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Correspondence between de Saint-Venant and Boussinesq. 1: Birth of the Shallow–Water Equations

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ABSTRACT

The Shallow–Water Equations (SWEs), also referred to as the de Saint-Venant equations, constitute the current governing mathematical tool for free-surface water flows. These include, e.g., flood flows in rivers and in urban zones, flows across hydraulic structures as dams or wastewater facilities, flows in the environmental fields, glaciology, or meteorology. Despite this attractiveness, the system of two partial differential equations has an exact mathematical solution only for a limited number of problems of practical relevance.

This historical work on the SWEs is based on a correspondence between two 19th-century scientists, de Saint-Venant and Boussinesq. Their well-known papers are thus commented from the point of development of their theory; the input of both scientists is evidenced by their writings, and comments of both to each other that led to what is commonly known as the SWEs. Given the age difference of the two of 45 years, the experienced engineer de Saint-Venant, and the mathematician Boussinesq, two eminent researchers, met to discuss not only problems in hydraulics, but in physics generally. In addition, their correspondence embraced also questions in ethics, religion, history of sciences, and personal news.

The results of the SWEs cease to hold if streamline curvature effects dominate; this includes breaking waves, solitary and cnoidal waves, or non-linear waves in general. In most other cases, however, the SWEs perfectly apply to typical flows in engineering practice; they are considered the fundamental system of equations describing open channel flows. This work thus provides a background to its birth, including lots of comments as to its improvement, physical meanings, methods of solution, and a discussion of the results. This paper also deals with the steady flow equations, gives a short account on the main persons mentioned in the *Correspondence*, and provides a summary of further developments of the SWEs until 1920.

1. Introduction

The de Saint-Venant or Shallow Water Equations (SWEs) currently represent the standard set for free-surface flows, both in one (1D) and two (2D) dimensions. The 1D equations have been proposed in 1871 by Adhémar Barré de Saint-Venant (dSV) thereby allowing for the first computations of unsteady free-surface flows; in addition to the system of equations, dSV solved these equations for a particular case, to be highlighted below. The SWEs are based on two assumptions, namely (1)

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the hydrostatic pressure distribution, apt for shallow flows, whose typical flow depth is much smaller than the typical wavelength, and (2) the uniform velocity distribution, describing the cross-sectional velocity distribution with its time-averaged local mean value. The SWEs constitute in the 1D setup a system of hyperbolic nonlinear partial differential equations for the two unknowns, flow depth h and cross-sectional average velocity V, as functions of the streamwise coordinate x and time t. This set may be complemented by additional equations describing sediment transport, or environmental quantities; these are excluded in the following, however, given the detailed literature on these topics.

The advance of free-surface equations started almost two centuries ago, when Navier [1] proposed a generalized form of the 3D Euler equations, thereby accounting for viscous effects. The complexity of the currently called Navier–Stokes equations is large. dSV's simplification thus was welcomed mainly by engineers although his set of equations was also inaccessible for practical purposes. The first insight into the Navier–Stokes equations was given by Prandtl [2] in his fundamental paper on the boundary layer theory. By restricting considerations to a thin layer along a solid boundary, and treating the remaining flow away from the wall with the Euler equations, the mystery of flows involving a boundary layer was solved for the first time. The final advance was made by Lighthill and Whitham [3] proposing the simplest set, namely the Kinematic Wave Theory. Whereas the mass conservation equation in all the above-stated flow theories is retained, the momentum equation of the kinematic wave theory only requests equality between the local bottom and friction slopes. This approach thus applies for extremely shallow flows, as, e.g., encountered in hydrologic rainfall/runoff processes. It will here not be considered any further. The relevance is, however, that the advance in the governing set of equations describing free-surface flows, is from top (Navier) to bottom (Lighthill and Whitham). The latter set was proposed just at the onset of the era of first computer usage, so that only few applications of all of these sets were attempted prior to this time.

dSV [4] derived the SWEs and their application to flood propagation in two notes of 17 and 24 July in the French journal *Comptes rendus hebdomadaires des séances de l'Académie des sciences, Paris* [referred to as the *Académie*], spanning over 10 pages. This paper remained largely unnoticed given that few realized the value of its content. For example, Ritter [5], known for the Ritter Dam Break Solution, re-derived the identical equations without any account on dSV. It thus appears pertinent to deal with the advance of the SWEs alternatively, namely the large correspondence between dSV and Joseph Valentin Boussinesq (JB), located at the Library of the *Institut de France*, to which the *Académie* belongs. The first author (WHH) spent weeks in this Library to take photos, then transcribing these letters (as far as possible due to some poor lettering quality and optical problems). WHH has by now a MS Word file containing the transcribed letters of more than 1,300 pages.

It appears evident that a complete translation from French into English of all these letters is out of the scope of this paper. Given the many topics addressed, only these relating to the SWEs shall be dealt with. Sketches usually drawn by dSV in his letters to JB are also reproduced after cleaning and re-lettering, as requested by the *Académie*.

An important restriction on the diffusion or publication of the original letters of the *Correspondence* was imposed by the *Académie*. Once WHH had received access to the Library, he was asked to sign a document in which strict personal use of the letters was requested. Accordingly, none of these letters can be published, and they remain property of the *Académie*. The only way to diffuse their contents thus was to transcribe them first into French, from where the English version was gained. The effort to obtain these transcriptions was heavy, but given their significance for the history of open channel hydraulics, the authors are convinced to add to this relevant topic of science.

The contents of this paper is a description of the *Correspondence*, the steady and unsteady forms of the SWEs followed by a resume of dSV's final paper on the topic. The many names of colleagues both from France and Europe mentioned in the *Correspondence* are highlighted in the Appendix for retrieval.

1.1. Correspondence

The correspondence between dSV and JB was made available by the nephew of JB, Marius Boussinesq, shortly after the death of his uncle in 1929, with whom he had a good personal acquaintance. JB likely discussed this procedure with Marius before passing away. The total number of letters kept at the Library is exactly 600, of which 247 are contained in Manuscript Ms 4226, and 353 in Ms 4227. These were photographed as double pages on 1,152 sheets. The quality of most letters is good, except for some which were written on poor paper or on which mainly dSV had problems of writing due to his advanced age. The letter pages are numbered consecutively, starting each year with number 1.

Mrs. Françoise Bérard, the Director of the Library, stated to WHH that, except for some isolated documents, the entire *Correspondence* has never been viewed. Given its importance, the authors decided to publish the major findings in paper form. Given the breadth of interests of both dSV and JB, letters dealing with SWEs are solely discussed. This restriction allows for a first insight into the world of 19th-century world-class scientists, who were in personal contact from 1868 to 1885.

When glancing through the *Correspondence*, it is soon evident that many letters are missing. Either a question, posed in a previous letter, is only answered in a later one, or important statements are missing; mostly letters written by JB to dSV are absent. Moreover, both have in addition had correspondences with many other scientists of the late 19th century. Given the length limitation of this journal, some letters are not contained in the paper but in a Complementary Data Section. To facilitate reading of the letters, these texts are presented in smaller lettering.

The first letter written by dSV to JB is not contained in the *Correspondence*. JB's answer (Letter 1) is dated 17 March 1868, written from Gap, where JB then was professor of mathematics at the High School [*Lycée*]. JB thanks dSV for his interest in his publication, and states:

> You can, with a look on the improved manuscript, observe that I have profited in all parts from your good advices, and realize at the same time that the majority of all passages on which you improve, astonish me mainly in terms of your comments because these are too short and need to be clarified. I hope that my thinking will be well explained in the developments given in the revised text. Have a look particularly at the first paragraph, which is, as you know, the most important. Here are the explanations which you have asked me in your letter dated 9 March.

Given that this letter treats questions in optics, it is not commented. JB closes this letter with:

> I thank you again for your goodwill, and the thoughts that you have for me. I wish from my very soul, and with lots of hope, that you will receive the title Member of the *Institut* [de France], Yours very devoted servant and disciple, JB.

This first letter reflects the close affection of JB to his mentor. Note that JB could have been the grandson of dSV. As will be evidenced below, dSV guided JB, whereas the latter investigated numerous scientific details, in which dSV was hardly able to advance. After years, the two have developed, apart from scientific exchanges, also a close friendship. Their relationship was a win-win situation for both. Note further that dSV was appointed in 1868 president of the Mechanics Section of the *Académie*, so that he was the key person not only heading it, but also in accepting or rejecting papers submitted to the journals issued by the *Académie*.

Letter 3, dated 24 April 1868: JB congratulates dSV for having been appointed president of the Mechanics Section. With this position, dSV was installed at one of the top seats of the French scientific system. These had previously served as platform of important scientists, including, e.g., Navier, Poisson, or Darcy.

Letter 4, dated 7 May 1868: dSV offers to send proofs of JB's manuscripts either directly to the printer (Gauthier-Villars, Paris), or to himself, so that dSV, could pass them to the printer and possibly give advice how to make corrections. dSV also indicates to JB that he can ask for a second proof if questions still are unsolved. Notice that dSV usually had to go each week to the *Académie* by train from his hometown Vendôme for three hours, to head the sessions of his Section.

Letter 5, dated 10 May 1868: JB thanks dSV for his suggestion to send papers to Prof. Liouville, editor of the *Journal de mathématiques pures et appliquées*, then a top journal. JB mentions that his papers include excellent ideas, so that they would open his way to become Lecturer at the *Faculté des Sciences*, Paris.

Letter 8, dated 13 June 1868: dSV provides to JB the following advice relating to his professional career:

> As to me, dear Sir, I can only make recommendations to you. If you would like to have in any case a chair or a Faculty position to become a member in a scientific center offering you resources for your studies, there exists nothing better than in Paris, where the Stanislas College makes a better stand as a Faculty than anything else in France. Professors there are well received and well payed. If this proposal would be made, you should accept.

Letter 9, dated 1 July 1868: JB thanks dSV for his advice. He continues:

> I would have to prepare details in physics, an ungrateful work if there is no laboratory: It is the more important for this preparation to give the practical proof of phenomena in physics. If these items will not advance next year, I will put it as part of my profession.

JB also tells he would spend his annual vacations at his hometown in southern France.

Letter 23 (dated 25 November 1868): dSV communicates to JB that he is about to prepare a Note on extremely small and continuous movements without vortex generation. dSV notes that the maximum velocity of such a flow would be 544 m/s, an enormous value under the equilibrium condition between the driving and resisting forces. Given the wall presence, this value appeared to be excessive. dSV thinks that even for extremely smooth wall properties, the flow patterns close to it will be highly rotational, thereby significantly increasing head-losses. dSV correctly realizes the significant wall effect on the flow properties, addressing the issues to be considered by Reynolds [6], and particularly by Prandtl [2].

These early letters at once reveal the widespread topics addressed by the two then of age 72 and 27. During the next two years, other topics were dealt with, so that it is proposed to continue here with questions in hydraulics and fluid mechanics. Note the enormous frequency of letters exchanged in the *Correspondence*. On average, more than one letter was written per week. These were on average three to four pages long, indicating the long preparation time apart from the daily business.

2. Steady open channel flows

From April 1871 to October 1872, dSV and JB discussed the equations of steady free-surface flows based on the energy and the momentum principles. Whereas dSV used energy conservation, focusing on the velocity correction coefficient α, JB favored the momentum approach, introducing the velocity correction coefficient α', currently referred to as β. Note that the cross-sectional average velocity was written by dSV as U, whereas JB denoted it by V; both are identical. In the following, this parameter will be written as V, to avoid confusion.

Letter 6, dated 20 April 1871, of JB:

> I have made a Note on the permanent movement of water. You can find it in the package sent to you, with a sheet of developments that I did for you, if you want to check details. I do not send you this Note with the intention to submit it for publication in a few days; keep it the time which you consider suitable. I know that you are interested in this topic, that's why I send it.

Based on earlier works of dSV on hydraulics, including for instance dSV [7–9], JB realized that his colleague would be a person of relevance when starting work on the equations of free-surface flows. dSV therefore was the most attractive 'reviewer' for JB, given that dSV not only had previously published outstanding works in this field, but also that he was much inclined to engineering practice. He realized that his future was not only in complicated formulations of theories in applied physics, but that he had also to attract readers from engineering practice who would support his professional career. It is true, JB was rather scientist than engineer, but was open to the latter branch because of the discussions with dSV, who posed the questions of relevance and hoped that JB would find answers that would be accepted by the engineering community.

Letter 7, dated 16 May 1871, from dSV to JB:

> I tell you now what prevented me to send to the printer your two articles on 'Varied permanent movement of water in channels [and pipes]'. I think they hardly will interest surveyors,[1] who have a minor esteem for results to the application of a little high or subtle analysis to problems of practice. I fear that submission in my request of the two new notes as they are, would be harmful to you.

dSV continues:

> These who read the two Notes will find, while you are based on similarities with the formula of Bazin, you are at odds with the usual formulas adopted. It was deduced from his data that RC'/V^2 is no constant, but a binomial $b_1 + (b_2/R)$, proposed by Darcy and Bazin [10].

They demonstrated (1) that the uniform flow formula applies both to open channel and pressurized pipe flows, (2) that the hydraulic radius R is the important length scale, (3) that the boundary roughness is relevant, and (4) that the quadratic velocity effect applies to turbulent rough flows.[2]

dSV further states:

> You say well, and very wisely, that [the constant] B in your 1870 Note depends on the average hydraulic radius, and in your 1871 manuscript, A and B vary also with the size of the cross-section. But, immediately after that, you assign constant values of 0.00065, and 0.00081; you rely on the crude formula $bi = 0.0004V^2$ proposed by engineers of Lombardy, or even Bazin's formula $Ri = 0.00028(1 + 1.25/R)V^2$.[3]
>
> The good form $b_1 + b_2/h$ given to hi/V^2 is not paying to your new formula giving u versus z.[4] Yet, the idea is not the same as a monomial of the exponent like what I proposed in 1864 (as also in 1868) to the Philomathic Society, and in my 1865 brochure on Du Buat, and which in 1867 was presented by Mr. Gauckler with roughly the real exponent ($-1/3$ instead of -0.3) based on a large number of experiences. I believe this factor $R^{-1/3}$ to the second member, or $R^{4/3}i$ to the first, is long accepted.
>
> If A and B depend on the cross-sectional shape, should we not keep these for both the wide channel and the circular pipe? One thing you will have to consider more strongly appears that you do like Darcy, Bazin or Dupuit: it is V^2 to the second power but not V and V^2 as a binomial. I believe nevertheless that a binomial in V, as according to Prony and Eytelwein, based on Coulomb and Girard, and even on [Daniel] Bernoulli, is much more in-line with what's going on. Darcy, Bazin, Gauckler would be forced to be convinced, because the first two recognized that for a small section and the third for slopes $i < 0.0007$, i.e. for all navigable rivers, V instead of V^2 is relevant. Such a jump is impossible; therefore, a binomial formula dominating one of the two terms, depending on the values of V or i, such as $R^{4/3}i = aV + bV^2$, or $1/V^2 = a'/(R^{4/3}i) + b'/(R^{4/3}i)^2$, has to be considered instead of $aV + bV^2$, to highlight the effect of velocity.[5]

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[1] Representing another Section of the *Académie*.

[2] dSV and JB considered two formulas for uniform channel flows, namely these of Chézy [11], and of Girard [12] and Prony [13], the latter of which was adopted similarly by Darcy and Bazin [10]. These read, with R as hydraulic radius and S as the constant bottom slope equal to the free-surface slope,

$$V = C(RS)^{1/2}, \qquad aV + bV^2 = RS.$$

The constants of proportionality a, b, C are dimensionally inhomogeneous, a fact only fully understood in the 20th century. An early review on uniform flow is provided by Mouret [14]. Chézy's modified formula involving $R^{2/3}$ instead of $R^{1/2}$ was first proposed by Gauckler [15].

[3] With i as the pressure gradient.

[4] The horizontal velocity distribution across the vertical $u(z)$.

[5] The above three equations are another attempt of dSV to access uniform flow by a simplified procedure. He correctly recognized that extremely small flow depths and high velocities have to be excluded. Currently, we know that the first may produce laminar flows, whereas the second lead to high-speed air–water flows. The discussion shall not be deepened here given that the time was not yet mature to detail this fundamental issue in hydraulics.

Fig. 1. Application of the uniform flow concept to a rectangular channel and a circular pipe.

As these formulas would become impossible when combined with $(-F)_{z=h} = \rho g \times$ (a binomial in u_0),[6] we could say that, without pretending to embrace extreme cases and, as you said, velocities neither too large nor too small, according to research, which you know since your Note of August 1870, you generalize to

$$F + \rho g \sin i = 0, \quad F = \rho g A u_0 h^{1+n}(\mathrm{d}u/\mathrm{d}z), \ (-F)_{z=h} = \rho g h^p \left(B'u\right)^m,$$

from where

$$u = (h + ai)^{1/m} / \left(B'h^p\right) + B'/(zA)\left[h^{p-n-1/m}\right]A'^{1-1/m}\left[1 - (z/h)^2\right],$$

simplifying for $p = 0$ and even more by taking $p = r$, because then

$$u = (h \sin i)^{1/m} / \left(B'h^n\right) + \left[B'/(zA)\right](h \sin i)^{1-1/m}\left[1 - (z/h)^2\right].$$

Due to two foreign scholars whose usage I did not support, we would have $n = 1/3$. As to m, for a long time equal to 2 or rather a little less, such as 21/11 or 1.9 according to Du Buat and Mr. Bazin for steep channels, since you overlooked flat rivers, which are the most useful to be considered; German hydraulicians advanced it even to 1.5; so 1.7 on average. Dear Sir and friend, edit your notes based on considerations as these.

Consider Mr. Bazin's proposal for velocity u at depth z with a parabolic formula $u = u_0 - (24 \text{ or } 22)(hi)^{1/2}(z/h)^2$ satisfying $\mathrm{d}u/\mathrm{d}z = 0$ at the surface $z = 0$. However, this gives according to me a discrepancy, because he noticed, according to Mr. Boileau (who gave the same application as Bazin, almost independent of air friction), and also Mr. Rancourt, for whom the maximum of u occurs below the water surface, either at 1/4 or 1/5 of depth. Consider also to include the rectangular channel, for which the maximum velocity is defined not at the half depth abcd, but at the 6/10 or 5/8 depth, viz., $a'b'cd$, or the conduit DCcd. (Fig. 1.)

Here are some other observations. Don't you want in general to use i, the ordinary notation, instead of α for the angle of the bottom slope, to be better understood? α has a meaning in hydraulics, it is $\int (u/V)^3(\mathrm{d}\omega/\omega)$ whose value is approximately 1.10, with ω as channel cross section. On page 2, it seems to me that Case 1 does not differ from Case 3 so that you mention only two, the only two for which you perform calculations. Do you mean by u the local averages (as also averages between components of the actual velocity at a certain point in space and for a time comprising more than one period), and the general average of V, so that the reader is not confused!

