C. R. Mecanique 347 (2019) 555–560

Contents lists available at ScienceDirect

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Asymptotic analysis of a thin linearly elastic plate equipped with a periodic distribution of stiffeners

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ARTICLE INFO

Article history:
Received 13 June 2019
Accepted 19 July 2019
Available online 23 August 2019

Keywords:
Hard abutting and rigidification of plates
Asymptotic modelling
Periodic homogenization
Reduction of dimension
Variational convergence

ABSTRACT

We derive several models of thin plates equipped with a periodic distribution of stiffeners. Depending on the orders of magnitude of the different parameters involved, diverse situations arise, from classical Kirchhoff–Love behaviour with additional energy term to full rigidification.

1. Introduction

To mathematically derive the Reissner–Mindlin model of thin linearly elastic plates (see [1]), we considered a periodic distribution of plates abutted together through thin and soft adhesive layers. So, it is natural now to examine the case when the adhesive layers are stiff. One of the main motivations is the rigidification of plates through a distribution of parallel stiffeners. But it is also a first step in the study of the optimization of plates. In that respect, a more distant goal lies in the relation between the design of the plate and the improvement of some selected aspects of its mechanical performances, without altering the total quantity of material employed (see [2] for example). We will present here six models indexed by $p = (p_1, p_2)$ in $\{1, 2, 3\} \times \{1, 2\}$, where p_1 is a geometric parameter linked to the stiffeners layout, while p_2 accounts for the order of magnitude of the rigidity of the stiffeners.

More precisely, as usual we make no difference between the Euclidean physical space and $\mathbb{R}^3$ with orthonormal basis $\{e_1, e_2, e_3\}$ and, for all $\xi = (\xi_1, \xi_2, \xi_3)$ in $\mathbb{R}^3$, we define $\hat{\xi} := (\xi_1, \xi_2)$ and denote the standard Euclidean distance by *dist*. Let $(\tau^k, \nu^k) := (e_{3-k}, e_k)$, for $k = 1, 2$ and $(\tau^3, \nu^3) := \dfrac{1}{\sqrt{2}}(e_1 + e_2, e_1 - e_2)$, (h, ε, η) three small positive real numbers and ω a domain of $\mathbb{R}^2$ with a Lipschitz-continuous boundary $\partial\omega$. For all k in $\{1, 2, 3\}$, we define:

$$\Sigma_i^k := \left\{ i\varepsilon\nu^k + \mathbb{R}\tau^k \right\} \cap \omega,\ i \in \mathbb{Z}, \quad I^k := \left\{ i \in \mathbb{Z};\ \Sigma_i^k \neq \emptyset \right\}, \quad \Sigma^{k,h} := \left(\bigcup_{i \in I^k} \Sigma_i^k \right) \times (-h, h), \quad \Sigma_{p_1}^h := \bigcup_{k \le p_1} \Sigma^{k,h}$$

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https://doi.org/10.1016/j.crme.2019.07.001

The following two subsets of $\Omega^h := \omega \times (-h, h)$ denoted by

$$B^h_{\eta,\varepsilon,p_1} := \left\{x \in \Omega^h; dist(x, \Sigma^h_{p_1}) < \eta\varepsilon\right\}, \quad P^h_{\eta,\varepsilon,p_1} := \Omega^h \setminus B^h_{\eta,\varepsilon,p_1}$$

are occupied by linearly elastic materials with strain energy densities μW_l and W respectively, where μ is a large stiffness parameter. Introducing $\mathbb{S}^N$ as the space of all $N \times N$ symmetric matrices equipped with the usual inner product and norm denoted as for $\mathbb{R}^N$ by $\cdot$ and $|\cdot|$, W and W_l are two positive quadratic forms on $\mathbb{S}^3$.

