed to the lowest nontrivial order in its Taylor expansion. As often stated, one expects that the nonlinearities will saturate the instability. This of course brings the question of computing the amplitude of the new solution (of double period) to the lowest nontrivial order in perturbation theory. This question was discussed first by Van der Pol [18], and then by Russian groups around Maldelstam, Papalexi, Andronov, Chaikin and Vitt, see [19] and references therein, and Bogoljubov, Krylov, see [20] and references therein. See also [21] and references therein. We refer the reader to [22] for a nice and detailed description of these works, their relations, and more references. See also [23] for the history of the Andronov school at Gorky.

In the 1930's and later, many experimental papers reported on the observation of double period, while several others discussed the theoretical aspects. It is impossible to list all these works, and I will just refer the reader to some of them. On the theoretical aspects, [24] is one of the few papers using the Poincaré map to analyze the problem of the existence of subharmonics. The stability of subharmonics is discussed in [25]. The paper [26] contains a figure of the domain of double period in a section of the parameter space from an earlier article of Andronov and Leontovich. See also [19] and references therein. The perturbation method of analysis of period doubling was exposed in several reference books like [27], [28], [29], [30] [31], [32].

Many experimental examples of subharmonics are described in the book [33]. The paper by Okumura [34] contains many early references about subharmonic oscillations in power circuits (mostly in Japanese). For other experimental observations, one can see, for example [35] for an iron-core system, [36] for a pendulum with vertically moving suspension, [37] for an experiment with cochlea, and [9] for the history of transducers.

Of course, one would like to go beyond perturbation theory and for example in the case of the system (1), one wants to prove the existence of the periodic orbit of period 4π, namely the occurrence of a period-doubling bifurcation of the cycle. Three (at least) general approaches were developed later on.

- Pursue the idea of perturbation theory and then control the equation for the remainder term (by contraction mapping principle or related techniques). This was done in Gambill Hale [38], see also [39].
- Use the time T map (time 2π map in our example (1)) following Poincaré, and prove that there is a period-doubling bifurcation. This is done by considering a center manifold (of dimension one). See [40], Theorem 3, page 576, [41], Theorem 3, page 781, and [42], page 575. The proof also follows by considering the time $2T$ map and applying the bifurcation theorem from simple eigenvalue [43], which was proved around the same time.
- One can also observe that in the case of the dynamical system (1) expressed as a system of two coupled one-dimensional ODEs, for $\gamma = \xi = g = 0$, the time 2π map is equal to $-\mathrm{Id}$. In other words, we have a double eigenvalue -1. One can unfold the three-parameter family of vector fields around this point and then study the bifurcations along a one-parameter family in the parameter space $\mathcal{P}$. See [44], [17], and [45] for the unfolding. This method can also be used to understand frequency demultiplication.

For example, consider a one-parameter family of equations (1), namely a curve $(\gamma(s), \xi(s), g(s)) \in \mathcal{P}$ crossing the surface (4) at a parameter value s_0. In order to conclude from the bifurcation theorem from simple eigenvalue applied to the Poincaré time 4π map, one should check first the hypothesis of Theorem 1.7 in [43]. They are all trivially satisfied in our case except hypotheses (d). With our notations, this hypothesis can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}s} H(\gamma(s), \xi(s), g(s))_{|s=s_0} \neq 0$$

This means that the one-parameter family crosses the surface (4) transversally and with non zero velocity in the parameter s. From now on, we will assume that

$$\frac{\mathrm{d}}{\mathrm{d}s} H(\gamma(s), \xi(s), g(s))_{|s=s_0} > 0$$

The final step is to check that a certain quantity Q (essentially a curvature term) is not zero (see [43] Theorem 1.18 with $n = 2$ for the details). This quantity relates the amplitude A of the bifurcated solution in the marginal (dominant) direction to the parameter s by

$$s - s_0 = Q\, A^2 + h.o.t.$$

In order to express this coefficient in a compact form, it is convenient to use polar coordinates for the parameters γ and ξ, namely

$$\gamma = 2\, r\, \sin(\theta)\,, \qquad \xi = r\, \cos(\theta)$$

with $r \geq 0$ and $\theta \in]0, \pi[$. In these coordinates, the function H is given by

$$H = \frac{16\, g^2}{9} - r^2 + h.o.t.$$

After some computations, one gets

$$Q = -240\, \pi^2\, r^5\, \cos\theta\, \frac{(1+\sin\theta)}{\frac{\mathrm{d}}{\mathrm{d}s} H(\gamma(s), \xi(s), g(s))_{|s=s_0}} + h.o.t.$$

We see that this coefficient can vanish only if $\theta = \pi/2$, namely zero detuning. Moreover if $\theta \in]-\pi/2, 0[$, the period-doubling bifurcation is supercritical ($Q > 0$) and the doubled cycle is stable, while for $\theta \in]0, \pi/2[$ the period-doubling bifurcation is subcritical ($Q < 0$) and the doubled cycle is unstable.