The term *driving force* for $M(\mathrm{d}v/\mathrm{d}t) = M(\mathrm{d}^2s/\mathrm{d}t^2)$ of a mass M is no longer used and has never been, I think, because we [ordinarily] refer to the *total force*. Today we rather call it *inertia*, or inertial force of negative sign. It is especially Poncelet who no longer needs to be invoked because we ask for a balance between the driving forces P, P', P'' etc. or F and the terms $-m(\mathrm{d}^2x/\mathrm{d}t^2)$, $-m(\mathrm{d}^2y/\mathrm{d}t^2)$.

The determination of $w = \int_z^H (\mathrm{d}u/\mathrm{d}z)\,\mathrm{d}z$ or $w = -(1/r)\int_0^R (\mathrm{d}u/\mathrm{d}r) r\,\mathrm{d}r$ is ingenious, as it is what you conclude in the second Note; it certainly deserves to be reserved for another paper. Nobody had considered to account for the word blossoming (*épanouissement*) h'. Nevertheless, I vaguely believe, if it is proven as on page 1 of the second Note, that we obtain the same formulas (6)[7] as in the uniform regime if $\mathrm{d}h/\mathrm{d}x$ is very small compared to 0.006, i.e. it is apparently only a thousandth; this means there was no need of all the calculation to be performed. Can't you then say that you are certain of all that?

At the end of the letter, dSV makes an extremely important statement:

Thank you to employ my definitions 'torrents' and 'rivers', which will be useful and are highly appropriate in my opinion, and no one employs them though I proposed these in 1852. You learned very well the condition of torrents $V^2 > gH$ [8] obtained if we adopt $Hi = b_1V^2$, that is to say a simple slope condition $i > gb_1$ (except for a slightly different coefficient of unity). Yet, I do not at all understand the end of your second Note, where the need for a hydraulic jump in the tailwater is proposed for the case of a river (as it is necessary that there is one upstream for a torrent). It would be good for new studies that you add to the essentials of the current knowledge.[9]

[6] With u_0 as the velocity at the bottom then considered different than 0.
[7] This formula was most probably stated in an attachment to the letter.
[8] dSV still describes here the flow depth with H instead of h.
[9] dSV was indeed the first to recognize the important distinction between rivers and torrents, or when using the Froude number Fr stating the ratio between the actual flow velocity V and the elementary wave propagation velocity c [$= (gh)^{1/2}$ in the rectangular channel] as $Fr = V/c$. Flows with $Fr < 1$ are characterized by the backwater effect, i.e. any flow perturbation is propagated both up- and downstream from the perturbation source, whereas for

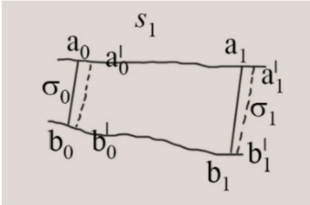

Fig. 2. Sketch explaining the energy conservation concept.

I'll send with Gauckler and Stapfer's notes the brochure *The hydraulic axis* of Boudin, who no doubt will quote you unless you yourself will provide the proof of the entire study. I will finally send you the 1828 Memoir of Bélanger [17], who since changed his views regarding the hydraulic jump: the only thing of interest is where he talks on the deduction of two kinds of water flows.

This first letter of dSV sent to JB dealing with open channel flows thus includes a rich source of information, by which the ensuing developments in open channel hydraulics were impressed. The response of JB to dSV is no less interesting to read. Letter 8 (dated May 19, 1871) states:

I just received your letter of 16 May; thank you a thousand times for all the devotion that you persistently invest in my interest. I see, reading your excellent comments on my Note 'Permanent movement of gradually-varied water flow', that this subject cannot be properly treated in a few pages, a short cut like I did; this leads to misunderstandings, and sometimes triggers further ideas in my thoughts.

JB continues:

Regarding the consistency of A and B, based on the experiences of Bazin on the 'varied movement', I thought to admit whether the role of the hydraulic radius along the considered current is not too strong, and, whether what I did by selecting $b_1 = 0.0004$ as Bresse is better. This would replace the functions A and B by simple numbers, and thus shorten the text so as not to exceed the maximum of 4 pages.[10] If I suggested for open channel flows that the maximum velocity is very close to the surface, it is that I confined myself to the case of a very wide bed and a large enough velocity, conditions for which this is true according to Mr. Bazin.

On page 1, I consider only uniform motion [or the case of a straight horizontal pipe of constant section], connecting two basins whose levels are different; if the relative velocities of the molecules were reduced, the pressure along the pipe would equally vary. This case study leads easily to one where the hydraulic radius varies strongly from one end to the other, but quite slowly so that the movement on a small length is uniform. I do not need to demonstrate that, for $\mathrm{d}h/\mathrm{d}x \leq 0.006$, the velocity distribution at any section differs hardly from that of the uniform regime, because it is not obvious that the absolute velocity for $h' = 0.006$ does not differ from that of the uniform regime.

I did not understand how others introduced the coefficient α. Bresse, the only reference available, does not seem to be clear on this point. This question is embarrassing enough to hardly reconcile with my formulas. The conclusion of my second article, too brief to be well understood, goes with what I said at its beginning, with the hydraulic jump I have sent to you; please keep it (because I have a good copy).

The remainder of this long letter includes other topics, not discussed here.

Letter 9 (dated 29 May 1871) gives the answers of dSV to JB as follows:

It is unfortunate that Bresse confused simple things by an innovative move into science to expose hydraulics. And in his backwater problem (p. 135 of the 1868 study), to ask for the equation of permanent movement relative to the channels. Finally, he saw the duty to simplify this corrective coefficient (which is slightly greater than 1), introduced by Coriolis and Poncelet, and retained by Bélanger (in the lithographed notes, which must affect the driving force due to the average velocity of a slice of water over time dt to have the real 'living force'). The simplest and clearest way to ask for the equation of non-uniform permanent water flow is based on the theorem of the active forces and work, applied to a channel portion, as previously considered by Coriolis or Bélanger.

Let $a_0b_0b_1a_1$ be a portion of the current, of length s_1, with boundary sections a_0b_0 and a_1b_1. These become after a time step dt sections $a_0'b_0'$ and $a_1'b_1'$, respectively, with V_0, V_1 as the average velocities across the first and the second section, with $V_0 = (1/\omega_0)\int u_0 \mathrm{d}u_0$, $V_1 = (1/\omega_1)\int u_1 \mathrm{d}u_1$, with i as the surface slope and ω as the cross-sectional area. (Fig. 2.)

$Fr > 1$ perturbations are propagated only in the flow direction. The so-called critical flow state $Fr = 1$ divides the above two regimes; it also is the state under which a flow has minimum energy head for given discharge, or maximum discharge for a given energy head.

Hager and Castro-Orgaz [16] wrote a paper on the Froude number, the most important notion in open channel hydraulics. They concluded that it is a misnomer, and should rather be referred to as the de Saint-Venant number, given his works on the topic. Froude, an outstanding naval engineer, was not involved in open channel hydraulics. The authors did not propose a name change of the Froude number, but tried to set things correctly. Note also that Froude's paper was only published in 1872.

[10] Publications in the *Comptes rendus* were limited to four printed pages.

The living power (i.e. the half living force[11]) acquired by $a_0b_0b_1a_1$ during dt will be reduced, because the branch output $a_1b_1b'_1a'_1$ equals the branch input $a_0b_0b'_0a'_0$; their masses for a stream tube are $\rho\,\mathrm{d}\omega_1u_1\,\mathrm{d}t$, $\rho\,\mathrm{d}\omega_0u_0\,\mathrm{d}t$. A half living force will thus be gained $(\rho/2)\,\mathrm{d}t\int u_1^3\,\mathrm{d}\omega_1-(\rho/2)\,\mathrm{d}t\int u_0^3\,\mathrm{d}\omega_0$, in which $\alpha_0=\int(u_0/V_0)^3(\mathrm{d}\omega_0/\omega_0)$, $\alpha_1=\int(u_1/V_1)^3(\mathrm{d}\omega_1/\omega_1)$, $\rho\,\mathrm{d}t\omega_1u_1\alpha_1(V_1^2/2)-\rho\,\mathrm{d}t\omega_0u_0\alpha_0(V_0^2/2)$. The work of gravity is equal to the weight $\rho g\omega_0u_0\,\mathrm{d}t=\rho g\omega_1u_1\,\mathrm{d}t$ of each of the two elements $a_0b_0b'_0a'_0$, $a_1b_1b'_1a'_1$, multiplied by the difference of the level of their gravity values G_0, G_1. Combining with the work of the pressures on sections a_0b_0, a_1b_1, which is $\omega_1\rho gG_1a_1V_1\,\mathrm{d}t-\omega_0\rho gG_0a_0V_0\,\mathrm{d}t$, yields $\rho gV\,\mathrm{d}t(\xi_1-\xi_0)$, in which $\xi_1-\xi_0$ represents the surface slope of a_0 to a_1, or the difference of the two ordinates $\xi_1-\xi_0$ of the two slopes relative to the horizontal. The resistance force is due to the presence of the bottom and the wall surfaces. With χ as the wetted portion of section ω, whose abscissa is s and where the average velocity is V, and F is the friction by meter squared, $\chi FV\,\mathrm{d}t$ expresses that this work involves the average velocity V. The equation of work is obtained by dividing by $\rho\omega_0u_0\,\mathrm{d}t=\rho\omega_1u_1\,\mathrm{d}t=\rho\omega V\,\mathrm{d}t$, so that

$$\alpha_1\big(V_1^2/2\big)-\alpha_0\big(V_0^2/2\big)=g(\xi_1-\xi_0)-\int(\chi/\omega)(F/\rho)\,\mathrm{d}s.$$

According to Prony and Eytelwein, $F/\rho=aV+bV^2$, $\chi/\omega=1/R$, R being the average hydraulic radius, with $F/\rho=0.00028[1+(1.25/R)]V^2$, according to Mr. Bazin, unless we replace this expression by $b_1R^{-1/3}$ according to Mr. Gauckler with 1.75 as the exponent of V, thereby transforming the last term on the right-hand side into $-\int(b_1/R^{4/3})V^{7/4}\,\mathrm{d}s$. (Lahmeyer and other Germans adopted 3/2 for the exponent of R). This relative friction term is the clearest way to present the equation of water flows, applicable to practice for a stream, even irregular, by dividing it into small portions of length. That is how we were taught at *École centrale*.[12,13]

I state in this way that Bresse should show the reason why the coefficient $\alpha=\int(u/V)^3(\mathrm{d}\omega/\omega)$ is always greater than unity. He found for it the value 1.08 by assuming a parabolic velocity distribution from the surface to the bottom, and a surface velocity of 5/4 of the average velocity V. I take it as 1.10 or 1.12 relative to the actual average velocity. Wouldn't this be equal to your 1.08? Verify this, dear Sir and friend! And if your factor is not the α of Coriolis [18], wouldn't you do well to say how it differs?

dSV continues:

Experimental data may be represented by different empirical formulae, as evidenced by the equations of Gauckler and the four proposals of Darcy and Bazin [10], respectively, on pages (1) 130, (6) 133, (7) 134, (8) 142, summarized on page 223 of Bresse [19]. It appears that in these four formulae the effect of R on b_1 considerably differs depending on the wall roughness, despite the fact that Gauckler represents it for all categories by $R^{-1/3}$. I think, when considering earthen banks, one would adopt Gauckler's proposal $R^{-1/3}$ in the second member, or $R^{-4/3}i$ in the first.

You accept that the term $V\,\mathrm{d}V$ of the dynamic equation has to be multiplied by $\alpha'=\int_0^\omega(u/V)^2(\mathrm{d}\omega/\omega)$, closer to unity than $\alpha=\int_0^\omega(u/V)^3(\mathrm{d}\omega/\omega)$. However, you base your approach on the momentum theorem. Note that you have to pay attention in doing so: its usage is much less safe when compared with energy considerations. This is caused by: (1) the energy approach involves forces as the wall reactions whose work is zero, whereas the momentum approach requires their knowledge, because the reactions have a component in the flow direction, that is required to formulate the momentum balance in the flow direction; (2) with the energy approach, one accounts for velocities in their real directions without any decompensation in certain directions.

For example, as is noted in the lecture notes of Bélanger to be sent, you will realize how simple it is to determine the energy equation by Torricelli's principle, $V=(2gH)^{1/2}$ or rather $V=[2gH(1-\Omega^2/O^2)]^{1/2}$.[14] By applying d'Alembert's momentum theorem to a section $abb'a'$, one would commit an error by stating an equation by not accounting for the fluid weight and the pressure on sections ab, $b'a'$, because one would omit the normal wall reactions of sections aa', bb'. Decomposed in the vertical direction, these reactions give an additional pressure from bottom to top similar to that if section $a'b'$ had the same area as ab. Poisson, who

[11] In today's terminology, the half living force is the flux of kinetic energy through a cross-sectional element.

[12] Then one of France's five top universities.

[13] In the language of modern physics, 'life power' is 'flux of kinetic energy'. Within an element of the trajectory at a certain position of a cross-section $b_0a_0(b_1a_1)$, see Fig. 2, this corresponds to

$$\frac{\rho}{2}\,\mathrm{d}t\int u_1^3\,\mathrm{d}\omega_1-\frac{\rho}{2}\,\mathrm{d}t\int u_0^3\,\mathrm{d}\omega_0$$

Denoting $\int u\,\mathrm{d}\omega$ by $V\omega$, where V is the cross-sectional average velocity, and introducing the abbreviation

$$\alpha=\int\left(\frac{u}{V}\right)^3\frac{\mathrm{d}\omega}{\omega}$$

the change of the kinetic energy, now for the element $(b'_0a'_0b'_1a'_1)\,\mathrm{d}t$, is

$$\rho\,\mathrm{d}t\omega_1u_1\alpha_1\frac{V_1^2}{2}-\rho\,\mathrm{d}t\omega_0u_0\alpha_0\frac{V_0^2}{2}$$

Note that u (u_0, u_1), v (v_0, v_1) are the horizontal and vertical Cartesian velocity components at the cross-sections $(b'_0a'_0)$ and $(b'_1a'_1)$, respectively. The work done by the hydrostatic pressure is again incorrectly written in the letter of dSV. It should read $\rho gVv(\xi_1-\xi_2)$, where v is missing if ξ_1 and ξ_2 are just the slopes of a_1 and a_2. The last sentence of the paragraph may be better written as follows. Writing the relative friction term in this form (in which Lahmeyer and other Germans adopted 3/2 for the exponent of V), the clearest way of presenting the equation of water flow [is reached] that is immediately applicable to practice for even irregular streams when dividing it into small portions of length.

[14] With H as the energy head and Ω, O as the efflux and reservoir cross-sectional areas.

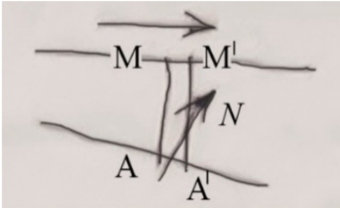

Fig. 3. Sketch explaining the differences between the application of the energy and momentum concepts.

overlooked the effect of wall reactions, had to accept equal pressures at the two walls by invoking the hydrostatic pressure assumption; these problems do not occur with the energy theorem.

Today, the energy and momentum equations are applied to gradually varied open channel flows. These principles are based on two notions of mechanics. Before the energy principle was securely established, dSV based all his hydraulic theories on the first principle, whereas JB applied the second. Yen [20] reviewed the analysis, finding no final answer. Both approaches are physically accurate if correctly applied. However, the knowledge on flow details requires different inputs, mainly on the velocity distribution and the source terms.[15]

dSV continues:

I am currently working on the theory of non-permanent fluid flow (observed during floods and their retreat, as also flows near river mouths of the sea). There, I have to include also normal bed reactions AA′ between two sections MM′, because I will else omit important information. Wouldn't you have a proposal how to include this detail into the equation of your last letter? Don't you equally have to decompose your velocities in a certain direction, given that you operate with the momentum approach? Wouldn't all this solve the differences between α' and α? I leave this for your comments. It would be difficult to accept concepts providing results different from those generally accepted, except you tell me how you established your approach. It is well possible that your way of accounting for the difference between the real velocity distribution across a section, and that of the uniform flow assumption, adds still an effect on the coefficient α. It would be good to describe this detail; I personally think that it is only a second-order effect. (Fig. 3.)

It is here that dSV announces his research on unsteady free-surface flows, to be discussed below; the topic of steady free-surface flows will be first terminated here. dSV apparently did not yet realize the difference between the energy and momentum approaches, given his critique on the two different velocity correction coefficients. This will be detailed below. He ends his letter with the statement:

Could you highlight an analogy between the superior and inferior hydraulic jumps in torrents and rivers, which I do not well understand yet, following the approach proposed by M. Boudin[16]? It appears that he had done a study on this question. You will receive, along with Bélanger's course notes, an article published in the *Comptes rendus* authored by Lt. Col. Boileau and a summary of his book *On the gauging of flowing water*, with an addition of my Note on the maximum velocity as compared with the surface velocity. As usual, I would like to express my affectionate feelings, dSV.

In Letter 10 dated 30 May 1871, the response of dSV is as follows:

Dear Sir,

The coefficient $\alpha = \int_0^{\omega} (u/V)^3 (\mathrm{d}\omega/\omega)$ by which you have to generally multiply the square of the average cross-sectional velocity, V^2, to obtain the mean of the individual squared velocities u^2 is motivated by a simpler approach than expressed in my letter dated 28 May. Indeed, by considering a surface element $\mathrm{d}\omega$, the flow passing across it during time $\mathrm{d}t$ is $u\,\mathrm{d}t$ such that the mean of the squared velocity u^2 requires the introduction of the two quantities $\int u^3\,\mathrm{d}\omega/[\int u\,\mathrm{d}\omega] = \int u^3\,\mathrm{d}\omega/[\omega V] = V^2 \int (u/V)^3 (\mathrm{d}\omega/\omega)$.

You mentioned in the first lecture notes of Bélanger his computation of the hydraulic jump in 1828, which is based on the energy theorem. He should have added the effect of unequal velocities by coefficient α. A reconsideration with the momentum theorem, as he did later, has changed this concept. When considering the square of individual velocities, don't you always have to use the same coefficient?