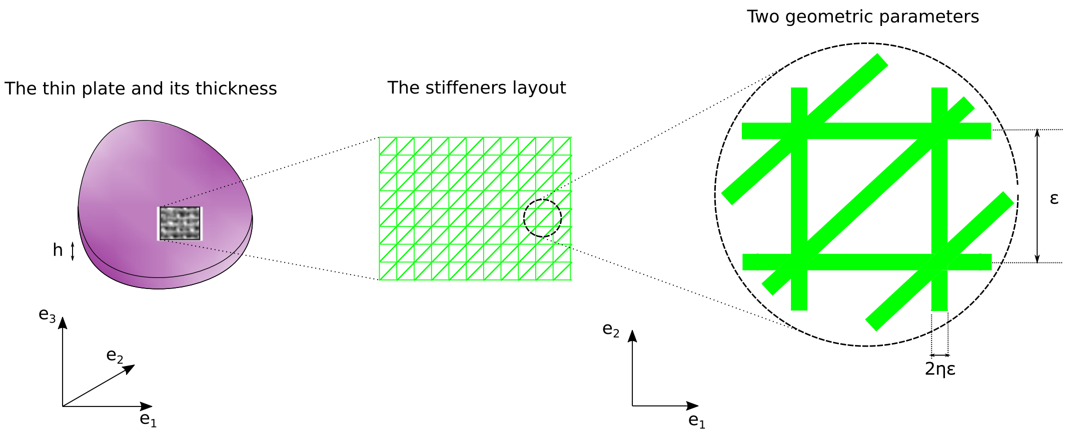

Fig. 1. The thin plate, the triplet of geometric parameters (η, ε, h) and the stiffeners' layout in the case $p_1 = 3$.

The structure made of these two parts perfectly bonded together is clamped on $\Gamma^h_D := \partial\omega \times (-h, h)$ and subjected to body forces and surface forces on $\Gamma^h_\pm := \omega \times \{\pm h\}$ of densities f^h and g^h. Hence, the equilibrium of the structure involves a quadruplet $s := (\mu, \eta, \varepsilon, h)$ of data and leads to:

$$\left(\mathcal{P}^s_p\right) \qquad \mathrm{Min}\left\{J^s_p(v); v \in H^1_{\Gamma^h_D}(\Omega^h; \mathbb{R}^3)\right\}$$

where, classically, for all domain G in $\mathbb{R}^N$ and all smooth part γ of its boundary ∂G, $H^1_\gamma(G; \mathbb{R}^{\mathbb{N}})$ denotes the subspace of the Sobolev space $H^1(G; \mathbb{R}^N)$ made of the elements with vanishing trace on γ,

$$J^s_p(v) := \int_{P^h_{\eta,\varepsilon,p_1}} W(e(v))\,\mathrm{d}x + \mu \int_{B^h_{\eta,\varepsilon,p_1}} W_l(e(v))\,\mathrm{d}x - L^h(v)$$

$$L^h(v) := \int_{\Omega^h} f^h \cdot v\,\mathrm{d}x + \int_{\Gamma^h_+ \cup \Gamma^h_-} g^h \cdot v\,\mathrm{d}\hat{x}$$

$e(v)$ being the strain tensor associated with the displacement field v.

Clearly, if (f^h, g^h) belongs to $L^2(\Omega^h \times (\Gamma^h_+ \cup \Gamma^h_-); \mathbb{R}^3)$, $\left(\mathcal{P}^s_p\right)$ has a unique solution u^s_p and, considering the data s as a parameter, we are interested in the asymptotic behaviour of u^s_p when s takes values in a countable set of $(0, +\infty)^4$ with $\bar{s} := (+\infty, 0, 0, 0)$ as a unique limit point. As in the mathematical derivation of Kirchhoff–Love theory of plates (*cf.* [3,4]), it is convenient to introduce the linear mappings Π^h and S_h:

$$\xi = (\hat{\xi}, \xi_3) \in \mathbb{R}^3 \mapsto \Pi^h \xi = (\hat{\xi}, h\xi_3) \in \mathbb{R}^3$$

$$v \in L^2(\Omega^h; \mathbb{R}^3) \mapsto S_h v \in L^2(\Omega; \mathbb{R}^3) \text{ s.t. } (S_h v)(x) = \frac{1}{h}\Pi^h(v(\Pi^h x)), \quad \forall x \in \Omega := \omega \times (-1, 1)$$

