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Short paper / Note

Blow-up, non-extinction and exponential growth of solutions to a fourth-order parabolic equation

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Abstract. This paper is concerned with the following fourth-order parabolic problem:

$$\begin{cases} u_t + u_{xxxx} = |u|^{p-1}u - \frac{1}{|\Omega|} \int_{\Omega} |u|^{p-1}u \, dx, & x \in \Omega, t > 0, \\ u_x = u_{xxx} = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0, & x \in \Omega \end{cases}$$

with $\Omega = (0, a)$ and $p > 1$. Here, $u_0 \in H^2(\Omega)$ is the initial function which satisfies $\int_{\Omega} u_0(x) \, dx = 0$ with $u_0(x) \not\equiv 0$. We prove that the solutions for the preceding problem blow-up at finite time t_* provided that:

$$\frac{\lambda(2(p+1)-m)}{(p+1)} |\Omega|^{-(p-1)/2} \|u_0\|_2^{p+1} + \frac{m-4}{2c^*} \|u_0\|_2^2 - mJ(u_0) \geq 0.$$

Here, $4 < m < 2(p+1)$, $0 < \lambda < 1$ and c^* is a positive constant related to the Poincaré inequality. Here, $J(u)$ is the energy functional. The above condition trivially holds for $J(u_0) \leq 0$. Thus, the blow-up result is valuable for arbitrary positive initial energy and suitable initial data. We also obtain upper and lower bounds for the blow-up time. Hence, the exact blow-up time is obtained under some conditions. Besides, if $J(u_0) > 0$ and the above condition holds in the strict sense or $J(u_0) \leq 0$, then for every $q \geq 2$, $\|u(t)\|_q$ grows exponentially for all $0 < t < t_*$, also, under the same conditions, the solution for this problem does not extinct in finite time if $\|u_0\|_2 > 0$. The non-extinction of solutions also holds in the equal sense of the above condition. These results extend the recent results obtained for this problem.

Keywords. Fourth-order equation, Blow-up, Exact blow-up time, Non-extinction, Exponential growth.

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1. Introduction

In this paper, we study the following fourth-order parabolic problem which describes

$$\begin{cases} u_t + u_{xxxx} = |u|^{p-1}u - \frac{1}{|\Omega|} \int_{\Omega} |u|^{p-1}u \, dx, & x \in \Omega, t > 0, \\ u_x = u_{xxx} = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0, & x \in \Omega. \end{cases} \quad (1.1)$$

Here, $\Omega = (0, a)$, $p > 1$ and $u_0 \in H^2(\Omega)$ is the initial function which satisfies $\int_{\Omega} u_0(x) dx = 0$ with $u_0(x) \neq 0$. The energy functional for this problem is as follows:

$$J(u(t)) = \frac{1}{2} \|u_{xx}(t)\|_2^2 - \frac{1}{p+1} \|u(t)\|_{p+1}^{p+1}. \quad (1.2)$$

The mathematical model of (1.1) is a simplified model of the fourth-order parabolic equation that is introduced in [1–3]. This model describes the evolution of the epitaxial growth of thin films at the nanoscale. The physical importance of this model has led many authors to study these types of equations, and the global existence and blow-up of solutions to these equations have been investigated. To view these results, we refer interested readers to [4–10] and the references therein. Also, to see the results of other fourth-order parabolic equations, we refer to [11–13]. We now state the results obtained for problem (1.1). For this problem, if $0 < p \leq 1$, it is proved that the weak solutions are global [8], also, if $J(u_0) := J_0 \leq 0$, then the solution does not extinct in finite time. For $p > 1$, in the case of $J_0 \leq 0$, it is proved that the solutions blow-up in finite time [9]. Moreover, if $d = \inf_N J(u)$, where N is defined as

$$N = \left\{ u \in H^2(\Omega) \mid I(u) = \|u_{xx}(t)\|_2^2 - \|u\|_{p+1}^{p+1} = 0, \|u_{xx}(t)\|_2^2 \neq 0 \right\},$$

then for $0 < J_0 < d$, if $I(u_0) > 0$, then the weak solutions are global and do not vanish in finite time, whereas for $I(u_0) < 0$, the weak solutions blow-up in finite time [9]. Also, the similar results with $0 < J_0 < d$ in the case $J_0 = d$ hold [9]. Recently, the blow-up of weak solutions with the positive initial energy with other constraints is proved in [10]. In fact, it is proved that the weak solutions blow-up in finite time provided that $J_0 < E_1$ and $\|u_{0,xx}\|_2 > \alpha$, where E_1 and α are positive constants which are introduced in [10]. Also, in [10], an upper bound for the blow-up time is obtained.