The Table at the end of your 'Note on hydraulic jumps' appears to be a justification of Bélanger's formula $h_1 = -(h_0/2) + [(h_0/2)^2 + (2/g)(Q^2/(b^2 h_0))]^{1/2}$.[17] Your proposal, by which the term in square brackets is increased by some 5%, generally gives a slightly better approximation (Bélanger's formula is included in my Note of 18 July 1870, 'On the propagation of the solitary wave'). This should be added to your Note. Would you, therefore, dear Sir, understand this 'Note on the hydraulic jumps' and examine once again the full background? Aren't you of the same opinion?

[15] Today's understanding is clear – and was already expressed by Euler – that the momentum and the energy balance equations express identical physical facts in 1D flow problems. Perhaps this may not have been aware by dSV and JB in their letters.

[16] Boudin [21].

[17] Bélanger [22].

The remainder of the letter refers to solitary waves of which JB sent a draft to dSV.
Letter 11, dated 3 June 1871, of dSV reads:

Dear Sir, before re-examining your improvements of the 'Notes on the permanent motion of water and the hydraulic jumps', and your letter dated 31 May, I would like to remind you that Coriolis committed an error affecting the square of the cross-sectional average velocity, V^2. His value $\alpha = \int (u/V)^3 (\mathrm{d}\omega/\omega)$ differs from yours $\alpha' = \int (u/V)^2 (\mathrm{d}\omega/\omega)$, due to the different account of interior work done by the fluid, namely the frictional work between the stream tubes. Indeed:

(1) Friction and its work are not included in this coefficient. If this would be the case in the free-surface equation by a term $\int_{s0}^{s} (\chi/\omega)(F/\rho g)\,\mathrm{d}s$ (where $F/\rho g$ refers to friction; it is given by $aV + bV^2$, or CV^m, or even $b_1 V^m R^{-0.3}$ per unit surface), it would also have to be added to an equation in which frictional effects are excluded. This was exactly made by Poncelet and Lesbros [23,24] resulting in $\int_0^{\omega} u^3\,\mathrm{d}\omega = \omega V^3 + 3V^2 \int_0^{\omega} \varepsilon^2\,\mathrm{d}\omega$ with $u = V(1+\varepsilon)$.[18]

(2) There is absolutely no omission of frictional effects when writing (Coriolis [18]) for the bottom friction $\chi(aV + bV^2)$, where V is the average velocity rather than the basal velocity V_0. In 1835, I sent to him a theorem, stating that the frictional work per unit time is equal to the product of the total friction along the bottom plus the walls of an infinitely wide rectangular channel, multiplied by the excess of the mean cross-sectional velocity V with respect to the bottom velocity V_0, a theorem contained in my Note published on 16 May 1846, later included by Bélanger in his Lecture notes.

Dear Sir, before presenting your three Notes,[19] I will study it along with your comments to see whether you consider Coriolis' formula $\Delta\alpha(V^2/2g)$ as exact, or whether you refer to yours, thereby committing an apparent omission. I intend to insert in your Note the statement that the principle of energy conservation is inexact. If you stick to it, you might be exposed to criticism because this principle currently reigns the industrial and applied mechanics. If you account for something that had been omitted by Coriolis, as the effect of convergent or divergent streamlines, all changes of velocity V from one section ω to another will result in transverse components w having their own kinetic energy contribution as the streamwise component u; wouldn't this ask for an increase of α instead of a reduction?

From 1834 onwards, I corresponded with Coriolis also on another topic by which α would be increased. Vortices, whose mean local values are of the order of u, are nothing else than velocities close to the value of v oscillating within a period of time T such that one has $v = u[1 + n\sin(2\pi t/T)]$. To retain $(1/T)\int_0^T v^3\,\mathrm{d}t$, one has to multiply u^3 by $1 + (3/2)\eta^2$ so that the unit kinetic energy would be $[1 + (3/2)\eta^2] \cdot \int_0^{\omega} (u/V)^3 (\mathrm{d}\omega/\omega) \cdot (V^2/2)$. I do not see how one would obtain from the momentum theorem, in which internal actions are overlooked, an increase, to be included in V^2 and thus in the flow equation.

I have told you in one of my letters that, if Bresse would not have talked of α as a man of precarious opinion, this is probably due to the late Dupuit who disagreed on the use of this coefficient. I reread Dupuit's *Études*.[20] I realize that he does not contest its existence, but he pretends as if a velocity increase would result in a reduction of the individual velocity differences, so that in $\alpha_1(V_1^2/2g) - \alpha_0(V_0^2/2g)$, one cannot accept both α_1 and α_0 to be equal to 1.10, because α_1 has to be significantly smaller than α_0. This is all he states including any other type of velocity distribution. I even admit when ignoring such a mismatch, you or he omit something. How do you think then, to exclude errors on the equation of energy conservation?

(3) Establishment of the gradually varied flow equation using an approach more rigorous than that of Coriolis, who did not accurately account for the external frictional effects, neglecting the effect of non-parallel streamlines on the velocity distribution. [...] You see, dear Sir, I am interested in the final solution to the problem of Coriolis' α, and the publication you intend to submit. You should closely study this, and I will engage you to do so. You could entitle this work 'On a comparison of the energy and momentum theorems in hydraulics', and 'On the Coriolis coefficient affecting the velocity square in the equation of permanent free-surface flow'.

I propose that you revise the text using clear statements as you teach your students at Gap, excluding repetitions to avoid that you have difficulties so that finally there will be no conviction. You best start with a simple case, considering a wide rectangular channel, for which $T_1 = P_2$ if frictional effects are excluded. Then you may pass to an arbitrary section along two friction forces T_3, T_2. If this will be slightly longer than a usual article for the *Comptes rendus*, I could send it to Liouville or to Belgium. You have particularly to pay attention to convincing clarity, to make it easily understood. And it would be good, by using an example of velocity distribution from the free surface to the bottom, as e.g. a cubic parabolic distribution, because the second order parabola gives no difference to the uniform distribution, thus indicating the error differences committed when the equation of Coriolis is applied. However, I repeat, take your time, to finally arrive at an improved work easy to be reviewed. Please use f for the friction between fluid layers, and F for the bottom friction. If my comments are an inspiration to you, then I will be happy to re-review your resubmission. Yours highly affected dSV.

This letter evidences the affectedness as well as the scientific quality applied by dSV to JB. dSV did not stop to ask for improvements in technical and editorial features, so that a product of outstanding quality would emerge. JB had to go through detailed advices and helps of suggested amendments, but finally was accepted by dSV both as an outstanding human as well as an excellent scholar. It was this feature that made their friendship living over such a long time.

Letter 17, written by dSV on 8 July 1871, continues with the issues on velocity correction coefficients. Given that it is similar to the previous letter, it is contained in the Supplementary Material. Note that dSV became interested in these coefficients, and the differences between the two mechanical conservation theorems.

[18] See also Poncelet [25] for a general account.
[19] To the *Académie*.
[20] Dupuit [26].

Letter 19, also dated 8 July 1871, contains an offer of JB to dSV for collaboration:

You told me in your last letter, dated 6 July, that you will have lots of things to do. I do not lament over this heavy work, because I do not feel well since years without work, and as soon as I stop to actively think or reflect to my friends' queries, this world becomes for me an illusion without consistency, a kind of abyss without base. As a curtain of clouds of all colors, mostly grey, representing what we see, and at the light surface I see myself staying for several days. However, don't we feel better when working hard? And couldn't you engage me strongly for contributions (myself, who can certainly spend more forces than you without getting tired), if you find on your way difficulties in analyses, or computations which are too hard. You would perhaps advance your beautiful edition on Navier. Without any mention of myself (given that I would only act as your handyman), couldn't you charge me with parts of your projects, passing to me some manuscripts to be improved, which you then improve as you like after having told me what you expect from me. I will well profit for the continuation of these relations of which I have already gained lots from you and your goodwill.

In the article to be published on 17 July, I denote the celerity ω as 'propagation velocity' [...]. In future, I will adopt this notation following your advice.[21]

Could you pass to me the address of Mr. Bazin as also of the members of the Section of Mechanics? I will send them a copy of my articles on hydraulics. I will also ask Mr. Bazin to pass to me information on the work of Mr. Lahmeyer, of whom he talks in his 'Study of free-surface profiles'[22] on experimental results relating to the resistance of river bends. If you can send me the three reports on the Memoirs of Mr. Joly, one relating to fluid friction, the other on earth pressure, and another of your report on my memoir concerning the internal friction of regular flows, I would appreciate. I will send you copies of my article 'On the solitary wave'.

My wife imposes on me that I demand at the Ministry for a position as Lecturer at a Faculty. The inspectors of the Academy, and without doubt the Rector of Grenoble, published in their annual report wishes in this regard. However, I know that all this has no impact in the Ministry of Public Instruction, and my candidature even less. I remain, with a deep respect, your thankful disciple, JB.

During the remainder of the year 1871, the correspondence did no longer pertain to open channel flows. Solitary waves, permanent waves, sea waves, curve flows, optics, ductile solids, or energy versus momentum principles were discussed. JB sought ways to obtain a faculty position in Paris, or even to become a Member of the Academy. It becomes evident from the letters that dSV considered time had not yet come to risk such a difficult endeavor.

In Letter 14, dated 10 March 1872, dSV returns to the SWEs:

Dear Sir, I do not want to give you new topics of study, which would increase your fatigue. Yet, perhaps I can propose a study on watercourses, to be prepared specifically within a large memoir.[23]

Vortices are, according to Poncelet [28], a means used by nature to moderate the speed of currents and, as you have noticed, a way to lower the differences in speed of adjacent fluid layers, which would result in huge differences, especially close to walls, or from the bottom of wide channels of a certain radius or depth according to the constant friction coefficient of 1/7488, or if the movements remained regular as in the small tubes of Poiseuille. Wouldn't this be a way to determine the value for each case close to walls, namely the condition that $\mathrm{d}u/\mathrm{d}r$ or $\mathrm{d}u/\mathrm{d}z$ does there not exceed a certain limit, representing the local average speed?

I feel it difficult to implement this idea. Is it indeed (by merely restricting to infinitely wide straight channels) $\mathrm{d}u/\mathrm{d}z$ to reach a maximum, or $(h/u_m)(\mathrm{d}u/\mathrm{d}z)$, with u_m as the maximum velocity and h as the maximum flow depth? In other words, isn't this a lottery of nature if large differences between the velocities of consecutive layers occur, when these velocities themselves are large, and the distances of the layers between which velocities should not exceed certain differences, can they not be larger if the total fluid mass is thicker? In a large watercourse, one has for the motion of a slice of depth z

$$\varepsilon(-\mathrm{d}u/\mathrm{d}z) = \rho g I z, \qquad u_m - u = \left(\rho g I z^2/2c\right), \qquad V = (1/h)\int_0^h u\,\mathrm{d}z = u_m - \left[\rho g I h^2/(6\varepsilon)\right].$$

If we adopt $hI = 0.0004V^2 = bV^2$ from Tadini,[24] and $(u_m/V) - 1 = 14(hI/V^2)^{1/2} = c(hI/V^2)^{1/2}$ from Bazin, i.e. $(u_m/V) - 1 = cb^{1/2}$, then $\varepsilon = [\rho g b/(\sigma c b^{1/2})]hV = (\rho g b^{1/2}/\sigma c)hV$. Thus, at the bottom $z = h$, $(-\mathrm{d}u/\mathrm{d}z)_{z=h} = \rho g I h/\varepsilon = \rho g V^2 b/[(\rho g b^{1/2}/\sigma c)hV] = (\sigma c b^{1/2})(V/h)$. According to the laws of Tadini and Bazin, one would get $(h/V)(-\mathrm{d}u/\mathrm{d}z)_{z=h}$ instead of $(-\mathrm{d}u/\mathrm{d}z)_{z=h}$, limited to a constant value of $\sigma c b^{1/2} = 6 \cdot 14 \cdot 0.02 = 1.68$. The value of $(h/u_h)(-\mathrm{d}u/\mathrm{d}z)_{z=h}$ would be 3.82 since $u_m = 0.44V$. My idea of limiting the relative speed or sliding of consecutive fluid layers is, as you see, the difficulty of proving to take $(h/u_m)(-\mathrm{d}u/\mathrm{d}z)$ instead of $(-\mathrm{d}u/\mathrm{d}z)$, given that the latter is limited. Can we admit this as a principle, or must we treat it as a computational basis, by deriving the indeterminacy in selecting the value of the internal friction coefficient ε?

[21] In contrast to dSV, denoting with ω the cross-sectional area.

[22] Lahmeyer [27].

[23] It is exactly here where dSV proposes to JB to write a memoir on open channel flows, based on his previous Notes in hydraulics, as well as his future researches directed mainly to flood flows close to river mouths into the sea. JB may have been astonished on this proposal, but set things up to finally present on outstanding work in open channel hydraulics, including the equations of steady and unsteady flows, as well as all aspects of free-surface flows such as the effects of streamline curvature, liquid friction, or flow stability. This project took JB more than five years, due to immense complications mainly from the side of the *Académie*.

[24] Tadini [29].

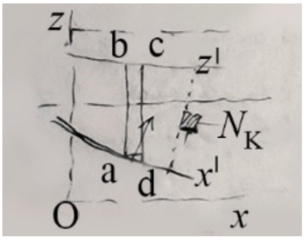

Fig. 4. Sketch illustrating the effect of sloping bottom on the governing flow equations.

dSV, who took already in 1843 a great interest in deriving the so-called Navier–Stokes equations, was continuously baffled by the enigma 'friction'. As many others before and after him, this question appeared essential to him, and by exposing his ideas to JB, he hoped that JB would be able to advance the issue.

The topic was taken up by dSV in Letter 23, dated 1 June 1872, in which he states:

> I will send you to Saint-André-de-Sangonis[25] your memoir on the permanent motion, but only after an even more detailed review in order to properly understand you. It is only today that I realize the origin of the coefficients $(1+\eta+\zeta)$ multiplied with the square of the average velocity, instead of Coriolis' α. I also inferred by your statement made in the Note for the *Comptes rendus*, (and it was impossible to give credit to you), that you consider $\zeta = 0$ for uniform flow, because streamlines then follow a different path than for the gradually varied flow, resulting in boundary friction proportional to u_0^2, a different multiple of V^2 than for uniform flow. However, I see that this is not at all the point. The term of your equation, representing the bottom friction, is $b_1 V^2 = 0.0004 V^2$ as for uniform flow, because what would be added to b_1 based on $\mathrm{d}h/\mathrm{d}x$ does not have to be accounted for; the velocity distribution is roughly that of uniform flow. It is the inertial term that is affected by $\mathrm{d}h/\mathrm{d}x = \mathrm{d}(q/V)/\mathrm{d}x = -(q\,\mathrm{d}V)/(V^2\,\mathrm{d}x)$, including $(1+\eta+\zeta)$, and I think that the difference with Coriolis imposes you to take $u' = u(\mathrm{d}u/\mathrm{d}x) + v(\mathrm{d}u/\mathrm{d}y) = -u(\mathrm{d}v/\mathrm{d}y) + v(\mathrm{d}u/\mathrm{d}y) = -u^2\,\mathrm{d}(v/u)/\mathrm{d}y$, whereas Coriolis refers to $u' = u(\mathrm{d}u/\mathrm{d}x)$. I have to understand this, and compare it with your mid-1871 correspondence.

dSV continues:

> I tell you, while waiting for your manuscript read on 15 April, that things must become clearer than in other of your texts; you need to improve the draft. It must not only be worthy for you and comparable to what you did best, but it still has to become readable, that even engineers understand you at each page which you develop. It must also include more fruitful results. So, please accept that I ask you to make me understand the setting of the centrifugal forces due to Coriolis from the bottom to the surface of the flow. This will cause a small difficulty of setting up the equation that you will easily overcome, because you can no longer take the bottom as the x-axis. You will probably take the horizontal line as x-axis. Parameters z and h include, therefore, the surface and the bottom a bit obliquely, which cannot be avoided. Yet you will have to do this like I did when deriving the equation for non-permanent flows. Recall that by taking the axis horizontally, I was obliged to balance the dynamic forces of a slice abcd, to account for the horizontal component of the normal bottom reaction ad, plus its tangential component, balancing all horizontal force contributions in the tetrahedron adcb. (Fig. 4.)
>
> You told me having taken the bottom as x-axis; there is no other possibility. Perhaps, to this end, you can always take the parallelepiped with parallel side to the bottom, of direction x', and perpendicular faces, of direction z', of slope i along x and x'. You will demonstrate that the equation is the same by placing this parallelepiped even in the direction of the streamlines, which is what you will have to present. Prepare all this and send it before your vacation to prepare your memoir. I will spend 36 hours in Paris for a discussion. Well, it was again my pleasure to entertain you. My family is sensitive to the good memory of Madame Boussinesq, pass her my regards. Your very affectionate dSV.

This letter thus includes the question of an optimum coordinate system to tackle open channel flows. And dSV was correct, because Cartesian coordinates appear better than curvilinear ones, as used until then by JB. He did not immediately take up this question, but dSV has another proposal in Letter 25, dated 14 June 1872, by stating:

> Dear Sir, here is my draft report on your last memoir. It presumes, that your memoir is clearer than other writings and announces a real improvement in the way you write and present things, but I say, it needs improved wording at various points.
>
> First, to allow engineers for the comparison between your equation (60) and that of Coriolis and Bélanger, namely $hI = b_1 V^2 + \alpha h(\mathrm{d}/\mathrm{d}x)(V^2/2g)$, such that it is the result of the theorem of the active forces, it must be said that replacing $\sin i - (\mathrm{d}h/\mathrm{d}x)\cos i$ or $i - (\mathrm{d}h/\mathrm{d}x)$ by the surface slope, and $(V^2/g)(\mathrm{d}h/\mathrm{d}x)$ by $(V^2/g)(\mathrm{d}q/V/\mathrm{d}x) = -(q/g)(\mathrm{d}V/\mathrm{d}x) = -h(\mathrm{d}/\mathrm{d}x)(V^2/2g)$; equation (60) is then written as

[25] The birthplace of JB, where he often spent his summer holidays.