We make the following assumption on the loading:

$$(H1) \begin{cases} \exists\, (f, g) \in L^2(\Omega \times (\Gamma_+ \cup \Gamma_-); \mathbb{R}^3) \text{ s.t.} \\ f^h(\Pi^h x) = h\Pi^h f(x) \text{ a.e. } x \in \Omega,\ g^h(\Pi^h x) = h^2 \Pi^h g(x) \text{ a.e. } x \in \Gamma_\pm \end{cases}$$

therefore, $u_{s,p} := S_h u^s_p$ is the unique solution to

$$\left(\mathcal{P}_{s,p}\right) \qquad \mathrm{Min}\left\{J_{s,p}(v); v \in H^1_{\Gamma_D}(\Omega; \mathbb{R}^3)\right\}$$

where

$$J_{s,p}(v) := \int_{P_{\eta,\varepsilon,p_1}} W(e(h,v))\,\mathrm{d}x + \mu \int_{B_{\eta,\varepsilon,p_1}} W_l(e(h,v))\,\mathrm{d}x - L(v)$$

$$L(v) := \int_{\Omega} f\cdot v\,\mathrm{d}x + \int_{\Gamma_+\cup\Gamma_-} g\cdot v\,\mathrm{d}\hat{x}$$

$$e_{\alpha\beta}(h,v) = e_{\alpha\beta}(v),\ e_{\alpha 3}(h,v) = \frac{1}{h} e_{\alpha 3}(v),\ 1\le\alpha,\beta\le 2,\ e_{33}(h,v) = \frac{1}{h^2} e_{33}(v)$$

with Γ_D the reciprocal image by Π^h of Γ_D^h and, similarly, index h is dropped for the image by $(\Pi^h)^{-1}$ of $\Gamma_\pm^h$, Ω^h, $B^h_{\eta,\varepsilon,p_1}$, $P^h_{\eta,\varepsilon,p_1}$, $\Sigma^{k,h}$ and $\Sigma^h_{p_1}$.

2. A convergence result

We assume that

$$(H2)\quad \begin{cases} \exists\,\bar{\mu}\in(0,+\infty]\ \text{s.t.}\ \bar{\mu} := \lim\limits_{s\to\bar{s}}(2\mu\eta), \quad \bar{\mu}\in(0,+\infty)\ \text{if}\ p_2=1, \quad \bar{\mu}=+\infty\ \text{if}\ p_2=2 \\ \lim\limits_{s\to\bar{s}} \dfrac{\eta\varepsilon}{h} = 0, \quad \lim\limits_{s\to\bar{s}} \dfrac{h^2}{\eta\varepsilon^2} = 0 \end{cases}$$

and introduce the space $V_{KL}(\Omega)$ of Kirchhoff–Love displacements vanishing on Γ_D:

$$V_{KL}(\Omega) := \left\{ v\in H^1_{\Gamma_D}(\Omega;\mathbb{R}^3)\ \text{s.t.}\ e_{i3}(v)=0\ \text{in}\ \Omega,\ 1\le i\le 3 \right\}$$

and the positive definite quadratic form on $\mathbb{S}^2$ defined by

$$W_{KL}(q) := \mathrm{Min}\{W(e);\, e\in\mathbb{S}^3\ \text{s.t.}\ \hat{e}=q\}$$

where $\hat{e}_{\alpha\beta} = e_{\alpha\beta}$, $1\le\alpha,\ \beta\le 2$, for all e in $\mathbb{S}^3$.

Let (τ,ν) in $\left\{(\tau^k,\nu^k), k=1,2,3\right\}$. We perform the change of coordinates

$$x=(x_1,x_2,x_3)\in\mathbb{R}^3 \overset{\theta}{\mapsto} \mathbb{R}^3 \ni y=(y_\tau,y_\nu,y_3) := (x\cdot\tau, x\cdot\nu, x_3)$$