In the present paper, we prove that the solutions to problem (1.1) blow-up at finite time t_* provided that:

$$\frac{\lambda(2(p+1) - m)}{(p+1)} |\Omega|^{-(p-1)/2} \|u_0\|_2^{p+1} + \frac{m-4}{2c^*} \|u_0\|_2^2 - mJ(u_0) \geq 0,$$

where $p > 1$, $4 < m < 2(p+1)$, $0 < \lambda < 1$, and c^* comes from the Poincaré inequality. The above condition trivially holds for $J(u_0) \leq 0$. Thus, the blow-up result is valuable for arbitrary positive initial energy and suitable initial data. We also obtain upper and lower bounds for the blow-up time. Furthermore, if $J(u_0) > 0$ and the preceding condition holds in the strict sense or $J(u_0) \leq 0$, then:

- for every $q \geq 2$, $\|u(t)\|_q$ grows exponentially for all $0 < t < t_*$;
- the solution of (1.1) does not extinct in finite time if $\|u_0\|_2 > 0$.

The non-extinction of solutions also holds in the equal sense of the above condition.

In the next section, we prove our results about the blow-up in finite time and in the last section, we show the exponential growth and non-extinction of solutions.

2. Blow-up in finite time

In this section, we prove the blow-up of solutions to problem (1.1) and obtain upper and lower bounds for the blow-up time.

Definition 2.1. A function $u(x, t)$ is called a weak solution to problem (1.1), if $u \in L^\infty(0, T, H^2(\Omega))$, $u_t \in L^2(0, T, H^2(\Omega))$ and $u(x, t)$ satisfies:

$$\int_0^t \int_{\Omega} \left[u_\tau \varphi + u_{xx} \varphi_{xx} - \left(|u|^{p-1} u - \frac{1}{|\Omega|} \int_{\Omega} |u|^{p-1} u \right) \varphi \right] dx d\tau = 0$$

for all $\varphi \in L^2(0, T, H^2(\Omega))$ with $\varphi_x = 0$ on $\partial\Omega$.

Throughout this paper, we assume that the local solution to problem (1.1) exists.

Lemma 2.2. *The energy functional $J(t)$ is non-increasing and satisfies:*

$$J(t) = J(0) - \int_0^t \|u_\tau(\tau)\|_2^2 \, d\tau \tag{2.1}$$

with

$$J(0) = \frac{1}{2} \|u_{0xx}\|_2^2 - \frac{1}{p+1} \|u_0\|_{p+1}^{p+1}.$$

Proof. Differentiating (1.2) and making use of (1.1), we obtain

$$\begin{aligned} \frac{d}{dt} J(t) &= \frac{d}{dt} \left(\frac{1}{2} \|u_{xx}(t)\|_2^2 - \frac{1}{p+1} \|u(t)\|_{p+1}^{p+1} \right) \\ &= \int_\Omega (u_{xx}(t)u_{xxt}(t) - |u(t)|^{p-1}u(t)u_t(t) \, dx) \\ &= - \int_\Omega u_t(t)(-u_{xxx}(t) + |u(t)|^{p-1}u(t)) \, dx \\ &= - \int_\Omega u_t(t) \left(u_t(t) + \frac{1}{|\Omega|} \int_\Omega |u(t)|^{p-1}u(t) \, dx \right) \\ &= - \int_\Omega (u_t(t))^2 \, dx - \left(\frac{d}{dt} \int_\Omega u(t) \, dx \right) \left(\frac{1}{|\Omega|} \int_\Omega |u(t)|^{p-1}u(t) \, dx \right). \end{aligned} \tag{2.2}$$