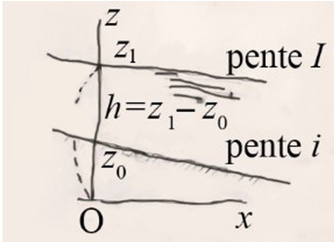

Fig. 5. Effet de la géométrie du fond courbe sur les équations d'écoulement.

$$hI = b_1 V^2 + (1+\eta+\beta)h\frac{\mathrm{d}}{\mathrm{d}x}\frac{V^2}{2g} + \left(\eta' - \beta'\right)\frac{V^2h^2}{g}\frac{\mathrm{d}^3h}{\mathrm{d}x^3}.$$

This is different from the Coriolis equation as follows:

(1) $h(\mathrm{d}/\mathrm{d}x)(V^2/2g)$ originates only from inertia multiplied by $(1+\eta)$ instead of $\alpha \cong (1+3\eta)$,

(2) the other term $h(\mathrm{d}/\mathrm{d}x)(V^2/2g)$ multiplied by β is much larger than η, resulting from the bottom friction, Bu_0^2, agreeing with its value b_1V^2 for the case of uniform motion; it is an important term that has in common only the shape of that and includes the inertia effect by putting $\alpha = \int_0^h (u/V)^3(\mathrm{d}z/h)$ instead of $(1+\eta) = \int_0^h (u/V)^2(\mathrm{d}z/h)$,

(3) there is still the term involving $(\mathrm{d}^3h/\mathrm{d}x^3)$ due to the centrifugal forces.

The nature of the coefficient B was in my contribution, I must tell you, the subject of extensive research with despair to form my ideas. I was unable to understand the details and only finally noticed that $Bu_0^2 - b_1v^2$ had a similar form as the term due to inertia, i.e. $h(\mathrm{d}/\mathrm{d}x)(V^2/2g)$ multiplied by a quantity which is β.

dSV continues:

It is important that your memoir[26] is not only a report, but equally addresses readers. [...] Engineers have kept few theoretical habits that you should attract to make you understood, and you must attract teachers as Collignon, Bresse and Phillips (of the Central School) who might include results in their lectures, regardless of Boudin, Rankine, Clebsch, who lecture outside France.

After a statement pertaining to JB's Note published in 1871, dSV continues:

I see no reason to say that, while we neglect streamline curvature, your and Coriolis' equations achieve similar statements indicating that $[(1+\eta)+\sigma]$ does not differ much from α. This depreciates without need your result to the point that I thought for a moment, in 1871, that the two items could be identical at the end. Yet this is not the case. Even Coriolis [18] provides an example resulting in $\alpha = 1.47$, that he considered a bit high, but then he finds on p. 330 $\alpha = 1.16$ considering this a bit low, thus concluding that $\alpha = 1.40$. It was Mr. Vauthier,[27] stating values between 1.03 and 1.10, yet without comment. According to your calculation, $(\alpha = 1+3\eta)$ equals only 1.054. Thus, there is no place to say that α gives about the same as $(1+\eta+\beta)$, consisting of two distinct parts with which this has absolutely nothing in common.

And, based on the periodicity of local motions, he tends to accept the Coriolis coefficient on the basis of the momentum equation. However, as this fortuitous quasi-coincidence between your result and that somewhat insecure approach of many engineers, there results a downgrading in the less conspicuous part of your work from practical relevance (even more because you find the passage from the uniform to the gradually varied states curious, and only of interest in theory). I consider it necessary, in your future memoir, to include the pressures due to the centrifugal force of the bottom curvature. Then, you will have to give up the bottom as x-axis as it is more natural to take the horizontal and vertical sections.[28] We have $\mathrm{d}h/\mathrm{d}x = \mathrm{d}z_0/\mathrm{d}x - \mathrm{d}z_1/\mathrm{d}x = i - I$, $\mathrm{d}^3z/\mathrm{d}x^3 = \mathrm{d}^3z_0/\mathrm{d}x^3 - \mathrm{d}^3z_1/\mathrm{d}x^3$. There will be nothing more complicated than to establish the limit velocity at the bottom, because the friction of the side 'ab' of 'abcd', equals the friction $F = Bu_0^2$; one will add the component on 'ab' of the section perpendicular to the bottom p_0, a reaction nearly equal to ρgh, plus the portion of the centrifugal force, if the bottom surface is curved. (Fig. 5.)

Make the exposure clear by providing a comprehensive analysis in chapter 1, assuming that there is neither curvature along the bottom nor at the surface, and taking, as you do, the horizontal bottom profile as reference for the x-axis. In this first chapter, the second of the conservation equations (16) yields $(1/\rho g)(\mathrm{d}p/\mathrm{d}x) = -\cos i$, or $\mathrm{d}p/\mathrm{d}z = \rho g$, and there is no comment needed except for the hydrostatic pressure but absolutely nothing on P, $\mathrm{d}^3z/\mathrm{d}x^3$ or even μ. It is only in a following chapter that you would enter the surface curvature, along with that of the bottom, and it would matter for μ and M; you would take your x-axis horizontal, or, better perhaps (to reduce the quantities of second order) use an intermediate fixed slope j, demonstrating that the results are independent of the somewhat arbitrary selection of this slope j along the x-axis. Nothing would inhibit that the coefficient B not only varies with the nature of the bottom roughness, but also with its curvature, concave or convex, and even perhaps with $\mathrm{d}h/\mathrm{d}x$, without yet

26 JB's *Essai* [31].
27 Vauthier [30], on which Coriolis [18] made a discussion thereby introducing the coefficient α.
28 '*Pente*' in the sketches means slope.

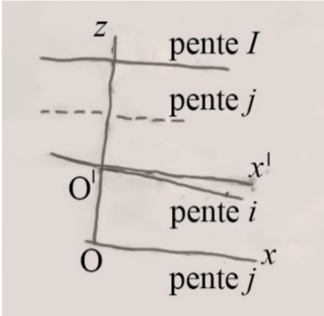

Fig. 6. Sketch highlighting the selection of the streamwise coordinates x and x'.

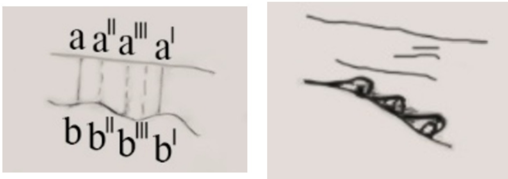

Fig. 7. Effect of bottom undulations on the free-surface profile.

engaging in assumptions their effects on vortices and on the numerical value of the coefficient B, given by the average value 0.00081 to determine η and β, which are only small numbers. (Fig. 6.)

Once η and β thus are set, as you did by supposing the average $b_1 = 0.0004$ of Tadini, it is good to recall that by applying the formula $hI = b_1 V^2 + (1 + \eta + \beta)h(\mathrm{d}/\mathrm{d}x)(V^2/2g) + \cdots$, one should replace $b_1 V^2$ by $0.0004V^2$, expressed as $0.00028[1 + (1.2b/h)]$ or $0.00035V^{1.9}h^{-0.3}$. This would improve the representation of experiments made in a canal of depth and section similar to that considered under these conditions.

And even though engineers often have to deal with artificial currents such as in ditches, or in wide rivers, I think it would be good to give them in this very memoir also hints on the second extreme case, the semi-circular canal. It could be put after chapter 1, excluding the effect of the centrifugal force in this chapter except for wide channels, and after having said that you can determine A and B with sufficient approximation to obtain η and β, starting from Tadini's $hI = 0.0004V^2$ and $u = u_0 - 24(hi)^{1/2}(z/h)^2$ from Darcy and Bazin [10], resulting in $V\beta/(2A) = 24$, $VB/(3A) = 16$, so that $A = 0.000643$, $B = 0.000869$. You would add: However, based on another formula of Mr. Bazin on semi-circular canals, as will be seen in chapter 2 relative to these types of channels, we take $B^{1/2}/(3A) = 21$ instead of 16, which gives for A and B the slightly different values $A = 0.00064$, $B = 0.00081$. However, in taking B identical for rectangular and semi-circular channels, don't you modify A, since it may affect something else?

The remainder of this letter deals with streamline curvature effects, not discussed here.

In Letter 26, dated 17 June 1872, dSV details his comments on JB's book project:

Dear Sir, here is my draft report on your manuscript. Do you want to annotate my statements in a margin? No doubt, you will have to change especially the end because I still do not understand the expressions 'fast' and 'moderate' torrents, nor even your 'between rivers and torrents'. From the bottom of p. 22, it seems that $i < i_0$ or $i > i_0$ distinguishes rivers from torrents, but there is more to say.

When you have completed your memoir accounting for the streamline curvature and bottom undulations, there is still to make a distinction, not because of more or less bottom slope, but by $V^2/(gh_1)<$, or >1, with h_1 as a local average flow depth resulting from what you refer to as a non-wavy line including bottom undulations, if present. If the regime is torrential, will small ripples on the bottom bumps not be felt on the surface once the uniform regime is established, or are they dampened under the river regime for which $[V^2/(gh)]$ or $[1 + \eta + \beta)V^2/(gh)] < 1$? However, is this not what you say, on pp. 25–26, the double effect of a simple pebble? As long as the bottom is undulated, even when not overly affected by short wave lengths, the vortex intensity will not strongly be increased in the torrential regime, so that A and B will have to be increased?

Taking into account the magnitude of the dynamic pressure terms due to the bottom undulations, does this not influence the extreme pressures on the two sections ab, $a'b'$ bounded by a finite portion $abb'a'$ of the current, so that the centrifugal forces generated along the intermediate sections $a''b''$, $a'''b'''$, etc. would have no influence on the movement of mass $abb'a'$? If it were so, then you should determine the motion in natural streams involving typical bottom undulations to model observations; you need to increase the turbulence intensity due to their presence. (Fig. 7.)

Isn't the condition $i < i_0$ or $i > i_0$, respectively, characterizing rivers and torrents, with $i_0 = (gb_1\varphi(h)/\alpha')[1 + (3/4)(g/\alpha')(12fb_1\varphi H)^{2/3}]$, nearly equal to $V^2/(gH) < 1$, or > 1? Why not quote Bélanger? To determine the height of the hydraulic jump, he explains it (as you say) by the loss of kinetic energy accompanying a sudden flow deceleration, or by accompanying turbulences; why the momentum theorem according to him is used? And couldn't you add something of your 1871 Note 'On the hydraulic jumps', to which I added a small table comparing calculation and experiments, that I judged at the time important enough to be printed in the *Comptes rendus*, since the non-modified formula of Bélanger seemed to give the same to ±1 or 2 cm. Yours truly dSV.

The above two letters again demonstrate the vivid interest of dSV in both the fundamental equations of fluid flow, and JB's exceptional researches. However, dSV realized that things had to be improved to become acceptable by the scientific community, proposing JB to reconsider his comments. JB explains in Letter 28, dated 10 August 1872, written at Saint-André-de-Sangonis:

Sir and dear Master, I plan for the end of August my main tour to my parents, able to spend eight days on the questions that you wish to see me study, i.e. those concerning the canal of curved bottom, and unsteady flows. I hope that the results achieved will enable you to solve the issue, despite being only approximate, but I did perform integrations, as for a plane bottom, by assuming a slightly different flow depth as that fitting the uniform regime.

Two differential equations for these problems are obtained by following the approach that led me to the steady motion in an entirely plane channel. Here is the equation of steady motion in a curved bottom channel, s denoting the coordinate taken along the longitudinal bottom profile, i the bottom slope, h the variable flow depth, V the average cross-sectional velocity, $q = Vh$ the unit discharge; then

$$q^2\left[\frac{\eta'-\beta'}{g}\frac{\partial^3 h}{\partial s^3}-\frac{\eta''-\beta''}{g}\frac{\partial^2 i}{\partial s^2}\right]+\left[h\cos i-\frac{\alpha' V^2}{g}\right]\frac{\partial h}{\partial s}=h\sin i-b_1\varphi(h)V^2+\frac{1}{2}h^2\sin i\left[1-\frac{B}{6A+2B}\right]\frac{\partial i}{\partial s}.$$

In this equation, the two new coefficients η'' and β'' are given by the expressions

$$\eta''=\left[(1/2)+(5/12)(B/A)+(11/120)(B/A)^2\right]/\left[1+(B/3A)\right]^2,$$
$$\beta''=(1/12)(B/A)\left[1+(2/3)(B/A)+(1/8)(B/A)^2\right]/\left[1+(B/3A)\right]^2,$$

with $A = 0.00064$ and $B = 0.00081$, $\eta'' = 0.58074$, $\beta'' = 0.07506$, from where $\eta' - \beta' = 0.3691$ instead of 1/3 for $B/A = 0$, $\eta'' - \beta'' = 0.4747$, differing hardly from 0.5 for $B/A = 0$.

At the points where the bottom has undulations of small amplitude or the regime differs slightly from uniform flow, the surface profile is the same as for a flat bottom, and this is close to a periodic term of the bottom undulation with identical period. This term communicates to the surface the same many undulations as has the bottom. If K divides the half-amplitude of the bottom undulations, S is the length of one of those undulations, K_1 is its half-amplitude of the surface, i_m is the average bottom slope, and $k^2 = (\eta' - \beta')/g$, $k_1^2 = [(\eta'' - \beta'') - (\eta' - \beta')]/g$, one has

$$\frac{K_1}{K}=\sqrt{\frac{1+[\frac{H^3}{3fb_1\varphi(H)}]^2[\frac{\alpha'}{gH^2}-\frac{4\pi^2k_1^2}{S^2}]^2\frac{4\pi^2}{S^2}}{1+[\frac{H^3}{3fb_1\varphi(H)}]^2[\frac{\alpha' i_m-b_1g\varphi(H)}{gi_mH^2}+\frac{4\pi^2K^2}{S^2}]^2\frac{4\pi^2}{S^2}}}.$$

For $i_m = 0$, $K_1 = 0$, in a channel of wavy bottom but small slope, bottom undulations are not felt on the surface. As i_m increases, the ratio K_1/K also does so. It is equal to 1 for

$$i_m=\frac{gb_1\varphi(H)}{2\alpha'[1+\frac{(K^2-K_1^2)g}{2\alpha'}\frac{4\pi^2H^2}{S^2}]}\cong\frac{0.001808H^{-0.3}}{1+4.233\frac{H^2}{S^2}}.$$

It is about half of the slope separating rivers from torrents. The ratio K_1/K increases until

$$i_m=\frac{\frac{gb_1\varphi(H)}{\alpha'}}{[1+\frac{gK^2}{\alpha'}\frac{4\pi^2H^2}{S^2}]}\cong\frac{0.003616H^{-0.3}}{1+13.43\frac{H^2}{S^2}}.$$

This is about the slope separating rivers from torrents. The maximum of K_1/K is

$$\frac{K_1}{K}=\sqrt{1+\left[\frac{H^3}{3fb_1\varphi(H)}\right]^2\left[\frac{\alpha'}{gH^2}-\frac{4\pi^2k_1^2}{S^2}\right]^2\frac{4\pi^2}{S^2}}\cong\sqrt{1+\left[52.65H^{0.3}\left(1-4.969\frac{H^2}{S^2}\right)\frac{H}{S}\right]^2}.$$

For, e.g., $H/S = 0.1$ and $H = 1$, this maximum value is about $K_1/K = 5$, and the resulting surface undulations caused by those at the bottom are much deeper. The K_1/K ratio decreases then without reaching 0 as i_m continues to grow. It is equal to 1 for

$$i_m=\frac{b_1\varphi(H)}{(K^2+K_1^2)\frac{4\pi^2H^2}{S^2}}\cong 0.0001965H^{-0.3}\frac{S^2}{H^2}.$$

However, I think that, if the ratio H/S is sufficiently small, the expression then becomes $[\alpha'/(gH^2)] - (4\pi^2k_1^2/S^2) > 0$; $S/H > 2$ corresponds to the ordinary case.

JB stepped with this outstanding theory into a completely new era of the hydraulic description of open channel flows. Even today, few have come to this point in understanding his eminent advances based on a thorough mathematical process description, the prominent application of the mathematical tools, and then the physical description of the governing phenomena associated with open channel flows in a complex environment. The ensuing considerations expressed in this Letter then pass to unsteady open channel flows, not further discussed here.

Letter 29, dated 10 September 1872; dSV first congratulates JB for having obtained the *Prix Poncelet* from the *Académie* for his outstanding papers, and then comes back to the steady flow equations as follows:

I hope to send you my new draft relating to a channel whose bottom has a slightly curved profile so that $\mathrm{d}^3h/\mathrm{d}x^3$ is not fully negligible, but where the surface curvature is assumed negligible; this occurs in all rivers, even in those whose bottom is extremely rough. I refer the surface profile to the x-axis, and the calculation is simple. I find [for slope I of the free surface]

$$I = (F_0/h) + (1+\eta+\sigma)(\mathrm{d}/\mathrm{d}x)\left(V^2/2g\right) + \left(\eta''' - \sigma'''\right)h^2\left(\mathrm{d}^3/\mathrm{d}x^3\right)\left(V^2/2g\right)$$

with

$$\eta''' = (1/6)\left[1 + (7/10)(B/A) + (19/142)(B/A)^2\right]/\left[1 + B/(3A)\right]^2,$$

$$\sigma''' = (2/45)\left[1 + (3/14)(B/A) + (2/121)(B/A)^2\right]/\left[1 + B/(3A)\right]^2.$$

Take your $\eta'' - \eta'$, $\sigma'' - \sigma'$. Perhaps I'll obtain the final solution, since the bottom curvature can cause similar undulations on the surface, to give the complete equation

$$I = \frac{F_0}{h} + (1+\eta+\sigma)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{V^2}{2g}\right) - \left(\eta' - \sigma'\right)h^2\frac{\mathrm{d}^3}{\mathrm{d}x^3}\left(\frac{V^2}{2g}\right) - \frac{hV^2}{g}\left(\eta'' - \sigma''\right)\frac{\mathrm{d}^2 i}{\mathrm{d}s^2},$$

transforming into

$$I + \left(\eta'' - \sigma''\right)\frac{hV^2}{g}\frac{\mathrm{d}^2 i}{\mathrm{d}x^2} = \frac{F_0}{h} + (1+\eta+\sigma)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{V^2}{2g}\right) - \left(\eta''' - \sigma'''\right)h^2\frac{\mathrm{d}^3}{\mathrm{d}x^3}\left(\frac{V^2}{2g}\right).$$

I would take the x-axis at the free surface instead of at the bottom, given that the surface is normally less curved, and that computations are then simpler.

I recommend clarity in your writing, which I do not recognize from the precision of the sentences. I wish to understand the basis of each item, each transformation, which you plan to do, and how you arrive there. And, please, give detailed evidence for each statement, without giving the reader the ultimate penalty to guess. Please reread it, put yourself in the same position as the poor attentive reader and start over without leaning back. I am absolutely unwilling to indulge on your new, so long and painful texts, and prospective thoughts, which you delivered to me; to decipher your first drafts, to understand what you want to say. I would also refer you to your manuscripts to redo the draft if not everything is clearly developed and lucid. This is the attitude that I take for all manuscripts, without restarting again to avoid too much trouble. I hope you find me clear! Well, take me as a guide if it is so; moreover, what is even better, to take many other guiding persons, especially little old persons. My respectful compliments and those of my family to Madam Boussinesq. Your very affectionate dSV.