and, for all v in $H^1(\Omega;\mathbb{R}^3)$, define v_τ by $v_\tau(y) = v(\theta^{-1}(y))\cdot\tau$. To shorten notations, we write $\partial_\tau v_\tau$ for the derivative in the sense of distributions $\partial_{y_\tau} v_\tau$. Note that $\partial_{\tau^1} v_{\tau^1} = e_{22}(v)$, $\partial_{\tau^2} v_{\tau^2} = e_{11}(v)$, $\partial_{\tau^3} v_{\tau^3} = \frac{1}{2}(e_{11}(v)+2e_{12}(v)+e_{22}(v))$. For all k in $\{1,2,3\}$, we define the real convex quadratic function W_l^k by:

$$W_l^k(t) := \mathrm{Inf}\left\{ W_l\left((Q^k)^\intercal\, e\, Q^k\right);\ e\in\mathbb{S}^3, e_{11}=t \right\}$$

where $Q^k = \tau^k\otimes e_1 + \nu^k\otimes e_2 + e_3\otimes e_3$ and $(Q^k)^\intercal$ denotes the transpose of Q^k.

Let

$$V_p := V_{KL}(\Omega)\ \text{if}\ p_2=1,\ V_{(p_1,2)} := V_{KL}(\Omega)\cap\left(\bigcap_{k\le p_1}\left\{\partial_{\tau^k} v_{\tau^k}=0\right\}\right)$$

$$\bar{J}_p(v) := \int_{\Omega}\left[W_{KL}(\hat{e}(v)) + (2-p_2)\,\bar{\mu}\sum_{k\le p_1} W_l^k(\partial_{\tau^k} v_{\tau^k}) \right]\mathrm{d}x - L(v)$$

Then we have the following result.

Theorem 2.1. *Under assumptions $(H1)$ and $(H2)$, as s goes to $\bar{s}$, $u_{s,p}$ converges strongly in $H^1(\Omega;\mathbb{R}^3)$ toward the unique solution u_p to*

$$\left(\bar{\mathcal{P}}_p\right) \quad Min\left\{ \bar{J}_p(v),\ v\in V_p \right\}$$

and

$$\bar{J}_p(u_p) = \lim_{s\to\bar{s}} J_{s,p}(u_{s,p}) \tag{1}$$

$$\int_\Omega W_{KL}(\hat{e}(u_p))\,\mathrm{d}x = \lim_{s\to\bar{s}} \int_{P_{\eta,\varepsilon,p_1}} W(e(u_{s,p}))\,\mathrm{d}x \tag{2}$$

$$\bar{\mu}\int_\Omega \sum_{k\le p_1} W_l^k(\partial_{\tau^k}(u_p)_{\tau^k})\,\mathrm{d}x = \lim_{s\to\bar{s}} \mu \int_{B_{\eta,\varepsilon,p_1}} W_l(e(u_{s,p}))\,\mathrm{d}x \ \textit{when}\ p_2 = 1 \tag{3}$$

The elementary proof is achieved in two steps through a standard method of variational convergence.

Step 1 (asymptotic behaviour of $u_{s,p}$)

Proposition 2.1. *When s goes to $\bar{s}$, $u_{s,p}$ (up to a not relabelled subsequence) weakly converges in $H^1(\Omega;\mathbb{R}^3)$ toward some u_p in V_p such that*

$$\bar{J}_p(u_p) \le \varliminf_{s\to\bar{s}} J_{s,p}(u_{s,p})$$

Proof. As, clearly, $u_{s,p}$ is bounded in $H^1(\Omega;\mathbb{R}^3)$, we deduce that there exists some u_p in $H^1_{\Gamma_D}(\Omega;\mathbb{R}^3)$ such that, up to a not relabelled subsequence, $u_{s,p}$ weakly converges in $H^1(\Omega;\mathbb{R}^3)$ toward u_p, which does belong to $V_{KL}(\Omega)$. Moreover, the very definitions of W_{KL}, W_l^k, $1\le k\le 3$, and Jensen's inequality imply:

$$\int_{P_{\eta,\varepsilon,p_1}} W(e(u_{s,p}))\,\mathrm{d}x \ge \int_{P_{\eta,\varepsilon,p_1}} W_{KL}(\hat{e}(u_{s,p}))\,\mathrm{d}x$$