We now integrate from (1.1) and use integration by parts to obtain

$$\frac{d}{dt} \int_\Omega u(t) \, dx = 0. \tag{2.3}$$

This inequality along with (2.2) gives

$$\frac{d}{dt} J(t) = - \int_\Omega u_t^2(t) \, dx. \tag{2.4}$$

Integrating of (2.4) results in

$$J(t) = J(0) - \int_0^t \|u_\tau(\tau)\|_2^2 \, d\tau.$$

This completes the proof. □

Lemma 2.3. *Assume that $p > 1$ and u is the weak solution of (1.1). Then for any $\mu > 0$, the following estimate holds:*

$$\frac{d}{dt} \|u(t)\|_2^2 \geq (k_1\mu + k_2) \|u(t)\|_2^2 + k_3 \|u(t)\|_2^{p+1} - k_4\mu^{(p+1)/(p-1)} - k_5 \tag{2.5}$$

with

$$\begin{aligned} k_1 &= \lambda \left(2 - \frac{m}{p+1} \right), \quad k_2 = \frac{1}{c^*} \left(-2 + \frac{m}{2} \right), \quad k_3 = (1 - \lambda) \left(2 - \frac{m}{p+1} \right) |\Omega|^{-(p-1)/2} \\ k_4 &= |\Omega| \left(2 - \frac{m}{p+1} \right) \left(\frac{2}{p+1} \right)^{(2/p-1)} \left(\frac{p-1}{p+1} \right) \lambda, \quad k_5 = mJ_0, \end{aligned} \tag{2.6}$$

where m and λ are some positive constants with $4 < m < 2(p+1)$ and $0 < \lambda < 1$, also, c^* comes from the Poincaré inequality.

Proof. At first, integrating (2.3), we are led to

$$\int_\Omega u(t) \, dx = \int_\Omega u_0 \, dx.$$

By assumption $\int_\Omega u_0 \, dx = 0$, we have

$$\int_\Omega u(t) \, dx = 0. \tag{2.7}$$

Making use (1.1) and (2.7), we obtain

$$\frac{d}{dt} \|u(t)\|_2^2 = 2 \int_{\Omega} u(t) u_t(t) \, dx = -2 \int_{\Omega} |u_{xx}(t)|^2 \, dx + 2 \int_{\Omega} |u(t)|^{p+1} \, dx. \quad (2.8)$$

From (2.8), (1.2) and (2.1), we can write

$$\begin{aligned} \frac{d}{dt} \|u(t)\|_2^2 &= -2 \|u_{xx}(t)\|_2^2 + 2 \|u(t)\|_{p+1}^{p+1} + mJ(t) - mJ(t) \\ &= -2 \|u_{xx}(t)\|_2^2 + 2 \|u(t)\|_{p+1}^{p+1} + \frac{m}{2} \|u_{xx}(t)\|_2^2 - \frac{m}{p+1} \|u(t)\|_{p+1}^{p+1} \\ &\quad - mJ_0 + m \int_0^t \|u_{\tau}(\tau)\|_2^2 \, d\tau \\ &= \left(-2 + \frac{m}{2}\right) \|u_{xx}(t)\|_2^2 + \left(2 - \frac{m}{p+1}\right) \|u(t)\|_{p+1}^{p+1} - mJ_0 + m \int_0^t \|u_{\tau}(\tau)\|_2^2 \, d\tau \end{aligned} \quad (2.9)$$

with $4 < m < 2(p+1)$. In the rest of the proof, we obtain a lower bound for the first term on the right hand side of (2.9). In order to do this, we apply the Poincaré–Wirtinger inequality to obtain

$$\left\| u(t) - \frac{1}{|\Omega|} \int_{\Omega} u(t) \, dx \right\|_2^2 \leq c_1 \|u_x(t)\|_2^2,$$

where c_1 is the best constant in the Poincaré–Wirtinger inequality. By considering $\int_{\Omega} u(t) \, dx = 0$, the preceding inequality becomes

$$\|u(t)\|_2^2 \leq c_1 \|u_x(t)\|_2^2. \quad (2.10)$$

Because of $u_x = 0$ on $\partial\Omega$, we can apply the Poincaré inequality to obtain

$$\|u_x(t)\|_2^2 \leq c_2 \|u_{xx}(t)\|_2^2,$$

where c_2 is the best constant in the Poincaré inequality. This inequality along with (2.10) yields

$$\|u(t)\|_2^2 \leq c^* \|u_{xx}(t)\|_2^2, \quad (2.11)$$

with $c^* = c_1 c_2$. Combining (2.11) with (2.9) gives

$$\frac{d}{dt} \|u(t)\|_2^2 \geq \frac{1}{c^*} \left(-2 + \frac{m}{2}\right) \|u(t)\|_2^2 + \left(2 - \frac{m}{p+1}\right) \|u(t)\|_{p+1}^{p+1} - mJ_0. \quad (2.12)$$