P.S. Why not leave, as I do, very wide margins to your manuscripts, to allow for draft changes? This takes just a few cents more for postage.

At this stage, dSV even had his own proposal for the equations of steady water flow. He proposed a number of practical issues to JB, from the improved paper editing to the point of writing not so densely in his letters. Let's see how the entire correspondence continues. Letter 30 of dSV, dated 17 September 1872, mainly deals with streamline curvature effects, and thus is available in the Supplementary Material.

Addition on 20 September:

I started reading your Memoir. It is very well written, clear, rigorous and deep. As to your annotations in the margins for printing, I would use Italic instead of English, so that the composers understand you better. You know that an underline is enough: $\underline{\mathrm{l}}$ $\underline{\mathrm{m}}$ $\underline{\mathrm{n}}$ $\underline{\tau}$ means l, m, n, τ in italic. Composers apply italic if you are not asking them for something else. In the final draft, add a notation, including:

- I as angle of the line taken by the x-axis with the horizontal axis,
- p_0 as pressure at a point of this axis indicating the section and repeated in several places so that I and p_0 are defined.

However, I will correct this when I re-review your memoir, unless something else stops me for a few days. Please accept my friendly compliments, dSV.

As previously, dSV not only proposes technical perfections to JB's manuscript, but also gives hints how to present the final work. JB, then only 30 years old, was certainly happy to receive this support from his experienced Master.

In Letter 31, dated 27 September 1872, JB deals with the open channel equations. He writes:

I already told you that I consider your third approximations useful, but it seems to me that what you propose for the expression $\partial/\partial x[\int_0^z (u^2/V^2)(z/h)(\mathrm{d}z/h)]$ is perhaps not sufficiently accurate; this derivative seems to me in fact to have to be expressed as $\partial h/\partial x$, because z is constant, while h is variable. It would be different if both z and h were variables, from one point to another along the x-axis, or even if you had to take the derivative in x of $\int_0^h (u^2/V^2)(z/h)(\mathrm{d}z/h)$.

I have almost finished things concerning curved bottoms, but I was unable to account for centrifugal forces, or rather the non-hydrostatic pressure distribution, by replacing terms depending only on u by V, which must be sufficiently approximate, simply rendering factors 1/3 and 1/2 instead of $\eta' - \beta'$, and $\eta'' - \beta''$.

I had already noticed that A, B, b have the dimension $(\text{length})^{-1}$ and $(\text{time})^2$; to put the equation of uniform flow $RI = bV^2$ is so evident that I thought to replace them with $1/A$, $1/B$, $1/b$, thereby introducing lots of fractions in the text. I indicated to set English lettering, so that I did not set A and B in italics, in order not to confuse this with other italics, having another meaning. So, please return to my original notation. I am with a profound respect and the most affectionate recognition, your devoted disciple, JB.

JB thus tries to follow the editorial advices of dSV, but still has clear ideas why notation should be selected as originally proposed. In Letter 32, dated 28 September 1872, dSV resumes his previous findings, namely:

Dear Sir, what would you say if, for a streamline element whose streamwise momentum equation is $\mathrm{d}h/\mathrm{d}z + \mathrm{d}p/\mathrm{d}x = \rho g I + (V^2/gh)(\mathrm{d}h/\mathrm{d}x)$, I would simply propose for $\mathrm{d}p/\mathrm{d}z = \rho g - \rho(z/h)(\mathrm{d}^2h/\mathrm{d}x^2)V^2$ as deduced from the radius of curvature r at depth z? With r_0 as the radius of the bottom curvature, and r_{00} the radius of the surface curvature, one can approximately assume (I thought of you already for a long time when I was asking for the equation by driving forces)

$$1/r = 1/r_{00} + (z/h)(1/r_0 - 1/r_{00}),$$

which reduces for a plane bottom, where $1/r_{00} = 0$, to

$$1/r = -(z/h)\left(\mathrm{d}^2h/\mathrm{d}x^2\right).$$

Adding to the elementary weight $\rho g\,\mathrm{d}x\mathrm{d}z$ its centrifugal force $\rho\,\mathrm{d}x\,\mathrm{d}z(V^2/r)$ gives

$$(\mathrm{d}p/\mathrm{d}z)\,\mathrm{d}x\,\mathrm{d}z = \rho g\,\mathrm{d}x\,\mathrm{d}z + \rho\,\mathrm{d}x\,\mathrm{d}z\left(V^2/r\right).$$

This is identical to what is stated in the equation

$$\mathrm{d}p/\mathrm{d}x = \rho g - \rho(z/h)\left(\mathrm{d}^2h/\mathrm{d}x^2\right)V^2.$$

This establishes your fundamental equation, including the third derivative of h as

$$\frac{u}{u_o} = 1 + \frac{B}{2A}\left(1 - \frac{z^2}{h^2}\right) + \frac{1}{gA}\frac{\mathrm{d}h}{\mathrm{d}x}\left[\int_z^h \frac{\mathrm{d}z}{h}\int_0^z \frac{\mathrm{d}z}{h}\left(\frac{u^2}{u_o^2} - \int_0^z \frac{u^2}{u_o^2}\frac{\mathrm{d}z}{h}\right)\right]$$
$$-\frac{h^2}{gA}\frac{\mathrm{d}^3h}{\mathrm{d}x^3}\left[\int_h^z \frac{\mathrm{d}z}{h}\int_0^z \frac{\mathrm{d}z}{h}\left(\int_0^z \frac{u^2}{u_o^2}\frac{z}{h}\frac{\mathrm{d}z}{h} - \int_0^h \frac{\mathrm{d}z}{h}\int_0^z \frac{u^2}{u_o^2}\frac{z}{h}\frac{\mathrm{d}z}{h}\right)\right]$$

Including all concerns relative to the determination of $1+\eta$, η', σ, σ', and neglecting all quadratic terms of the products of $\mathrm{d}h/\mathrm{d}x \times \mathrm{d}^2h/\mathrm{d}x^2$, can you really neglect the second derivative of h? Is there any principle to ensure that assuming $\mathrm{d}^3h/\mathrm{d}x^3$ is negligible and therefore the effect of the centrifugal force? Would the formula also apply for a slightly curved water surface? Because $(1/r_0 - 1/r_{00}) = -\mathrm{d}^2h/\mathrm{d}x^2$, or $1/r = -\mathrm{d}I/\mathrm{d}x - (z/h)\mathrm{d}^2h/\mathrm{d}x^2$, then $\mathrm{d}p/\mathrm{d}z = \rho g - \rho u^2(\mathrm{d}I/\mathrm{d}x) - \rho u^2(z/h)(\mathrm{d}^2h/\mathrm{d}x^2)$, and therefore

$$p = p_{oo} + \rho q^2 - \rho\frac{\mathrm{d}I}{\mathrm{d}x}\int_0^z u^2\,\mathrm{d}z - \rho\frac{\mathrm{d}^2h}{\mathrm{d}x^2}\int_0^z u^2\frac{z\,\mathrm{d}z}{h}.$$

Moreover, one must add to u/u_0 only a single term of the form $\mathrm{d}^2I/\mathrm{d}x^2$, having in its final equation both β' and $\mathrm{d}^3h/\mathrm{d}x^3$. Yet, the part of your memoir that remains for me to review, will give me the answer. Well, save your health, Your very affectionate dSV.

P.S. You could solve the problem of a poor manuscript by writing less tightly. With a post stamp of 15 cents more, you give yourself the necessary wideness. I say this especially with regard to the compositions for the printers.

In Letter 33, dated 29 September 1872, dSV continues to comment on JB's draft. He writes:

Dear Sir, at the end of your memoir there are excellent things yet difficult to understand, and I do not support them with full conviction. I agree with you perfectly on equation (108)[29] $\sin I = (Q^2/\sigma^3)[b'\chi - (\alpha'/g)(\mathrm{d}\sigma/\mathrm{d}s)]$, which is equal to $\sin I = F_0/(\sigma/\chi) +$

[29] Equation numbers agree in most cases with those stated by JB [31].

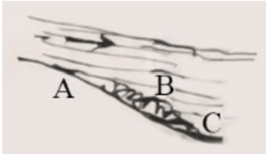

Fig. 8. Effect of the bottom roughness on the flow pattern.

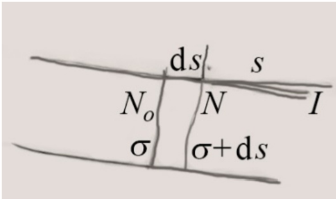

Fig. 9. Effect of bottom slope on the flow equations.

$(1+\eta+\delta)\,\mathrm{d}/\mathrm{d}s(V^2/2g)$ if $F_0 = b'V^2$, $V = Q/\sigma$ and $1+\eta+\delta = \alpha'$. However, I cannot follow what you deduce from (109), $\mathrm{d}\sigma/\mathrm{d}s = l(I_0 - I)$, etc. I_0 corresponds to the slope under which $\mathrm{d}\sigma/\mathrm{d}x = 0$, so that from eq. (108) one would have $\sin I_0 = (Q^2 b'\chi/\sigma^3)$, by replacing the respective terms, namely $(I_0 - I) = (\alpha' Q^2/g\sigma^3)(\mathrm{d}\sigma/\mathrm{d}s)$. I do not see that this is identical to (109).

You obtain from (110^{bis}) $[1-(\alpha' Q^2 l/g\sigma^3)](1/l)(\mathrm{d}\sigma/\mathrm{d}s) = I_0 - (b'Q^2\chi/\sigma^3)$, a deduction which I do not understand. As previously demonstrated, the method of approximation which you employ to establish the equation of gradually varied flows does not provide any trustworthiness with respect to the degree of approximation, because $(1/l)(\mathrm{d}\sigma/\mathrm{d}s)$ (equal to $\mathrm{d}h/\mathrm{d}x$ for a wide rectangular section) exceeds 0.01, and because the second member $I_0 - (b'Q^2/\sigma^2)(\chi/\sigma)$ is of the order 1/1000. You conclude that if $1-(\alpha'/g)(Q^2 l/\sigma^3) = [I_0 - (b'Q^2/\sigma^2)(\chi/\sigma)]/[(1/l)(\mathrm{d}\sigma/\mathrm{d}s)] > 0.001/0.01 = 0.1$, the gradually varied flow regime is necessarily destroyed (to allow without doubt place for the regime of hydraulic jumps, thus for momentum conservation). I do not at all see this consequence.

If a computational method is unable to provide with certainty the approximation which one is looking for, I see no reason that we wish to hold back owing to lack of a gradual variation of parameters. Your inference would be important, before concluding anything on the establishment of hydraulic jumps, below eqs. (124^{bis}).

As you do not leave anything at random, I claim that your conclusions are correct. Perhaps, as soon as your analysis of slowly gradually varied flows becomes applicable, for example if the bottom slope is larger than 0.01, with sensitive curves, then perhaps, I said, that the formation of these highly turbulent fluid portions ABC will start. In such a case, the momentum equation will be used such that it leads to the theorem of Borda as in abruptly diverging pipes. If you can demonstrate this, and as a result present equations for water flow with this abruptly varied bottom geometry, you will render a great service; but it must be well checked in all small speculations by experiments that you propose to be done. Yet, your present analysis does not yet establish things sufficiently clear, and there may be place to redesign few parts of the draft at the end of your memoir, which has already required so much work from you. (Fig. 8.)

In view of these comments, dear Sir, be not discouraged. What I find not sufficiently clear will not stop me to present a favorable report.[30] What I am telling you should rather give you courage and confidence. What you did to advance the theory of water flows is already a lot, and you find the time to complete your manuscript. Your very affectionate dSV.

In Letter 34, dated 1 October 1872, dSV asks for additional items to be included in the Memoir. He writes:

Dear Sir, I finally understand $\mathrm{d}\sigma/\mathrm{d}s = l(I_0 - I)$. It would have been easier to understand it with a sketch showing that $\mathrm{NN}_0 = \mathrm{d}s(I_0 - I) = \mathrm{d}\sigma/l$. I understand that I_0 is the bottom slope at N_0. When Laplace said: It is clear, then there is always something confusing to realize. (Fig. 9.)

Yet, I do not understand eq. (127). The cause is the long sentence that you have even lengthened by an addition on the margins, and that you, no doubt, intend to cut it into several portions in the final draft. I neither understand the sentence that follows. And I confess to understand hardly anything of this §XIV. I do not doubt its correctness, even though you are talking there several times about the limit above which gradually varied flows are not established; this limit, seems to me again, is not yet clearly established. Yet I hope, dear Sir, that once you have completed the remainder of your memoir that I will receive and consider it with pleasure; you will certainly once more improve the draft of this §XIV. Reproduce in the course of writing once again Equation (110a), saying from where it originates, what it represents, what is increasing, and repeat what σ', σ'', I'_0, etc. stand for. Is σ' the section of uniform flow?

I see that σ'' is eliminated by $1-(\alpha' V^2/gh'') = 0$, with h'' as average flow depth σ''/l''. This is the distinction between rivers and torrents, and the statements (130) indicate that $(\sigma''/\chi'')J''$ is smaller or larger than $b'V''^2$, i.e. the hydraulic radius multiplied by the bottom slope is smaller or larger than its value at uniform flow. Would this not become clearer if being expressed in simpler

[30] dSV, president of the Section of Mechanics, had to furnish a report in which he and his commission agree on its acceptance or rejection.

language? I would like that you will have readers, successors or perhaps disciples, that your memoir may become the subject of a book instead of staying buried in the *Savants étrangers*, not being consulted by anyone.

§XV is much clearer.

When you return to my draft report, do neither overlook the comments in the margins, nor remove them by cancelling because I know that this would have to be completely redone. Moreover, you do not have to explain the assumptions, because your proposals along with your improved writings give adequate reasons. There will be at most parts of the first pages that can currently be used.

I assumed originally to take into account the centrifugal forces, as you do now, based on equal average velocity V, as that seems to be sufficient. Yet, couldn't you say that instead of adopting the hypothesis $u = V$ to approximate these forces, one could adopt any other assumption, for example that u varies from the surface to the bottom as if there would be no curvature, or even that u may vary as in uniform flow, but then we would find $\eta' - \sigma' = 0.3691$, and $\eta'' - \sigma'' = 0.50568$ which would not greatly alter, given that 0.333 and 0.500 applies for $u = V$; moreover one would not be sure to have gained anything by this procedure for flows with a sensitive streamline curvature, which form the basics of analysis leading to the final equation. You could put this into the note.

My 3rd approximation, that I studied painfully and the evaluation of terms involving $(\mathrm{d}h/\mathrm{d}x)(\mathrm{d}^3h/\mathrm{d}x^3)$ and $(\mathrm{d}^3h/\mathrm{d}x^3)^2$ would have been reduced if I would have given up, to what was on the note I sent you. Your very affectionate dSV.

As in many other instances, dSV asks JB to improve the technical writings by using shorter and clearer sentences. However, sentences of dSV are also extremely long, so he also could have accepted critiques how to edit technical texts. dSV further states that he does not intend to improve the technical contents of JB's research, only the writing style, so that the final document is easier to access. When looking at the final text published in 1877, this has only partly been done. JB's writing style was always complicated, and the excessively long sentences do not detract from this comment. This may be a reason why JB never became a popular author in the engineering community.

Note that the idea of writing a 'book' came in the year 1872. It took JB five years to have his work published. The final result was based on a report of dSV written in 1872, concluding that JB's work was worthy to be included in the *Memoirs* of the *Académie* (Boussinesq [31]). The 'book' is currently considered as the basis of the fundamental ideas of 1D and 2D depth-averaged channel flow models.

Letter 35, dated 2 October 1872, was written by JB from his hometown, Saint-André-de-Sangonis, during his summer holidays. It reads:

I finished the two parts of my memoir concerning steady water flows, thus accounting for the centrifugal forces (circumstances for the establishment and destruction of the uniform regime, torrents of moderate slope, direct and undular hydraulic jumps, effects of bottom undulations on the surface profile, curved bottom geometry rendering the free surface of the same forms as a flat bottom), and also the unsteady gradually varied flows, thereby excluding streamline curvature effects. Propagation of waves and surges in a channel with uniform flow regimes were also accounted for, with consequences that are confirmed by experiments of Bazin (Darcy and Bazin [10]), and all questions I have dealt with in the memoir on the theory of waves and surges for the case of water originally at rest.

I was unable to account for terms stemming from streamline curvature by replacing in these terms u by its average value V, which is acceptable for gradually varied flows (since this is only to replace α' by 1) and what it really is *a fortiori*, if this assumption concerns only the generally small terms due to streamline curvature. This approach relates only to terms for which $u/V = 1$ is assumed, applying also without the expression of w/u, whose first term is substantially $(z/h)(\mathrm{d}h/\mathrm{d}s)$ (but having other terms assigned to z^2, z^3, etc., and higher derivatives $\partial^2 h/\partial s^2$, $\partial^3 h/\partial s^3$, etc., i.e., higher powers of $\partial h/\partial s$, as also products of all these quantities), neglecting those more complicated terms in comparison to the significant first terms, as long as the square of $\partial h/\partial s$ and the higher derivatives of h remain negligible, and if the ratio u/V would strongly differ from unity.

For the same reason, if either the ratio u/V hardly differs from 1, or if streamline curvature effects are small, both are linear functions of z, as you assumed in your former work on the implementation of the centrifugal forces; otherwise, neither the streamline slopes w/u do vary linearly with z, nor, *a fortiori*, their derivatives in s or x, as soon as the streamline curvature becomes strong. The demonstration is in the long Note, which I ask to implement in the Memoir. Upon assuming $u/V = 1$ for the terms depending on the streamline curvature, I found for steady water flow

$$I = \frac{bV^2}{h} + \alpha' \frac{\partial}{\partial s}\frac{V^2}{2g} - h^2\left[\frac{1}{3}\frac{\partial^3}{\partial s^3}\frac{V^2}{2g} + \frac{1}{2}\frac{V^2}{gh}\frac{\partial^2 i}{\partial s^2}\right],$$

or

$$I + \frac{h^2}{2}\frac{V^2}{gh}\frac{\partial^2 I}{\partial s^2} = \frac{bV^2}{h} + \alpha' \frac{\partial}{\partial s}\frac{V^2}{2g} + \frac{h^2}{6}\frac{\partial^3}{\partial s^3}\left(\frac{V^2}{2g}\right).$$

Integrating the effect of unsteady gradually varied flow leads to

$$I = \frac{bV^2}{h} + \alpha' \frac{\partial}{\partial s}\frac{V^2}{2g} + \left(\frac{1+2\eta}{g}\right)\frac{\partial V}{\partial t} + \left(\frac{\eta' + \beta''}{g}\frac{V}{h}\right)\frac{\partial h}{\partial t}.$$

Here, $\eta' = 0.02253$, $\beta'' = 0.003979$, and η is the number $\int_0^h (u^2/h^2)\partial z/h \cong 0.01761$. An expression easier to integrate would be, with $\alpha'' = (\alpha' - \eta' - \beta'')/(1 + 2\eta)$,

$$h\frac{\partial V}{\partial t} - \alpha'' V\frac{\partial h}{\partial t} + \frac{gh - \alpha' V^2}{1+2\eta}\frac{\partial h}{\partial s} = \frac{g(hi - bV^2)}{1+2\eta}.$$

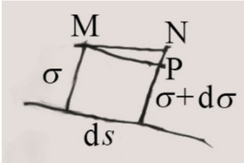

Fig. 10. Effect of free-surface slope on the flow equations.