3. The beginning of the cascade

To go beyond the first period doubling, there are several difficulties in observing subsequent ones. As we now know, one looses almost one digit of precision at each bifurcation requiring a high precision in the experimental setting or in numerical computations using analogical or digital computers.

Moreover, most experiments until the seventies were done with systems having periodic forcings (recall that the period doubling of a cycle is impossible in a plane system). These systems exhibit in general frequency demultiplication [1]. It is then easy to misinterpret a period $4T$ arising by a second period doubling and the period $4T$ of the frequency demultiplication, which has a very different origin. Many experimental efforts also concentrated on the period $3T$. The sequence $1, 2, 3, \ldots$ being of course more intuitive than the sequence $1, 2, 4, \ldots$.

An experiment with loudspeakers was performed by Pedersen [11] and [46] in 1933. Besides containing a discussion on the occurrence of the double period (using perturbation techniques), he mentions the experimental observation of an oscillation with a period four times the forcing period (see Fig. 1 in [11]) inside the domain of existence of the double period. Although the figure is presented as a "sketch", this may be the first example of an experiment where a quadruple cycle occurred.

We will mention few other "early" papers dealing with higher subharmonics. Figure V-11 on page 57 in [47], studying nonlinear control systems, seems to describe an overlapping domain of periods 2 and 4. Numerical simulations of voltage regulators in [48] seem to show transitions 2 to 4, tables 2-1, 5-1 and 5-3 with some period 8 harder to interpret. Numerical computations in [49] using the Poincaré map shows clearly up to period 4, see figure 6 on page 248. The book [50] shows period 4 page 167, 173 and describes experiments (page 176, figure 7.26, and page 177, figure 7.27). See also sections 7.6 and 7.7 and references therein. For a review of subharmonic observations in acoustic systems, we refer the reader to [51].

The above list of references is certainly incomplete, since higher period doublings must have been observed in many experiments and perhaps also in numerical simulations. As mentioned at the beginning of this section, it is sometimes difficult to discriminate from the content of the publication if there is a secondary period doubling or a period four coming from frequency demultiplication.

4. The infinite qualitative cascade

We already mentioned that, in the early days, the experimental setup had to be cleverly designed to overcome the lack of precise measurement techniques. From a theoretical point of view, hand-made calculations of perturbation series were rather limited. The advent of computer simulation of dynamical systems [52] and later of computer algebra systems (CAS) opened the road to a much deeper understanding and allowed one to easily perform "computer experiments".

In the 1960's and 1970's, the infinite cascade of period doubling was proved in theoretical works and also observed in numerical computations.

Myrberg, in the late 1950's [53], proved the existence of infinitely many doubling bifurcations in the quadratic family by using an interspersed argument between bifurcations and superstable periods.

Around the same time, Sharkovsky proved his famous result on the ordering.

Theorem 4.1 *(Sharkovsky). Define an order on the integers by*

$$\begin{array}{ccccc} 1 & \prec 2 & \prec 4 & \prec 8 & \prec \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \ldots & \prec 4*9 & \prec 4*7 & \prec 4*5 & \prec 2*3 \\ \ldots & \prec 2*9 & \prec 2*7 & \prec 2*5 & \prec 2*3 \\ \ldots & \prec 9 & \prec 7 & \prec 5 & \prec 3 \end{array}$$

Then, if $q \prec p$ and a continuous map of the interval has a periodic orbit of period p, it has a periodic orbit of period q.

See [54], [55], and [56].

The infinite cascade and combinatorial structure of the orbits of dimensional maps was studied by Metropolis Stein and Stein in [57] and later in [58]. This developed later on in the kneading theory of Milnor and Thurston [59].

Hayashi, Ueda and Akamatsu [60] based on Levinson–Massera relations argued for the existence of 2^n unstable periodic points for the Poincaré return map of a periodically forced nonlinear second-order ODE.

According to [61] and [62], Shapiro [63] observed the infinite period-doubling cascade in an ecological Ricker model. Around the same time, May [64] [65] and May and Oster [66] saw clearly in numerical experiments the infinite cascade and discovered that it was present in many one-parameter families of maps of the interval modeling, in particular the evolution of ecological systems.