$$\mu\int_{B_{\eta,\varepsilon,p_1}} W_l(e(u_{s,p}))\,\mathrm{d}x \ge 2\mu\eta \int_{\mathbb{R}^2\times(-1,1)} \sum_{k\le p_1} W_l^k(\partial_{\tau^k}(<u>^k_{s,p})_{\tau_k})\,\mathrm{d}x$$

where

- $<u>^k_{s,p} := \sum_{i\in I^k} \frac{1}{2\eta\varepsilon}\int_{-\eta\varepsilon}^{\eta\varepsilon} \widetilde{u_{s,p}}\Big((x\cdot\tau^k)\tau^k + (i\varepsilon+t)\nu^k\Big)\,\mathrm{d}t\ \chi^k_{\varepsilon,i}$
- $\chi^k_{\varepsilon,i}$ is the characteristic function of $\Big\{(i+t)\varepsilon\nu^k, 0<t<1\Big\}\times\mathbb{R}\times(-1,1)$
- $\widetilde{v}$ is the extension by 0 to $H^1(\mathbb{R}^2\times(-1,1);\mathbb{R}^3)$ of all v in $H^1_{\Gamma_D}(\Omega;\mathbb{R}^3)$

As $<u>^k_{s,p}$ has the same strong limit $\widetilde{u_p}$ in $L^2(\Omega;\mathbb{R}^3)$ as $\widetilde{u_{s,p}}$, a standard lower semi-continuity argument yields the result. □

Step 2 (identification of u_p)

Proposition 2.2. *For all v in V_p, there exists a sequence v_s in $H^1_{\Gamma_D}(\Omega;\mathbb{R}^3)$ such that*

$$\varlimsup_{s\to\bar{s}} J_{s,p}(v_s) \le \bar{J}_p(v)$$

Proof. It is straightforward by using test functions like

$$w_s(x) = h\Pi^h\rho(x) + \varepsilon\sum_{k\le p_1}\Big(\delta_\eta(x\cdot\nu^k/\varepsilon)\Pi^h\rho(x) + \varphi_\eta(x\cdot\nu^k/\varepsilon)(\psi^k,0)\Big)$$

with

(ρ,ψ^k) in $C^\infty(\overline{\Omega};\mathbb{R}^3\times\mathbb{R}^2)$ vanishing on Γ_D,
δ_η,φ_η the 1-periodic functions such that

$$\delta_\eta(t) = \begin{cases} 0 \ \text{if}\ t<\eta \\ (t-\eta)/\eta \ \text{if}\ \eta\le t\le 2\eta \\ 1 \ \text{if}\ 2\eta\le t\le 1/2 \\ \delta_\eta(1-t) \ \text{if}\ 1/2\le t\le 1 \end{cases}, \quad \varphi_\eta(t) = \begin{cases} t \ \text{if}\ |t|<\eta \\ \dfrac{\eta(1-2t)}{1-2\eta} \ \text{if}\ \eta<t<1-\eta \end{cases}$$

to check that v_s defined by

$$v_s \in H^1_{\Gamma_D}(\Omega;\mathbb{R}^3)\,;\ \int_{P_{\eta,\varepsilon,p_1}} DW(e(h,v_s))\cdot e(h,w)\,dx + \mu \int_{B_{\eta,\varepsilon,p_1}} DW_l(e(h,v_s))\cdot e(h,w)\,dx =$$

$$= \int_\Omega DW_{KL}(\hat e(v_s))\cdot \hat e(w)\,dx + (2-p_2)\,\bar\mu \int_\Omega \sum_{k\le p_1} DW^k_l(\partial_{\tau^k}(v_s)_{\tau^k})\cdot \partial_{\tau^k} w_{\tau^k}\,dx,\ \forall w\in H^1_{\Gamma_D}(\Omega;\mathbb{R}^3)$$

satisfies the assertion.