When accounting for streamline curvature, it is replaced by

$$h\frac{\partial V}{\partial t}-\alpha''V\frac{\partial h}{\partial t}+\left[\frac{gh-\alpha'V^2}{1+2\eta}\right]\frac{\partial h}{\partial s}+\frac{V^2h^2}{3}\left[\frac{\partial^3 h}{\partial s^3}+\frac{2}{V}\frac{\partial^3 h}{\partial s^2\partial t}+\frac{1}{V^2}\frac{\partial^3 h}{\partial s\partial t^2}\right]-\frac{1}{2}\frac{\partial^2 i}{\partial s^2}=g\left[\frac{hi-bV^2}{1+2\eta}\right].$$

My formula (109), $\partial\sigma/\partial s = l(I_0 - I)$, is not a consequence of the former; it results from the definitions of I, I_0, and the assumed smallness of the streamline curvature. I call indeed l the cross-sectional width at the surface, I_0 the free-surface slope so that section σ conserves the same value by passing from s to $s+\partial s$, and by this definition of I_0, one had $\partial\sigma/\partial s = 0$ for $I = I_0$. Given the small difference between I and I_0, the angle NMP of the free-surface profile MN relative to MP, if the surface slope would be I_0, is below MP, $I_0 - I$, and the section $\sigma + \partial\sigma$ is above section σ by a small amount of width l and height NP $= (I_0 - I)\partial s$. Thus, one has $\partial\sigma = l(I_0 - I)\partial s$ or $\partial\sigma/\partial s = l(I_0 - I)$. In eq. (108), you can neither simply set $I_0 = I$, nor $\partial\sigma/\partial s = 0$, because these equations apply to evaluate the values of I_0 and $\partial\sigma/\partial s$, satisfying also $\partial\sigma/\partial s = l(I_0 - I)$. Putting $I_0 = I$ and $\partial\sigma/\partial s = 0$ makes it in general incompatible. (Fig. 10.)

I now turn to the difficulty I have with my eq. (111) as a necessary condition to make gradually varied flows possible. Please make sure from the beginning of §VI that the x-derivatives of v/u, w/u are small compared with their last values at x as functions of y and z. This allows for the definition of this motion that I start to study, and I think that everyone would have given it after having considered this question.

I call a steady flow *gradually* varied if its streamline curvature and the square of its streamline inclination is negligible. However, the equation of steady gradually varied flows that I present is not only approximate (at least in two cases of wide rectangular and circular sections), but still considerably accurate (whenever the square of $\partial h/\partial s$ and streamline curvature effects are negligible). When my equations cease to apply with sufficient accuracy, I am correct to state that the scheme ceases to be gradually varied. You define it as gradually varied by the possibility to neglect $(\mathrm{d}h/\mathrm{d}x)^2$, $\mathrm{d}^2h/\mathrm{d}x^2$.

The remainder of this letter refers to the energy equation, not dealt with here.

In summary, the year 1872 was a highly fruitful period for both dSV and JB, because the gradually varied flow theory was established based on the momentum equation. Its physical background was clear at that stage, yet JB had to improve his formulation so that it would be evident to the hydraulic community.

In Letter 17, dated 26 May 1880, JB writes:

Since you would like that I publish in the *Annales des ponts et chaussées* a paper on permanent flows, would it not be better to take yours, whose manuscript I have here? It would be sufficient, unless you were to take your time again, to shorten what would take too long and then, to correct the proof that you only see, because your name would naturally be the only on the title. As to myself, my head and stomach need two years of rest.[31] Agree, Sir and dear Master, in the new expression of my unalterable devotion and gratitude. JB.

It is also worthwhile to state that the approach of dSV as opposed to that of JB has survived until today. The main difference between the two is the large number of coefficients related mainly to the velocity distribution when expressed by the 1D approximation. Consider for instance the dynamic term $(1/g)\,\mathrm{d}[\beta VQ]/\mathrm{d}x$ as expressed by JB. Except for uniform flow, β remains unknown, as does the derivative $(\mathrm{d}\beta/\mathrm{d}x)[VQ]$, so that this correction cannot be retained in the governing equations when applied in hydraulic practice. Even though not proved, the effects of α and β are small and currently not generally included in hydraulics, because their impact is considered small when compared to those of the friction coefficient (difficult to assess mainly for rivers and tunnels), and to the bottom slope, particularly at small values in lowland rivers. The approach of JB is physically correct, but it does not provide explicit solutions for practice, so that the fundamental free-surface flow equation of dSV is preferred over JB's complicated approach. Note that if $\alpha \equiv \beta = 1$, the energy and momentum equations provide the identical statement for the free-surface profile. The merit of JB is in his formulation of the equation of the free-surface profile under non-hydrostatic pressure conditions.

3. Theory of unsteady flows

The mathematical formulation of unsteady water flows was achieved in the year 1871. Despite the works of dSV over decades on the topic, he had so far never reached this point, and most probably was happy to have in JB an adequate

[31] JB often complained on these health problems in his letters to dSV.

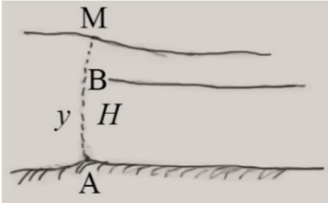

Fig. 11. Sketch explaining unsteady flow generation in a channel.

colleague who would comment on his approach. However, in a way strangely, this approach preceded that of the steady water-flow theory, although the basic governing equations were available to Coriolis [18], and that Bresse [19] had given a full review for hydrostatic pressure and uniform velocity distributions.

At the end of Letter 10, dated 30 May 1871, dSV writes to JB:

As to your remarkable Note entitled 'Theory of the liquid intumescence referred to as solitary wave or translation wave', I intend to send the manuscript to the *Comptes rendus*, once the communication with Paris will be re-established.[32] I have the following comments:

(1) I do see neither the sense of the sub-title 'Computation of the speed of a counter-current', nor the meaning of this counter-current.

(2) At the end of your Note, you state that lines 5 and 6 could be dropped if space is not available, starting with: If the still water depth H is larger than $h_0 \ldots$. I would rather propose to drop all because these are merely citations of Bazin, because our theories support essentially that H is much larger than h_0.

In a Report entitled 'On fluvial tides and floods', I explain that Bazin's results relating to a large h_0 [still water depth] would affect the theoretical formula $[g(H+h_0)]^{1/2}$ differently because of its limitation to small h_0.[33] I attach this effect to 'accumulated intumescences'. This is the core of my theory 'On the unsteady motion of water'. Do you think that this is correct?

Consider a channel whose still water depth is H; the water depth increases gradually at section AB, e.g. because of a supply from the sea during tidal action or by an increase of discharge. A first perturbation of extremely small height h or $\mathrm{d}H$ is propagated with celerity $(gH)^{1/2}$. The water added from the stagnant portion gives rise to a small velocity $\mathrm{d}V$. The sea water would increase at point A, generating a second extremely small perturbation, whose celerity would be $k=[g(H+h_0)]^{1/2}+V$. This water then mixes with that of velocity V, increasing its velocity. The question is, during water rise to the height y at the river mouth, what are: (1) The channel water velocity V, and (2) the celerity k of a novel small perturbation? One would have for the latter $k=(gy)^{1/2}+V$. As to the condition of mass conservation after a novel discharge increase and a new depth increase of $\mathrm{d}y$, we have $\mathrm{d}(Vy)=k\,\mathrm{d}y$, so that

$$\mathrm{d}V=\left[(k-V)/y\right]\mathrm{d}y=(g/y)^{1/2}\,\mathrm{d}y.$$

Integrating with $y=H$ as the initial condition yields

$$V=2(gy)^{1/2}-2(gH)^{1/2},\quad \text{from which } k=3(gy)^{1/2}-2(gH)^{1/2}.$$

This corresponds to the general equations of unsteady flows. I found for these in a water flow a layer ABba of extent $\mathrm{d}s$ becoming A′B′b′a′ in a short duration θ, because of surface height reduction, the two equations

$$\frac{\mathrm{d}\xi}{\mathrm{d}s}=\frac{1}{g}\frac{\mathrm{d}V}{\mathrm{d}t}+\frac{V}{g}\frac{\mathrm{d}V}{\mathrm{d}s}+\frac{\chi}{\omega}\frac{p}{\rho g},$$
$$\frac{\mathrm{d}\omega}{\mathrm{d}t}=\frac{\mathrm{d}(\omega V)}{\mathrm{d}s}.$$

Here, ξ is the ordinate of the free water surface below the horizontal, and $(\mathrm{d}V/\mathrm{d}t)$, $(\mathrm{d}\omega/\mathrm{d}t)$ are the total partial derivatives of the average velocity V, and the cross-sectional area ω with respect to time t. This is to say that the two above variables, ξ and ω, change at a certain location relative to their changes of location.[34] The second equation expresses conservation of mass. Dupuit [26] presented it, yet instead of the first equation, his alternative was $\mathrm{d}\xi=(\mathrm{d}H/\mathrm{d}s)\,\mathrm{d}t+V(\mathrm{d}V/g)+(\chi/\omega)(F/\rho g)\,\mathrm{d}t$. The term F originates from the frictional force as in the non-uniform but steady flow equation. I obtained this by accounting for the normal bottom force reaction Bb, decomposed into the direction of the line GG′ along the gravity center, so that no other terms have to be considered. I assume for these computations a rectangular[35] channel. (Fig. 11.)

The system of the above two equations is known as the Saint-Venant equations. It relates the two unknowns, the cross-sectional average velocity V and the flow depth h (along with the cross-sectional area ω and the frictional effect χ) to the

[32] Due to the Franco-Prussian War 1870–1871, the French State was in disorder, and only reached normality after the end of the 'Paris Commune.'

[33] This formula relates to the celerity of a surge.

[34] This statement indicates again that dSV considered mathematically partial derivatives expressed as 'd', in contrast to the current notation ∂.

[35] And prismatic.

variables location s, and time t. Currently, the term $S_f = (\chi/\omega)(p/\rho g)$ is referred to as the friction slope. Note the non-linearity of this system, mainly entering the velocity head and the friction terms of the first equation, and in the discharge term (ωV) of the second equation. These mathematical complications do not allow for an analytic integration of the system, except for one case, as presented below.

These two equations simplify by excluding the frictional effect, and for a horizontal rectangular channel, so that $-(\mathrm{d}\chi/\mathrm{d}s)$ replaces $(\mathrm{d}\xi/\mathrm{d}s)$, and y stands for ω, with y as the flow depth. Integrating by assuming that the velocity V depends only on y, gives

$$(\mathrm{d}y/\mathrm{d}x)\big[g + (V\,\mathrm{d}V/\mathrm{d}y)\big] + (\mathrm{d}V/\mathrm{d}y)(\mathrm{d}y/\mathrm{d}t) = 0,$$
$$(\mathrm{d}y/\mathrm{d}x)\big[V + y(\mathrm{d}V/\mathrm{d}y)\big] + (\mathrm{d}y/\mathrm{d}t) = 0.$$

The term $\mathrm{d}y/\mathrm{d}t$ is eliminated by multiplying the second equation by $\mathrm{d}V/\mathrm{d}y$,

$$(\mathrm{d}y/\mathrm{d}x)\big[g - y(\mathrm{d}V/\mathrm{d}y)^2\big] = 0, \quad \text{so that } (\mathrm{d}V/\mathrm{d}y) = (g/y)^{1/2}, \text{ from which } V = 2(gy)^{1/2} + \text{constant}.$$

As above, one has $k = 3(gy)^{1/2} - 2(gH)^{1/2}$. The celerity measured during an experiment, to propagate an extremely small perturbation of height $h = y - H$ comparable to depth H, is a mean value including all internal layers which have contributed to its formation. Upon letting k_m be this average celerity, one has

$$k_\mathrm{m} = \frac{1}{y-H}\int_H^y (3\sqrt{gy} - 2\sqrt{gH})\,\mathrm{d}y = \frac{2gy}{\sqrt{gy}+\sqrt{gH}} = \sqrt{gH}\frac{2(1+\frac{h}{H})}{1+\sqrt{1+\frac{h}{H}}}.$$

With $k' = [gH(1 + (3/2)(h/H))]^{1/2}$ as celerity observed by Bazin, the ratio of the two is

$$\frac{k'}{k_\mathrm{m}} = \frac{\sqrt{1+\frac{3}{2}\frac{h}{H}}\cdot(1+\sqrt{1+\frac{h}{H}})}{2(1+\frac{h}{H})}.$$

Note that $k'/k_\mathrm{m} = 1$ if $h/H \to 0$, $k'/k_\mathrm{m} = 0.989$ if $h/H = 1/3$, and 0.970 if $h/H = 2/3$. It is obvious why Mr. Bazin believed in his expression $k' = [g(H + h')]^{1/2}$, with $h' = (3/2)h$ as the front wave height; it is the height of the wave body following the head, and applies if h is no longer small as compared to H. This formula should include k_m. If you support this anticipation, then our two researches will excellently combine.

When your baggage is small for the next presentation, why not taking a rest, having certainly enough work to prepare your students for their exams. Does Mrs. Boussinesq not support my proposal? I am always at your disposal. Sincerely yours, dSV.

Letter 20, dated 18 July 1871, was written by dSV at a railway station in Paris:

Dear Sir, I have corrected the last proof of your modified Note, as also your last additions of your work 'On permanent movement and on the intumescence.' I resubmitted my memoir 'On the unsteady flow' to the *Comptes rendus* of 17 July. I will send you the proof, as also the Memoir 'On river tides' of M. Partiot.

As to my memoir, I reread your letter dated 4 June, in which you find things simpler than I did in two different ways. I have not completely followed your proposal. I think it is excellent to start by presenting the two governing differential equations, then to assume the average velocity V exclusively being a function of the flow depth y at the section considered. I first had to account for the results of Mr. Partiot, who determined the longitudinal free-surface profile as the geometric location where, at a certain instance, the extremes of the perturbations are located, all starting at the same location at different instants and with different velocities given by $(gy)^{1/2} + V = 3(gy)^{1/2} - 2(gh)^{1/2}$.

You did well in proposing to exclude the front wave observed by Bazin, given that the equation $S = \varphi(y) + [3(gy)^{1/2} - 2(gh)^{1/2}]t$ does not apply once it reaches a position resulting in two flow depths y for each value of the horizontal abscissa s. Your explanation in your last article 'On the wave or propagated intumescence' is better suited to this topic. The case of double flow depths y is nothing else than a bore.[36] A breaking wave ABC as that sent by the tide into a canal MN develops into A′B′C′ because higher portions take up due to higher speed and pass lower portions. There is no theoretical mistake in this formulation, but as the wave becomes multi-valued at B′C′, there occurs wave breaking.[37] (Fig. 12.)

Would you think it possible to extract from my equations of unsteady free-surface flows in a rectangular and horizontal, frictionless channel, viz.,

$$(\mathrm{d}y/\mathrm{d}t) + \mathrm{d}(Vy)/\mathrm{d}s = 0, \qquad g(\mathrm{d}y/\mathrm{d}s) + (\mathrm{d}V/\mathrm{d}t) + V(\mathrm{d}V/\mathrm{d}s) = 0, \tag{a}$$

other features, if the average velocity V would not directly depend on the flow depth y? More important, what can be done, if the second equation in (a) reads

$$-g(\mathrm{d}\xi/\mathrm{d}s) + (\mathrm{d}V/\mathrm{d}t) + V(\mathrm{d}V/\mathrm{d}s) + g(\chi F/\omega) = 0, \quad \text{or} \tag{b}$$

[36] In French, *mascaret*.
[37] In French, *déferlement*.

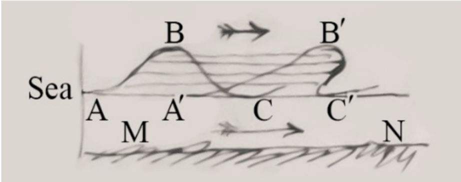

Fig. 12. Wave breaking associated with multi-valued flow depths.

$$g(\mathrm{d}y/\mathrm{d}x) + (\mathrm{d}V/\mathrm{d}t) + V(\mathrm{d}V/\mathrm{d}x) - g(i - \chi F/\omega) = 0, \tag{a}$$

if the bottom slope $i \neq 0$, and if you include friction effects ρgF, assumed to depend on y and V such as $\rho gbV^2 y^{-0.3}$? Could one successively design the advances of a water flow if the initial condition would be specified, and if a lateral discharge addition would also be known, for a certain sea water level as a function of time? I am embarrassed to neglect friction if V depends only on y, resulting after the elimination of $(\mathrm{d}y/\mathrm{d}t)$ in

$$(\mathrm{d}y/\mathrm{d}x)\left[g - y(\mathrm{d}V/\mathrm{d}y)^2\right] - gi = 0. \tag{c}$$

Set $y = y' + ix$ to drop the second term, yet I do not see what can be done then.

Equations (a) have a solution if V is only a function of y, namely $x = t \cdot \mathrm{d}(Vy)/\mathrm{d}y + \mathrm{f}(y)$. However, this does not furnish a value for $y(x)$ inserted in eq. (b), not even for term F, if $(\mathrm{d}\xi/\mathrm{d}x) = -(\mathrm{d}y/\mathrm{d}x) + i$ instead of only $-(\mathrm{d}y/\mathrm{d}x)$. My elementary demonstration relating to the formula of Lagrange $k = (gy)^{1/2} + V$ as celerity of an infinitesimally small perturbation, appears to apply also to water flows over small bottom slopes. The experiments of Bazin confirm this finding (if the friction slope is about half of the bottom slope). I concluded from the above differential equations, that the processes in tidal rivers or floods in a sloping channel are similar. Then, $x = [3(gy)^{1/2} - 2(gh)^{1/2}]t + \varphi(y)$.[38]

I think that until I find a better result, I will contend myself with the equation resulting by the elimination of $(\mathrm{d}y/\mathrm{d}t)$, completed from (c) to

$$(\mathrm{d}y/\mathrm{d}x)\left[g - y(\mathrm{d}V/\mathrm{d}y)^2\right] - g(i - \chi F/\omega) = 0.$$

I will drop the last term, which becomes zero for uniform flow, given that it is defined by $\omega/\chi = F/i$, a state that can be regarded as oscillating around uniform flow at least for floods. Once the tidal river and the flood problems are dealt with

$$\left[g - y(\mathrm{d}V/\mathrm{d}y)^2\right] = 0, \quad \text{or } V = 2(gy)^{1/2} - 2(gh)^{1/2},$$

one has[39]

$$x = \left[3(gy)^{1/2} - 2(gh)^{1/2}\right]t + \varphi(y).$$

I have corrected today the proof of my article to be published in the *Comptes rendus* which I will send you. Your affectionate dSV.