Mira and Gumowski developed a technique similar to renormalization to understand the successive bifurcations, see [67], [68] and references therein.

5. The quantitative universal cascade

The quantitative properties of the infinite cascade were first reported in the papers [69], [70] and [71], see also [72]. The universal properties, stability (openness) of the "road to chaos", other universality classes, experimental observations have been studied in many publications, and I will not review them here.

One can refer more generally to papers on the history of dynamical systems like [73], [74], [75] [76], [77], [78], among many others. See also [79] for an occurrence in the field of literature.

An important property of the period doubling "road to chaos" is its openness. In the space of all (regular) one-parameter families of dynamical systems (of fixed dimension), there is an open set such that any one-parameter family inside this set will present the qualitative and quantitative universal properties of the accumulation of period doubling. In particular, any not too large perturbation of a one-parameter family in the set will still be in the set and will have the same universal properties. Of course, this set is far from including all the one-parameter families of dynamical systems. It is easy to construct (by regular surgery) one-parameter families having n period doublings and then a Hopf bifurcation (or other "roads to chaos").

References

[1] B. Van der Pol, J. Van der Mark, Frequency demultiplication, Nature 120 (1927) 363–364.
[2] H. Bremmer, The scientific work of Balthasar Van der Pol, Philips Tech. Rev. 22 (2) (1960) 36–52.
[3] J. Van der Mark, An experimental television transmitter and receiver, Philips Tech. Rev. 1 (1936) 16–21.
[4] M. Faraday, On a peculiar class of acoustical figures; and on certain forms assumed by groups of particles upon vibrating elastic surfaces, Philos. Trans. R. Soc. Lond. 121 (1831) 299–340.
[5] F. Savart, Recherches sur les vibrations longitudinales, Ann. Chim. Phys. 65 (1837) 337–402.
[6] A. Terquem, Etude des vibrations longitudinales des verges prismatiques libres aux deux extrémités, PhD thesis, Faculté des sciences de Paris, 1859.
[7] R. Koenig, Quelques expériences d'acoustique, Paris, 1882.
[8] C. Raman, Experimental investigations on the maintenance of vibrations, Bull. Indian Assoc. Cultiv. Sci. 6 (1912) 1–40.
[9] W. Mumford, Some notes on the history of parametric transducers, Proc. IRE 48 (1960) 848–853.
[10] T. von Kármán, The engineer grapples with nonlinear problems, Bull. Am. Math. Soc. 46 (1940) 615–683; Gibbs lecture 1939.
[11] P.O. Pedersen, Subharmonics in forced oscillations in dissipative systems. Part i, J. Acoust. Soc. Am. 6 (1935) 227–238.
[12] L. Rayleigh, Maintenance of vibrations by forces of double frequency, and propagation of waves through a medium with a periodic structure, Philos. Mag. 24 (1887) 145–159.
[13] L. Rayleigh, The Theory of Sound, vol. I, MacMillan and Co., 1894.
[14] A. Stephenson, On a class of forced oscillations, Q. J. Math. 148 (1906).
[15] H. Poincaré, Les méthodes nouvelles de la mécanique céleste, vol. III, Gauthier-Villars, 1899.
[16] E. Coddington, N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill, New York, NY, 1955.