Thus u_p is the unique minimizer in V_p of $\bar J_p$ and satisfies (1) and, consequently, (2) and (3). Hence, the whole sequence $u_{s,p}$ weakly converges in $H^1(\Omega;\mathbb{R}^3)$, but also strongly because

$$\overline{\lim_{s\to\bar s}} \int_{P_{\eta,\varepsilon,p_1}} W_{KL}(\hat e(u_{s,p}))\,dx \le \overline{\lim_{s\to\bar s}} \int_{P_{\eta,\varepsilon,p_1}} W(e(u_{s,p}))\,dx = \int_\Omega W_{KL}(\hat e(u_p))\,dx \le \underline{\lim}_{s\to\bar s} \int_{P_{\eta,\varepsilon,p_1}} W_{KL}(\hat e(u_{s,p}))\,dx$$

and

$$\int_{B_{\eta,\varepsilon,p_1}} |\hat e(u_{s,p})|^2\,dx \le \frac{C}{\mu},\quad \lim_{s\to\bar s}\int_\Omega |e_{i3}(u_{s,p})|^2\,dx = 0,\quad 1\le i\le 3 \quad \square$$

As quoted in [5,6], to make more precise the asymptotic behaviour of u^s_p, we develop a variant of Theorem 2.1. As no ambiguity ensues, we use the same symbol $\hat e$ for an element e of $\mathbb{S}^3$ such that its non vanishing entries are $e_{\alpha\beta} = \hat e_{\alpha\beta}$, $1\le\alpha,\beta\le 2$, and let $e^\perp := e - \hat e$. Then we have the following theorem.

Theorem 2.2. *There exists a unique z_p in $L^2(\Omega;\mathbb{R}^3)$ such that $e(h,u^s_p)$ converges strongly in $L^2(\Omega;\mathbb{S}^3)$ towards $\hat e(u_p) + z_p \otimes_s e_3$. Moreover,*

- $\Big(DW(\hat e(u_p) + z_p\otimes_s e_3)\Big)^\perp = 0$
- $\int_\Omega W\Big(\hat e(u_p) + z_p\otimes_s e_3\Big)\,dx = \int_\Omega W_{KL}(\hat e(u_p))\,dx$
- *(u_p, z_p) is solution to*

$$\Big(\mathcal{Q}_p\Big)\quad Min\left\{\int_\Omega \Big[W\Big(\hat e(v) + z\otimes_s e_3\Big)\,dx + (2-p_2)\,\bar\mu \sum_{k\le p_1} W^k_l(\partial\tau^k v_{\tau^k})\Big]\,dx - L(v)\,;\ (v,z)\in V_p\times L^2(\Omega;\mathbb{R}^3)\right\}$$

Proof. As $e(h,u_{s,p})$ is bounded in $L^2(\Omega;\mathbb{S}^3)$, it converges weakly toward some $\hat e(u_p) + z_p\otimes_s e_3$ up to a not relabelled subsequence. Moreover (u_p,z_p) appears as the unique solution to $\Big(\mathcal{Q}_p\Big)$ because for all element z of $C^\infty(\bar\Omega;\mathbb{R}^3)$ vanishing on Γ_D, $e(h,Z_h)$, with $Z_h = h\Pi^h Z$, $Z(\hat x, x_3) := \int_0^{x_3} z(\hat x,t)\,dt$, converges strongly in $L^2(\Omega;\mathbb{S}^3)$ toward $z\otimes_s e_3$. Hence, the whole sequence $e(h,u_{s,p})$ converges weakly in $L^2(\Omega;\mathbb{S}^3)$, but also strongly because

$$\int_{B_{\eta,\varepsilon,p_1}} |e(h,u_{s,p})|^2\,dx \le \frac{C}{\mu},$$

$$\overline{\lim_{s\to\bar s}} \int_{P_{\eta,\varepsilon,p_1}} W(e(h,u_{s,p}))\,dx = \int_\Omega W_{KL}(\hat e(u_p))\,dx \le \int_\Omega W\Big(\hat e(u_p) + z_p\otimes_s e_3\Big)\,dx \le \underline{\lim}_{s\to\bar s} \int_{P_{\eta,\varepsilon,p_1}} W(e(h,u_{s,p}))\,dx \quad \square$$