In this letter, dSV includes the effects of bottom and friction slopes for the first time. He realized that his solution was related exclusively to a particular case, i.e. provided that the two slopes are identical. In practice, however, mainly over large reaches, their individual effects need to be considered, an issue attacked over the next decade, yet this problem was not solved, neither by dSV nor JB.

Letter 23 of JB to dSV, dated 6 August 1871, contains additions to the SWEs:

Sir and good Master, I would not have waited until today to thank you for your gracious invitation,[40] if I would not have been absorbed by corrections of exams for the admission to the *École des arts et métiers* at Aix. Finally, I am here again, more free, given that I have no longer lectures until the return of classes. Vacations start after tomorrow until 9 October, and I would be happy to meet you at Vendôme, to finally see you and hear you. I also do not refuse your offer to use your room in Paris. I would only first like to go during the second half of August to Saint-André de Sangonis, to take baths in the clear waters of River Hérault which always rendered great relaxation, and which would be this year important due to nose bleedings. These are the cause why Madame Boussinesq does not want to leave me alone, particularly during the long voyage to Vendôme. I would have her with me, but I hope that this will not cause too much work for you. She will be so happy to also meet you and Madame de Saint-Venant, and your daughters! We will depart to Saint-André on 9 August, arriving there on Saturday because we will spend two days in Montpellier with my uncle (Mr. Cavalier), priest of the Sainte-Eulalie Church. We wish to go in September to Lozère, meeting there the parents of my wife, whom I do not yet know.

Thank you for sending the Abstract of the memoir 'On river tides' by M. Partiot. He did not sufficiently state that his formula, excluding surface curvature, does not apply to regions of large curvature, as, e.g., at the wave front. He also finds that wave breaking

38 With φ as arbitrary function defined by the initial and boundary conditions.
39 From the expression above.
40 To dSV's castle at Vendôme.

occurs only in sufficiently long canals. However, the equations of Mr. Bazin and my article published on 24 July prove that this phenomenon depends on the perturbation height relative to the still water depth, thus they are only trustworthy in shallow water.

I received what you call the bundle Lahmeyer.[41] I think that his formula

$$RI = \left[b_1 + K(l/r)^{1/2}\right]V^{3/2}$$

is doubtful in terms of the effect of bottom curvature l/r, given that he finds for the average coefficient $K = 0.0002881$, a value differing almost by a factor of two from the Weser and Saale Rivers to the Elbe and Mulde Rivers, from 0.00023 to 0.00041. However, what is of more interest to me would be your results relating to the effect of the centrifugal force on the pressures. By applying specifically the method published in my memoir dated 3 and 10 July, yet without considering now $h\partial^2 h/\partial x^2$ to be small as compared to $\partial h/\partial x$, yields both terms rather small along the curve so that the effects of vorticity causing rapid streamline divergence remain also small. Squared terms and products of these are dropped, resulting in an expression involving h, $\partial h/\partial x$, $\partial^2 h/\partial x^2$.

I hope to be able to study the undular hydraulic jump [French: *ressaut allongé*] produced in rectangular channels whose bottom slope is slightly larger than the maximum slope of a river, or around the minimum slope of torrents. I remain, with my deepest respect, Mister Count and dear Master, Yours all devoted and grateful JB.

I forgot to tell you that I have been appointed Corresponding Member of *Faculté des sciences naturelles*, Cherbourg. I have sent to the president, along with my letter of thanks, a memoir published in the *Journal de mathématiques*.

The next letters (a total of 14) written between 13 August and 13 November 1871, deal with sea waves, optics and elasticity. Letter 38, dated 4 December 1871, is reproduced in the Supplementary Material. The next letter concludes with best wishes for the New Year 1872:

Hello dear Sir, happy new year, patience and courage. I wish you success, because I think that you are not one of those whose successes paralyze. I hope that you appreciate these successes, and you have not turned to arrogance. As to myself, I do not know what I would have become if I had had no success, perhaps I would have remained there, while constant failures, setbacks, disappointments, the humiliations even, gave me the tone, and the courage. Whom would I have become, above all, so early, as was my wish, after having been scientifically asked, I had managed to be one of the defenders of religion with science. Maybe my pride would have been at the peak, I wouldn't be as more infallible than the Pope. Oh that God was good to preserve me. Respectful congratulations and my best wishes for a happy New Year to Mrs. Boussinesq. Your affectionate dSV.

The year 1871 brought the appearance of the SWEs almost entirely based on the research of dSV. A strong limitation next to the assumption of hydrostatic pressure and uniform velocity distributions was the neglect of the bottom and friction slopes. dSV thus pursued this task, involving JB and his mathematical talents. The final success was only partial, however, as described below.

Letter 44, dated 20 November 1872, states:

I send you, dear Sir, new attempts to account for friction and the velocity distribution in an unsteady motion. My question is to ask you whether you will in your final document reach the goal that I have proposed. This is, believe me, (1) the establishment of the equations of unsteady flows, because your eqs. (258) seem to fully account for friction and the deviation of the velocities under gradually varied motion; (2) the calculation of the propagation speed of small waves, which are given by your eqs. (265), and of better approximations by (289), (295), (296), which also account for friction and the cross-sectional variation of the velocity. These formulae for the celerity lead to $\omega = V + [(gH)/(1+2\eta)]^{1/2}$, from which follows $V = 2[(gH)/(1+2\eta)]^{1/2} + \text{const}$. This is what I found in my unsteady flow equation,

$$g(\mathrm{d}h/\mathrm{d}s) + (\mathrm{d}V/\mathrm{d}t) - V(\mathrm{d}V/\mathrm{d}s) = g\left(i - bV^2/h\right),$$

when assuming that V is only a function of h, and by dropping the last term, if the second and the third were altered by $(1+2\eta)$. Then, the square root of $(1+2\eta)$ enters as denominator the integrand of $\int_0^{\omega}(u/V)(\mathrm{d}u/\mathrm{d}V)(\mathrm{d}\omega/\omega)$.[42]

Expanding the formulas of the propagation speed and flow velocity V to cases of large flow depths, where bottom layers would only slowly adapt to increased velocities due to the successive addition of waves, so that there are large velocity differences between these at the surface and the bottom, I have resorted to an empirical formula. Your exact analysis cannot give these formulas. However, will your analysis give results on the upper curved wave surfaces above their concavity and have as asymptote the non-disturbed fluid surface, whose height decreases due to friction when moving away from the point of generation. (You have written to me that you would look for this issue). Isn't this about what you find at the end of your memoir when introducing, below formula (328), the gravity centers of waves approaching persistently from the current surface during propagation? It is true that you speak here only of isolated fronts and rear waves. Yet, doesn't this equally apply to continuous waves or surges, which are relevant in the theory of unsteady water movement?

This is what I wish to elaborate on the dual problem of river tides and floods. What you say in §XXVII, below eqs. (299) and (300) and already at the end of your Note printed in Liouville, would be the problem of a Mediterranean tide, that is of extremely small height compared to the canal depth entering the sea, or also the problem of a very small flood. Yet, it is for rivers running into the ocean, or for strong floods, natural or artificial; the problem is of vital interest for navigation. How, therefore, can we solve it? If this

[41] Lahmeyer [32].

[42] What is likely meant here is that ω in the integrand is replaced by $\omega = 3\{gH/(1+2\eta)\}^{1/2}$.

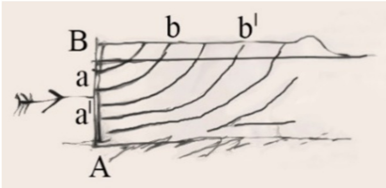

Fig. 13. Sketch describing the generation of a small water wave.

is not achieved by successive approximation to integrate the two equations in h, V for non-permanent flows, wouldn't the approach suggested to me by Mr. Partiot be a good method? We know the law of the variation of the flow depth at a given place, by assuming that each infinitely small depth increase produces a continuous and infinitely small wave; these waves overlap while spreading first on the stillwater elevation and then on a series of surfaces turning above their more pronounced concavity. Can't we find these surfaces by small secants to integrate the equation?

Doesn't this amount to a graphical integration of two simultaneous equations for h, V, as functions of s and t? These surfaces of elementary wave propagation, don't they play a similar role in the phenomena of unsteady flows. Don't they have a special relation with undular trajectories? And, as trajectories are concerned, couldn't you limit yourself even to confine to waves or surges of extremely small height, determine the trajectories of the fluid particles during the generation of a solitary wave or a continuous wave on a fluid surface assumed initially as stagnant?

Which streamline ab, or a′b′, produces the same geometrical shape if a piston acts on section AB, lifting the fluid involved, or does the tide at this location simply lift fluid particles? You excellently and completely determined the shape of the solitary wave; thus, both straight and elliptic trajectories of various inclinations result at various depths of the wave; moreover, your equations will probably also provide trajectories in the wave or the small wave train, thereby elucidating the theory of the unsteady wave movement. (Fig. 13.)

Am I correct in judging from your memoir that the centrifugal force has hardly an effect on the non-permanent motion for the flood or tidal bore? It is not this, seems to me, but rather the search for the answer to the objection. Partiot observed that it is at full sea where the water entrance velocity in the river is smallest, while according to $V = 2(gh)^{1/2} - 2(gh_0)^{1/2} + V_0$, this velocity should be the largest possible. Do not leave to our nephews the merit to solve these issues. What is the future of science after a time of back steps? Europe will not be in full decadence? Your very affectionate dSV.

This letter 44 contains numerous questions addressed to JB. Even though the SWEs do not look extremely difficult, dSV was unable to solve problems related to them. In Letter 46, dated 30 November 1872, dSV returns to this issue, stating:

Dear Sir, I am writing again to Saint-André risking that my letter will be returned to me if you are already on the way back to Lille. If you wish to have copies of my report on the Poncelet Award (I have not read the Report, which will perhaps be abbreviated), the easiest way will be to ask for prints published by Gauthier-Villars. I do not know whether it is sold apart from the number of the minutes containing the session awards.

I wish [in my report] that you:

(1) discuss in §XXIX the effect of the term $g(hi - bV^2)/(1 + 2\eta)$ instead of simply neglecting its second part,

(2) determine the upward concavity of continuous waves, which I attempted to prove,

(3) try to solve by approximate integration the two simultaneous differential equations for h, V, as functions of s, t for non-permanent flows or indicate the way to do the integration numerically if data are available. Show the way to solve the dual problem of river tides and floods that I addressed in my 1870 paper, following Mr. Partiot.

I do not think that this requires to consider a discontinuous motion, or the passage from one steady state condition to another through a mobile projection as you say in your letter dated 26 November, because we can track the spread of the tide in the river up to when it has degenerated into a tidal wave. Its rise takes 12 hours. For floods, we only consider those that arrive slowly enough to not break, as do all, as I observed.

Any equation(s) of a single (and several) unknown(s) is (are) solved by reasoned groping, whatever the degree, even if it is transcendent; similarly, aren't all differential equations integrable? Doesn't calculus propose procedures, such as the use of Taylor series? Can't we give a first approximation by dropping some secondary terms, and then substitute in this restored term the results from this temporary solution? Can't we also ask the two unknowns h and V to equal entire series in s and t with undetermined coefficients, and can't we establish from these equations or these coefficients, by restricting these terms to a small number of the series? Isn't your process to obtain u/u_0 for permanent flow, in the form of an entire expression in z/h, based on this method of undetermined coefficients? You would make a great service in solving this problem, as you did for the permanent motion. This will take time, because it will certainly require several opinions before printing your large memoir plus time to make and read the Report. Forward to hearing from you. Your very affectionate dSV.

It is here where dSV 'invites' JB to think of any feasible way to find solutions to the complete SWEs. So far, JB never addressed this issue because he too saw no way of solution. dSV, in turn, was so much involved in this very question that he took the last possibility to find it, namely by involving JB and his outstanding mathematical genius. Let us therefore see how the story continues.

In Letter 47, dated 1 December 1872, dSV states:

I gave you yesterday a somewhat absurd question, which I could answer myself, as I shall do to avoid you the penalty. Unable to do it analytically, it is imperative to numerically integrate the two nonlinear first order differential equations in s (longitudinal axis), t (time), for h (flow depth), V (average velocity) in an open channel whose flow is not permanent, because at a given point, for example at $s = 0$, h, or where the unit discharge hV varies with time as a function of t. So, I was asking if there was a way to integrate numerically these two equations by a step-by-step method by putting Δ instead of 'd'? The answer is easy, as it is easy to integrate the equation of Vauthier or Coriolis and yourself,

$$\mathrm{d}\zeta = I\,\mathrm{d}s = \left(bV^2/h\right)\mathrm{d}s + \alpha'\,\mathrm{d}\left(V^2/2g\right)$$

to obtain the ordinates ζ of the water surface below a horizontal reference elevation for permanent flow. Take here for simplicity the mean cross-sectional velocity V as

$$g(\mathrm{d}h/\mathrm{d}s) + (\mathrm{d}V/\mathrm{d}t) + V(\mathrm{d}V/\mathrm{d}s) = g\left[I - \left(bV^2/h\right)\right],$$

$$(\mathrm{d}h/\mathrm{d}t) + V(\mathrm{d}h/\mathrm{d}s) + h(\mathrm{d}V/\mathrm{d}s) = 0.$$

Take successively

$$t = 0, 60, 120, 180, \ldots \ (\mathrm{s}), \quad \text{and} \quad s = 0, 10, 20, 30, \ldots \ (\mathrm{m}).$$

Let $\mathrm{d}t = 1$ minute, and $\mathrm{d}s = 10$ m. For $t = 0$, $s = 0$ one knows h, V, $\mathrm{d}h/\mathrm{d}t$ (if $h = \mathrm{f}(t)$ for $s = 0$), I and V, and also $\mathrm{d}h/\mathrm{d}s$ for a given bottom slope and the initial free-surface slope at these locations. The second equation provides $\mathrm{d}V/\mathrm{d}s$. Substituting this into the first then gives $\mathrm{d}V/\mathrm{d}t$. Because the two derivatives have nearly equal values apart from second order quantities, for $t = 60''$, $s = 0$, as for $t = 0$, $s = 10$ m, one deduces the values of V for these new values of t and s. One has already h for these equal systems. The two equations give you, still for these systems ($t = 60''$, $s = 0$), ($t = 0$, $s = 10$ m) the values of $\mathrm{d}V/\mathrm{d}t$, $\mathrm{d}V/\mathrm{d}s$, then $\mathrm{d}h/\mathrm{d}t$, $\mathrm{d}h/\mathrm{d}s$, thereby assuming that they remain nearly equal when further augmenting t and s from $t = 60''$ to $s = 10$ m, furnishing V and h for this new system values of t and s. By continuing this procedure, you will succeed in assets, for each value of s, the rest of the water surface, providing the surface of Mr. Partiot.

It is understood that we have to try whether substantially the same results follow from 30 s by 30 s to 60 s, and from 5 m by 5 m instead of 10 m, to shorten the computations; we will also try if the results remain similar by increasing the time to half hour by half, and distances of 100 m by 100 m. It is an operation that can be made by the employees of an engineering office, just as the Bélanger–Vauthier computations. This applies to an arbitrary bottom shape. It meets the quadratures of curved areas by rectilinear trapezoids. It might be possible to proceed similarly with Simpson quadratures by parabolic trapezoids, which would get the same approximation by taking many more large intervals of time and space. However, it would be complicated, and it is good only if you had to do several sets of similar calculations so that you would have the advantage to develop a method demanding for continuous learning.

Make computations only on the two differential equations with the indirect method of Mr. Partiot, amended by a first trial, a network of curves, intended to generate continuous, infinitely small disturbances to be superimposed. Yours sincerely, dSV.

Even though dSV mentions the letters received by JB, these are not contained in the *Correspondence*. Once dSV noted the reservation of JB against his proposal, dSV put forward ideas how the SWEs could be integrated using approximate methods. He noted that the value of his 1871 Paper could be greatly increased if he were able to deal with more general, and significantly more realistic problems concerning unsteady free-surface flows. Let us have a look how this continued.

Letter 2, dated 30 January 1873, and Letter 3, dated 2 February 1873, contain further ideas of dSV, detailed in the Supplementary Material.

In Letter 9, dated 1 May 1873, the topic of numerical integration of partial differential equations is reconsidered. JB states:

Sir and dear Master, Mr. Puiseux has answered your question of numerical integration from his first impression, the most natural, but that would change the reflection. I indeed cannot find anything more difficult than to solve a partial differential equation in finite differences, by reducing differences of the partial derivatives to allow for convergence to the limit, because a large number of equations with partial finite differences, whose integrals seem to converge in the limit, to similar equations of partial derivatives, of which some are likely to converge, whereas the integrals of others tend to nothing determined. I do not claim that it is impossible to numerically integrate equations involving partial derivatives by transformation to equations in partial finite differences, but I think we have to find an equation of finite differences whose integral tends toward that which one wants to have, as the differences tend to zero. I am with deep respect, your devoted and grateful JB.

All through 1874, no problems in free-surface hydraulics were discussed. Questions on simplified flood hydraulics based on the approach of Graeff [33], earth pressure, the definition of mechanical energy, the effect of surface tension, molecular mechanics, elasticity, thermodynamics, Liouville's integration method, problems with the publication of JB's work in the *Savants étrangers*, lecturing at Lille University, and ethics were considered; this is not further discussed here.

The correspondence of 1875 starts with comments on the works of Tait, Thomson (Lord Kelvin) and Airy on composed movements, followed by a discussion on the theory of Stokes related to the Navier–Stokes equations, and a consideration

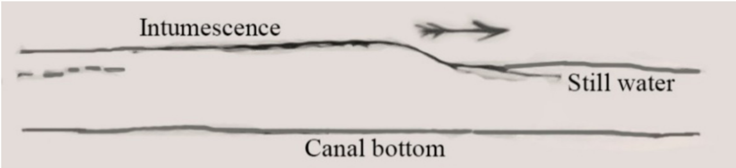

Fig. 14. Propagation of wave in still water.