[17] V. Arnol'd, Loss of stability of self-oscillations close to resonance and versal deformations of equivariant vector fields, Funct. Anal. Appl. 1 (1977) 1–10; For an extended version, see Geometrical Methods in the Theory of Ordinary Differential Equations, Springer, 1988.
[18] B. Van der Pol, A theory of the amplitude of free and forced triode vibrations, Radio Rev. (London) 1 (1920) 701–710, 754–762.
[19] L. Mandelstam, N. Papalexi, A. Andronov, S. Chaikin, A. Witt, Report on recent research on nonlinear oscillations, Tech. Phys. USSR 2 (1935) 81–134, Original title: "Exposé des recherches récentes sur les oscillations non linéaires". Original English Translation: Techtran Corporation, NASA TT F-12,678 (59 pages), November 1969.
[20] N. Kryloff, N. Bogoliuboff, Introduction to Non-Linear Mechanics, Princeton University Press, 1947.
[21] B. Van der Pol, The nonlinear theory of electric oscillations, Proc. Inst. Radio Eng. 22 (1934) 1051–1086.
[22] J.-M. Ginoux, History of Nonlinear Oscillations Theory in France (1880-1940), Springer, 2017.
[23] A. Dalmedico, I. Gouzevitch, Early developments of nonlinear science in soviet Russia: the Andronov school at Gorḱiy, Sci. Context 17 (2004) 235–265.
[24] G. Reuter, Subharmonics in a non-linear system with unsymmetrical restoring force, Q. J. Mech. Appl. Math. 2 (1949) 198–207.
[25] C. Hayashi, Stability investigation of the nonlinear periodic oscillations, J. Appl. Phys. 24 (1953) 344–348.
[26] L. Mandelstam, N. Papalexi, On the parametric excitation of electric oscillations, Zh. Tech. Fiz. 4 (1934) 5–29.
[27] N. Bogoliubov, Y. Mitropolsky, Asymptotic Methods in the Theory of Non-Linear Oscillations, Gordon & Breach, 1961.
[28] A. Nayfeh, D. Mook, Nonlinear Oscillations, John Wiley & Sons, Inc., 1995.
[29] V. Bolotin, The Dynamic Stability of Elastic Systems, Holden-Day, 1964.
[30] A. Andronov, A. Vitt, S. Khaikin, Theory of Oscillators, Pergamon Press, 1966.
[31] N. Minorsky, Introduction to Non-Linear Mechanics, J.W. Edwards, 1947.
[32] J. Stoker, Nonlinear Vibrations, Pure and Applied Mathematics, vol. II, Interscience Publishers, 1950.
[33] N.W. McLachlan, Ordinary Non-Linear Differential Equations in Engineering and Physical Sciences, Oxford at the Clarendon Press, 1950.
[34] K. Okumura, Nonlinear oscillations in power circuits: short history and recent research, in: Chaos Memorial Symposium in Asuka: Selected Papers Dedicated to Professor Yoshisuke Ueda on the Occasion of His 60th Birthday, 1997, pp. 25–34.
[35] I. Travis, C. Weygandt, Subharmonics in circuits containing iron-cored reactors, AIEE Trans. 57 (1938) 423–431.
[36] R. Skalak, M. Yarymovych, Subharmonic oscillations of a pendulum, J. Appl. Mech. 27 (1960) 159–164; Comments by T. Caughey, J. Appl. Mech. 27 (1960) 754–755.
[37] P. Dallos, C. Linnell, Subharmonic components in cochlear-microphonic potentials, J. Acoust. Soc. Am. 40 (1966) 4–11.
[38] R. Gambill, J. Hale, Subharmonic and ultraharmonic solutions for weakly nonlinear differential systems, J. Ration. Mech. Anal. 5 (1956) 353–394.
[39] J. Hale, Oscillations in Nonlinear Systems, Dover, 1963.
[40] P. Brunovsky, On one-parameter families of diffeomorphism, Comment. Math. Univ. Carol. 11 (1970) 559–582.
[41] P. Brunovsky, On one-parameter families of diffeomorphisms ii, Comment. Math. Univ. Carol. 12 (1971) 765–784.