Now these mathematical results can immediately be rephrased in terms related to the genuine physical problem $\Big(\mathcal{P}^s_p\Big)$, which will supply our asymptotic model. Let

$$V_{KL}(\Omega^h) := \Big\{v\in H^1_{\Gamma^h_D}(\Omega^h;\mathbb{R}^3)\,;\ e_{i3}(v) = 0, 1\le i\le 3\Big\}$$

$$V^h_p := V_{KL}(\Omega^h) \text{ if } p_2 = 1,\ V^h_{(p_1,2)} := V_{KL}(\Omega^h)\cap\Big(\bigcap_{k\le p_1}\{\partial_{\tau^k} v_{\tau^k} = 0\}\Big)$$

$$\bar{u}_p^h := S_h^{-1} u_p$$
$$\bar{z}_p^h(x) := z_p\Big((\Pi^h)^{-1} x\Big) \ a.e.\ x \in \Omega^h$$

then we have

Theorem 2.3. *The fields $\bar{u}_p^h$ and $\bar{z}_p^h$ are solutions to*

$$\left(\bar{\mathcal{P}}_p^h\right) \quad Min \left\{ \int_{\Omega^h} \Big[W_{KL}(\hat{e}(v)) + (2-p_2)\,\bar{\mu} \sum_{k \le p_1} W_l^k(\partial_{\tau^k} v_{\tau^k}) \Big] \mathrm{d}x - L^h(v)\,;\ v \in V_p^h \right\}$$

$$\left(\bar{\mathcal{Q}}_p^h\right) \quad Min \left\{ \int_{\Omega^h} \Big[W\Big(\hat{e}(v) + z \otimes_s e_3\Big) + (2-p_2)\,\bar{\mu} \sum_{k \le p_1} W_l^k(\partial_{\tau^k} v_{\tau^k}) \Big] \mathrm{d}x - L^h(v)\,;\ (v,z) \in V_p^h \times L^2(\Omega^h;\mathbb{R}^3) \right\}$$

and

$$\lim_{s \to \bar{s}} \frac{1}{h^3} \int_{\Omega^h} |\hat{e}(u_p^s) - \hat{e}(\bar{u}_p^h)|^2 \,\mathrm{d}x = 0, \ \int_{\Omega^h} |e^\perp(u_p^s)|^2 \,\mathrm{d}x \le C h^3 \tag{4}$$

$$\int_{\Omega^h} |e^\perp(u_p^s) - h\bar{z}_p^h \otimes_s e_3|^2 \,\mathrm{d}x = o(h^3) \tag{5}$$

3. Physical interpretation

The second line of hypothesis (H2) refers to the design of the stiffeners but also to their layout. The condition $\dfrac{\eta\varepsilon}{h} \to 0$ is clear: the stiffeners have to be slender. As to $\dfrac{h^2}{\eta\varepsilon^2} \to 0$, it encloses various information. On the one hand, because $\dfrac{h^2}{\eta\varepsilon^2} = \dfrac{h}{\varepsilon} / \dfrac{\eta\varepsilon}{h}$, it says that the slenderness of the microscopic plates constituting the genuine plate Ω^h (see Fig. 1) is lesser than the one of the stiffeners. On the other hand, the thickness h of the plate being given, it yields that the distance between two nearest parallel stiffeners has to be large enough (more precisely, the condition is $\varepsilon \gg \dfrac{h}{\eta\varepsilon}$).

Theorem 2.3 tells us that, when the order of magnitude of the rigidity of the stiffeners is $\dfrac{1}{2\eta}$, the asymptotic behaviour of the structure is the one of Kirchhoff–Love type. The stiffeners supply an additional term $\bar{\mu} \sum_{k \le p_1} W_l^k(\partial_{\tau^k} v_{\tau^k})$ to the classical term $W_{KL}(\hat{e}(v))$ stemming from the sole W. When the rigidity is of an order of magnitude larger than $\dfrac{1}{2\eta}$, the periodic distribution of stiffeners of direction τ^k implies a vanishing stretch in this direction. Hence, to get a full rigidity, it suffices to use three families of stiffeners, in our case a fourth direction like $\dfrac{e_1 - e_2}{2}$ is not necessary.

To go to the essential we assumed that the stiffness of each family of layers was the same, it is easy if not tedious to consider $p_2^k \in \{1,2\}$, $k = 1,2,3$!...

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