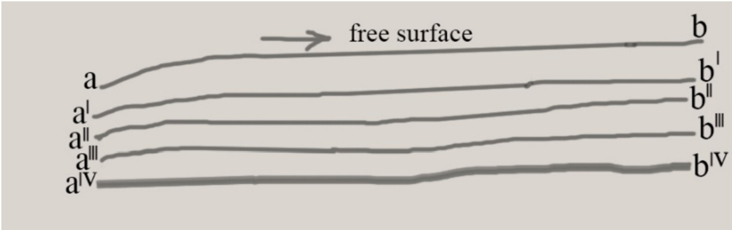

Fig. 15. Sketch of temporal paths of five selected particles.

of uniform flow based on the data of Bazin. It is in this year that dSV starts asking well-defined questions on one side of the letter paper, leaving the other side to JB for his answers. In Letter 14, dated 9 May 1875, JB comes back to the SWEs, stating:

> Dear Master, these days I work on nothing else than my courses and lectures, for I have resolved to take some rest to catch up on sleep. That's why I am eagerly waiting for your letters for which I already have the pen in my hand to respond (it is a distraction pleasing me a lot when it comes from you).

JB takes up his comments on questions of dSV relating to the SWEs:

> As to the complication, which you find hard, the constancy of velocity along a vertical line with its gradual change when moving from a vertical section to another seems to me not difficult, however. In your equations of non-permanent motion of 1871, you assume a constant velocity, equal to the average velocity V, which did not prevent its value to vary gradually within very wide limits *along* the channel from zero up to any value. It is as in my analysis as you assume in your demonstration, equal to its average velocity V, generally quite variable, when relating to very long backwater reaches, where friction is involved (although it does not entail but a small error in the equation of gradually varied unsteady flows); the same assumption, $u = V$, for moderately long backwater propagating from a still water zone, has nothing arbitrary: To first approximation, we have indeed $u = V$ at any section (see No. 126bis, p. 280, or No. 132, pp. 288–289).[43]
>
> You ask for the trajectories of a continuous surge. The eqs. (269bis) (p. 295) define them once we specify a free-surface profile. For an intumescence having roughly the frontal shape of a half solitary wave, followed by an undefined horizontal profile, the trajectories are composed first of parabolic rings, continued upstream by a horizontal indefinite straight line, as shown in the sketch. (Fig. 14.)
>
> Five particles a, a′, a″, a‴, a′V, located, respectively, at the free surface, at a quarter, half, three-quarters of the flow depth, and at the bottom, will all at once rise proportional to their distance to the arrival of the surge, without ceasing to be all four in the same vertical plane. Then soon, and once arrived at the heights which they need to keep, they will continue to move along a straight line, until friction has acquired a sensitive effect on them, putting the particles in the back behind those of highest elevation. From that moment, my analysis of your 1871 memoir and my current memoir developing your ideas no longer applies; you then have to consider the analysis which I give in §XXXVII (not yet printed)." JB then details the differences between the solitary wave and periodic waves, not discussed here. He continues: "You ask me whether the problem of solitary waves relates to that of gradually varied unsteady flows. It is closely connected to it, so that I only made the first approximation of the solitary wave theory and the equation of gradually varied unsteady flows. This theory is the subject of §XXVII, once the equation of gradually varied unsteady flows is established, serving as its basis. (Fig. 15.)
>
> I will work on the memoir that you ask me to do (if necessary after the publication of your work, popularizing these beautiful theories among us) on simple cases which you consider, especially if you include the solitary wave as a desired problem. However, if I had followed in my Essay on flowing waters and had processed questions (which may be found interesting), the analytical way you advise me to take, namely to deal with the most important special cases (prior to the general case), and its extent would have consented me to consider not only one volume for the *Savants étrangers*, but three. For the remainder, I am careful not to present what I find difficult. One thing surprises me that I find to be straightforward: I am surprised, but unfortunately, I am apparently not well understood and unable to express myself more clearly.

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[43] Note that page numbers, equations, paragraphs relate to Boussinesq [31,34].

> In your letter of 25 April (containing a summary of English works on the swell), you ask me whether my judgment on the analysis of the 2nd approximation of Stokes agrees with that of Rankine. From what you tell me, I find that Rankine is correct. In any case, it is the opinion I expressed to you. Yet, perhaps (as much as I am able to judge from your letter), one does not have to sufficiently emphasize that the effect of friction is canceling and invalidating the findings of Stokes (2nd approximation). Without the effect of friction, the laws of Gerstner would never apply, they may apply only to a swell having existed for all time. In contrast, the case of a swell that would have started (for which $\mathrm{d}w/\mathrm{d}x - \mathrm{d}u/\mathrm{d}z = 0$), the laws of Stokes appear more preferable. However, it is useless to question whether there is absolutely no fluid friction, since it is an impossible hypothesis. I am, Sir and dear Master, Yours true and respectful disciple, JB.

Letter 25, dated 10 July 1875, written by JB, deals with the velocity correction coefficients, and fluid friction. Letter 1, dated 23 January 1876, written by dSV, returns to the flood flows under large wave heights, see the Supplementary Material.

As in similar previous cases, JB does not at all respond to the questions of dSV relating to the SWEs. Instead, he exposes on the friendship between the two. Only in Letter 18, dated 9 July 1876, dSV returns to his problem, stating:

> Thank you for your explanation demonstrating[44] the need for a relation between the heights denoted by a and H_1, or between a and the bottom slope i so that the problem of fluvial tides, even of very small height, is solvable by accounting for this slope and friction. That will simplify my task, as I said, in my memoir (which should not only be historical), given that we find this problem treated in your *Essai* for a particular case, which is a useful addition. If I say everything I would fear that engineers and readers find most of the comments of Poisson in the works of Cauchy, dear Sir, so that I reduce all this part of your *Essai*, where things are discussed once you have made my recommended improvement. Yet, I am still inclined to believe that the problem of fluvial tides (with backwater present instead of only small waves on the river) cannot analytically be solved, not even by successive approximation, and not even for a Mediterranean tide. It would be nice to solve numerically or graphically step-by-step several examples of fluvial tides and floods based on the general equations. You deserve this because of your sagacity, your dedicated work, your explosive sense to earthy realities, concrete and useful things already (and I mean useful in this high sense, as opposed to industrial usage), to overcome the degree you have, namely for long purely numerical computations. You also have been creative in constructing numerical solutions of equations in partial differentiations, a kind of solution so far available only for simple differential equations. Your very affectionate, dSV.

In Letter 19, dated 10 July 1876, dSV returns to his mission, receiving in Letter 31, dated 20 December 1876, a partial answer from JB. Both letters are contained in the Supplementary Material. In Letter 33, dated 31 December 1876, JB repeats the previous issues on the frictional effect of waves. He also notes that his *Essai* is still not published, asking dSV to contact the printer during his next stay in Paris.

Despite the enormous correspondence in 1877, not a single letter addresses the above problems, and no letter relates to the publication of the *Essai* appearing in that year. The authors of this paper suppose that several letters are not included in the *Correspondence*, because the arrival of JB's main work in hydraulics would have led to a small comment, at least. The correspondence in the years 1878 and 1879 is similar, equally not advancing the problem of the SWEs. Only in Letter 10, dated 10 May 1880, JB returns to the SWEs, see Supplementary Material.

From 1881 to 1885, no additions were made to the SWEs. dSV wrote his last letter on Dec. 10, 1885, and passed away within less than a month on 6 January 1886. A great mechanical engineer and an outstanding human had closed his eyes. At this point, the authors would like to draw the readers' attention to historical publications dealing with the SWEs. These include, among others, Dooge [35], reviewing both dSV's and JB's approaches for steady and unsteady flows, including the classification of non-uniform flows. As to the SWEs, he incorrectly attributes the system of equations valid for the wide rectangular and horizontal channel to Dupuit [26] and Kleitz [36,37], whereas dSV [4] is credited to have presented the generalized 1D SWEs. JB's theories are evidently cited.

4. Further developments of the unsteady flow theory

Despite the strong contribution of both dSV and JB to the SWEs, others have also studied the issue. The following deals with a short review of these works, starting in 1871 when dSV published his SWEs, based on Miller and Yevjevich [38].

Partiot [39] describes the celerity of tidal waves based on observations, including a coefficient in the term $(gH_0)^{1/2}$ accounting for frictional effects. His method was not successful to solve the problem. Partiot [40] summarizes his various results in a book, overlooking, however, the fundamental theory of JB.

Based on the SWEs of dSV, and by accounting for streamline curvature effects, Boussinesq [41] finds his formula for wave celerity including the curvature term of the free-surface profile. The movement of the center of gravity of gradually varied waves is analytically explored, culminating in the description of the solitary wave. Given that JB's *Essai* was only published in 1877, his 1872 paper was eminently important for his priority in the development of the solitary wave theory. Boussinesq [42] enlarges his previous research to continuous flows of arbitrary media, heavy liquids, periodic liquid waves, including first- and second-order approximations. Special cases include waves propagating in an infinite reservoir or in a

44 Not available in the *Correspondence*.

rectangular channel. Diffraction processes in liquid waves are also studied. Boussinesq [43] further improves his theory of unsteady flows by including the cross-sectional velocity distribution, deriving an improved equation for wave propagation based on the momentum equation.

Graeff [44,33] studies unsteady flows by the reservoir storage equation, an early approach of the flood routing procedure, thereby neglecting dynamic effects. He applies the method to the Pinay Dam in France.

Herschel [45] investigates waves generated by rapidly opening a gate that controls a reservoir. Based on the test data of Darcy and Bazin [10] and own data, both the wave celerity and the free-surface profile versus time are detailed.

Rayleigh [46] presents an alternative to the solitary wave theory previously described by JB. He realized the work of JB only at the proof reading stage.

Kleitz [37] discusses the propagation of flood waves, with results similar to those of dSV. Frictional effects are included by a coefficient. His paper was an answer to JB [31], thereby mentioning an unpublished work [36]. Based on this paper, the credit for the SWEs definitely goes to dSV.

Comoy [47] details the various aspects of tides in rivers, their movement and the bores they create. This work is largely descriptive.

Lechalas [48], author of a book on river engineering, discusses flood flows by a modified reservoir storage equation. He details the approach of Kleitz [37].

Bazin [49] details the coefficients provided by the theory of JB using his experimental data. He simplifies JB's method to allow for easy access in practice.

Note that until dSV's death in early 1886, no significant addition to the knowledge of both dSV and JB has been provided. In the ensuing survey, only additions pertinent to the topic of the SWEs and their solution are listed.

Basset [50] summarizes the wave theories available at the time, in discussing assumptions and derivations. His results relate to wave propagation and the shapes of long waves. The solitary wave and other wave types are also addressed.

Massau [51,52] introduces a novel principle to graphically integrate partial differential equations, such as the SWEs. His principle is currently known as the Method of Characteristics. For arbitrary curves along which the derivatives exist, flow depths h and velocities V are determined successively. Discontinuities are propagated along the characteristic curves. For unsteady flows, the characteristics represent the movement of two infinitesimally small waves, including the waves propagating up- and downstream of the point considered. These waves may become of finite height once bores are developed.

Ritter [5] considers the dam break wave, i.e. the instantaneous and abrupt release of a water body in a rectangular, prismatic, horizontal and frictionless channel, thereby determining with the simplified SWEs the velocity and flow depth advances in terms of streamwise location and time. He based his analysis on the SWEs derived by dSV, yet neither citing this paper, nor the work of JB.

Wheeler [53] presents a book on tidal rivers, including the history, the hydraulics, effects of friction, the disturbances and mixing of fresh and salt water, the tidal bars at river mouths, the littoral drift, and examples of tidal rivers. This book is mainly descriptive, but counts as one of the first on this topic. The names of dSV and JB are not mentioned.

Lévy [54] verifies the SWEs devised by dSV. Tides in canals are considered, thereby accounting for frictional effects. A wave extinction coefficient is defined to include the latter effect. This semi-theoretical approach results in improved knowledge as compared with that of dSV or Airy. The results were applied to the Suez Canal, with the computed wave celerities much lower than those observed.

Haerens [55] appears to be the first satisfying dSV's wish to express the SWEs in finite difference form. Two examples are considered, namely (i) determination of the wave profiles and the hydrographs for a given discharge hydrograph, and (ii) the waves upstream of a river junction for a given tide.

Bourdelles [56], in his PhD thesis, considers the theory of wave propagation of translation waves, to check how closely it matches prototype observations.

Flamant [57] successfully summarizes the complicated works of, mainly, JB. This book is still considered an excellent source to retrieve facts before reading the original texts of JB.

Gwyther [58] studies long waves analytically, including the first approximation in the most general manner. The steady motion of long waves is also considered by finding the differential equation to be satisfied by the velocity potential.

Seddon [59] is the author of a paper relating to river hydraulics; he considers wave propagation in arbitrary river sections, and propagation of flood waves. His theory is also known as the Kleitz–Seddon approach in hydraulics.

Cornish [60] describes waves without details to their hydraulics. The book includes the size and celerity of deep-sea waves, sediment transport by sea waves, and steady and progressive waves in rivers. This book includes photos of waves.

Cisotti [61] is the first to consider SWEs with a perturbation analysis, thereby restricting himself to small wave heights and uniform velocity distribution.

Levi-Cività [62] presents a theoretical study on wave motion developing the concept of absolute and relative discharges by introducing complex variables. He generalizes the Stokes–Rayleigh theorem.

Bakhmeteff [63] considers in his book free-surface flows, providing the main formulas for surge propagation and wave height.

von Mises [64] outlines a method to generally study the gradual damping of translation waves subjected to the effect of roughness. A perturbation analysis yields results that were previously obtained by Boussinesq [31].

Lord Rayleigh [65] finds that a long wave of finite height is able to propagate in still water only by a change of shape, because it is impossible otherwise to satisfy the requirement of a free surface for a steady long wave, if the wave height above the original water level is determined, unless its square term is neglected.

Palatini [66] studies analytically the effect of a non-horizontal bottom on wave propagation. The integration of the governing wave equation is reduced to a single fourth-order partial differential equation.

Alibrandi [67] offers an analytic treatment of wave propagation, comparing his results with those of JB. By considering flood waves both in rivers and in straight channels, the latter are found to be less difficult to integrate.

Johnson [68] compares the surge propagation with the hydraulic jump in a frictionless rectangular and horizontal channel, finding the corresponding equations.

Schoklitsch [69] provides the first experimental data to the dam break wave, and expands the theory of Ritter [5] by accounting for frictional effects. He also demonstrates that the latter have a significant influence on the wave front, whereas the remainder of the dam break wave is hardly affected by wall friction.

Maillet [70] neglects the two dynamic terms $[\partial V/\partial t + V\partial V/\partial x]$ and compares the ratio of free-surface slope to the bottom slope of many rivers, concluding that all flood peaks attenuate; sharp peaks are found to attenuate even more strongly. Additional elements of flood flows are derived based on the simplified model.

From 1920 to the advent of computers, hundreds of papers on the topic were published; these are not reviewed here. Miller and Yevjevich [38] list a total number of publications of nearly 500 up to 1955. As a result of this review, the Method of Characteristics would have indeed been a general approach to solve the SWEs, whereas most other approaches are limited by, e.g., small wave heights, omission of roughness and slope effects, and expansion to typical bathymetries as encountered for tidal rivers. The SWEs proposed by dSV and JB thus constitute the important basis for all hydraulic computations in which streamline curvature effects are negligible. The SWEs currently constitute the foundation of open channel hydraulics if curvature effects and spatial features are ignored.

5. Conclusions

The technical papers of both de Saint-Venant and Boussinesq are well known by experts in technical physics and applied mathematics, despite the French language used, and the complicated writing styles of both authors. The correspondence between the two is so far not available. The intention here is that this commented translation may be considered a worthwhile addition to their finally published papers. The thoughts expressed in the many letters shed additional light on the development of particularly open channel hydraulics, dealing in this work with the formation of the one-dimensional steady and unsteady equations based on both the energy and the momentum principles. The great wish of de Saint-Venant to expand his findings on unsteady flows including the effects of friction and bottom slopes, both important parameters when long distances or long times are considered, as, e.g., for flood waves, was not addressed by Boussinesq. The mathematical barriers were too high to provide at this era an adequate solution. In turn, the effect of the velocity correction coefficients was adequately dealt with by both, based on the energy and the momentum conservation equations.

One astonishing issue is that de Saint-Venant was in almost all relations the demanding person, whereas Boussinesq hardly had any questions to his colleague. More astonishingly, de Saint-Venant became never tired of the partly complicated answers of his correspondent, he normally accepted the proposals of Boussinesq, but still tried to further consider in depth a certain problem. It may even be stated that de Saint-Venant hardly ever was satisfied with an answer, despite he could personally also not advance a certain issue. He greatly profited from Boussinesq, who himself was certainly often astonished by the depth of problems considered. One may even state that Boussinesq's success, particularly until he moved in 1872 from Gap to Lille, largely rested on the shoulders of de Saint-Venant. It is not clear to the authors whether Boussinesq would have survived without his mentor, who fatherly cared for his pupil first, and then his colleague, whose genius was gradually realized by him. The *Correspondence* was therefore a great luck not only for the two writers, but also for the scientific world, profiting from a combined and thorough attack of the relevant problems in physics and engineering. The submissiveness of Boussinesq toward de Saint-Venant was typical in the 19th century in Mid-Europe, posing yet no problems to the collegial way the two profited largely from each other. The *Correspondence*, in addition to this human closeness, evidences how a problem may be better attacked by a combined approach, in which either of the two parties contributes to the successful solution to, say, here, the formulation of the SWEs.

The texts of the letters evidence the close personal relationship between the two writers. De Saint-Venant, 45 years older than Boussinesq, sometimes reacts like a father, but he is the person asking from his younger colleague persistently further answers to his questions. Boussinesq, in turn, tries to solve the problems posed by his colleague, but does not advance the problem of unsteady flows including the effects of bottom slope and friction, despite the many proposals of de Saint-Venant on how this could be achieved. Except for this particular problem, the two advance the theory of open channel flows to almost the point of current knowledge, so that both of them may be considered pioneers of open channel hydraulics. Additional items covered in the *Correspondence* include the further development of the unsteady flow theory until 1920, and a short review of de Saint-Venant's posthumously published paper on gradually varied flow. Given the many persons mentioned in the *Correspondence*, the main individuals are shortly introduced below with their life years and their professional positions.

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Appendix A. Supplementary material

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.crme.2019.08.004.

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