[42] J. Sotomayor, Generic bifurcations of dynamical systems, in: M. Peixoto (Ed.), Salvador Symposium on Dynamical Systems, Academic Press, 1973.
[43] M. Crandall, P. Rabinowitz, Bifurcation from simple eigenvalues, J. Funct. Anal. 8 (1971) 321–340.
[44] F. Takens, Forced oscillations and bifurcations, in: G. Vegter, H. Broer, B. Krauskopf (Eds.), Global Analysis of Dynamical Systems, Institute of Physics Publishing, 2001, pp. 1–62.
[45] G. Ioos, Bifurcation of Maps and Applications, North-Holland, 1979.
[46] P.O. Pedersen, Subharmonics in forced oscillations in dissipative systems. Part ii, J. Acoust. Soc. Am. 7 (1935) 64–70.
[47] E. Carag Tangan, Investigation of Subharmonic Oscillations in Nonlinear Second Order Systems, PhD thesis, Naval Postgraduate School, 1970.
[48] M. Ali Güler, Subharmonic Resonance in a Limit Cycling Voltage Regulator, Thesis, 1970.
[49] C. Hayashi, Y. Ueda, H. Kawakami, Transformation theory as applied to the solutions of non-linear differential equations of the second order, Int. J. Non-Linear Mech. 4 (1969) 235–255.
[50] C. Hayashi, Nonlinear Oscillations in Physical Systems, McGraw-Hill Book Company, 1964.
[51] N.C. Yen, Subharmonic Generation in Acoustic Systems, Contract N00014-67-A-0298-0007 Technical Memorandum No. 65, Office of Naval Research, 1971.
[52] P. Stein, S. Ulam, Non-linear transformation studies on electronic computers, Rozprawy Mat. 39 (1964) 1–66.
[53] P. Myrberg, Sur l'itération des polynômes réels quadratiques, J. Math. Pures Appl. 41 (1962) 339–351.
[54] A. Sharkovsky, Coexistence of cycles of a continuous map of a line into itself, Ukr. Math. J. 16 (1964) 61–71.
[55] J. Guckenheimer, On the bifurcation of maps of the interval, Invent. Math. 39 (1977) 165–178.
[56] K. Burns, B. Hasselblatt, The Sharkovsky theorem: a natural direct proof, Am. Math. Mon. 118 (2011) 229–244.
[57] N. Metropolis, N. Stein, P. Stein, Stable states of a non-linear transformation, Numer. Math. 10 (1967) 1–19.
[58] N. Metropolis, M. Stein, P. Stein, On finite limit sets for transformations of the unit interval, J. Comb. Theory 15 (1973) 25–44.
[59] J. Milnor, W. Thurston, On iterated maps of the interval, in: J. Alexander (Ed.), Dynamical Systems, in: Lecture Notes in Mathematics, vol. 1342, Springer, 1988, pp. 465–563.
[60] C. Hayashi, Y. Ueda, N. Akamatsu, H. Itakura, On the behavior of self-oscillatory systems with external forcing, Electron. Commun. Jpn. 53-A (1970) 31–38.
[61] Y. Kuznetsov, Elements of Applied Bifurcation Theory, Applied Mathematical Sciences, vol. 112, Springer, 1998.
[62] V. Arnol'd, V. Afrajmovich, Y. Il'yashenko, L. Shil'nikov, Bifurcation Theory, Encyclopedia of Mathematical Sciences, vol. 5, Springer, 1993, pp. 7–206, chapter I.
[63] A. Shapiro, Mathematical models of competition, Control Inf., DVNC AN SSSR, Vladivostok 10 (1974) 575.
[64] R. May, Biological populations with nonoverlapping generations: stable points, stable cycles, and chaos, Science 186 (1974) 645–647.
[65] R. May, Simple mathematical models with very complicated dynamics, Nature 261 (1976) 459–467.
[66] R. May, G. Oster, Bifurcations and dynamic complexity in simple ecological models, Am. Nat. 110 (1976) 573–579.
[67] I. Gumowski, C. Mira, Accumulations de bifurcations dans une récurrence, C. R. Acad. Sci. Paris, Sér. A–B 281 (1975) A45–A48.
[68] C. Mira, Systèmes a dynamique complexe et bifurcations de type « boites emboitées », RAIRO Autom. Syst. Anal. Control 12 (1978) 63–94.
[69] P. Coullet, C. Tresser, Itération d'endomorphismes et groupe de renormalisation, J. Phys., Colloq. C5 (1978) 25–28.
[70] C. Tresser, P. Coullet, Itérations d'endomorphismes et groupe de renormalisation, C. R. Acad. Sci. Paris, Ser. A 287 (1978) 577–580.
[71] M. Feigenbaum, Quantitative universality for a class of non-linear transformations, J. Stat. Phys. 19 (1978) 25–52.
[72] B. Derrida, A. Gervois, Y. Pomeau, Universal metric properties of bifurcations of endomorphisms, J. Phys. A 12 (1979) 269–296.
[73] R. Abraham, Y. Ueda, The Chaos Avant-Garde: Memoirs of the Early Days of Chaos Theory, World Scientific, 2000.
[74] D. Aubin, A. Dalmedico, Writing the history of dynamical systems and chaos: Longue durée and revolution, disciplines and cultures, Hist. Math. 29 (2002) 273–339.
[75] P. Holmes, A short history of dynamical systems theory: 1885-2007, in: V. Hansen, J. Gray (Eds.), History of Mathematics, EOLSS, 2010.
[76] P. Coullet, Y. Pomeau, History of chaos from a French perspective, in: C. Skiadas (Ed.), The Foundations of Chaos Revisited: From Poincaré to Recent Advancements, Springer, 2016, pp. 91–101.
[77] C. Mira, Les héritiers de Poincaré, in: E. Falcon, C. Josserand, M. Lefranc, C. Letellier (Eds.), Résumés des exposés de la 15^{e} Rencontre du Non-Linéaire, Paris 2012, Institut Henri-Poincaré, 2012, pp. 1–18.
[78] J.-M. Ginoux, C. Letellier, Des mouvements récurrents de birkhoff aux régimes quasi-périodiques, in: Rencontre du non-lineaire 2011, 2011, pp. 75–80.
[79] G. Schreiber, R. Umansky, Bifurcations, chaos, and fractal objects in Borges "garden of forking paths" and other writings, Rev. Centro Estud. Doc. Jorge Luis Borges 11 (2001) 61–80.