We now write the second term on the right hand side of (2.12) as

$$\left(2 - \frac{m}{p+1}\right) \|u(t)\|_{p+1}^{p+1} = \lambda \left(2 - \frac{m}{p+1}\right) \|u(t)\|_{p+1}^{p+1} + (1-\lambda) \left(2 - \frac{m}{p+1}\right) \|u(t)\|_{p+1}^{p+1},$$

with $0 < \lambda < 1$. Thus the estimate (2.12) is written as

$$\frac{d}{dt} \|u(t)\|_2^2 \geq \frac{1}{c^*} \left(-2 + \frac{m}{2}\right) \|u(t)\|_2^2 + \lambda \left(2 - \frac{m}{p+1}\right) \|u(t)\|_{p+1}^{p+1} + (1-\lambda) \left(2 - \frac{m}{p+1}\right) \|u(t)\|_{p+1}^{p+1} - mJ_0. \quad (2.13)$$

We apply the Hölder inequality to obtain

$$\int_{\Omega} |u(t)|^{p+1} \, dx \geq |\Omega|^{-(p-1)/2} \left(\int_{\Omega} u^2(t) \, dx \right)^{(p+1)/2}. \quad (2.14)$$

Combining the preceding inequality with (2.13) gives

$$\begin{aligned} \frac{d}{dt} \|u(t)\|_2^2 &\geq \frac{1}{c^*} \left(-2 + \frac{m}{2}\right) \|u(t)\|_2^2 + (1-\lambda) \left(2 - \frac{m}{p+1}\right) |\Omega|^{-(p-1)/2} \|u(t)\|_2^{p+1} \\ &\quad + \lambda \left(2 - \frac{m}{p+1}\right) \|u(t)\|_{p+1}^{p+1} - mJ_0. \end{aligned} \quad (2.15)$$

In order to obtain a lower bound for the third term on the right hand side of (2.15), we apply the Young inequality to obtain

$$\int_{\Omega} |u(t)|^{p+1} dx \geq \mu \int_{\Omega} u^2(t) dx - \mu^{(p+1)/(p-1)} \left(\frac{2}{p+1}\right)^{2/(p-1)} \left(\frac{p-1}{p+1}\right) |\Omega|, \tag{2.16}$$

where μ is some positive constant to be determined later. This inequality along with (2.15) yields

$$\frac{d}{dt} \|u(t)\|_2^2 \geq (k_1\mu + k_2) \|u(t)\|_2^2 + k_3 \|u(t)\|_2^{p+1} - k_4\mu^{(p+1)/(p-1)} - k_5$$

with

$$k_1 = \lambda \left(2 - \frac{m}{p+1}\right), \quad k_2 = \frac{1}{c^*} \left(-2 + \frac{m}{2}\right), \quad k_3 = (1 - \lambda) \left(2 - \frac{m}{p+1}\right) |\Omega|^{-(p-1)/2}$$

$$k_4 = |\Omega| \left(2 - \frac{m}{p+1}\right) \left(\frac{2}{p+1}\right)^{2/(p-1)} \left(\frac{p-1}{p+1}\right) \lambda, \quad k_5 = mJ_0.$$

Thus, we obtain the desired result. □

Lemma 2.4. *Assume that $p > 1$ and u is the weak solution of (1.1). Then the following estimate holds:*

$$\frac{d}{dt} \|u(t)\|_2^2 \geq \left(k_6 \|u_0\|_2^{p+1} + k_2 \|u_0\|_2^2 - k_5\right) e^{(k_7 \|u_0\|_2^{p-1} + k_2)t} + k_3 \|u(t)\|_2^{p+1}, \tag{2.17}$$

with

$$k_6 = \frac{\lambda(2(p+1) - m)}{p+1} |\Omega|^{-(p-1)/2} \quad \text{and} \quad k_7 = \frac{\lambda(2(p+1) - m)}{2} |\Omega|^{-(p-1)/2}, \tag{2.18}$$

where k_2, k_3 and k_5 are defined in (2.6) and m and λ are some positive constants with $4 < m < 2(p+1)$ and $0 < \lambda < 1$.

Proof. By considering the inequality (2.5), we can write

$$\frac{d}{dt} \|u(t)\|_2^2 - (k_1\mu + k_2) \|u(t)\|_2^2 \geq -k_4\mu^{(p+1)/(p-1)} - k_5.$$

This inequality yields

$$\begin{aligned} \|u(t)\|_2^2 &\geq \|u_0\|_2^2 e^{(k_1\mu + k_2)t} + \left(\frac{k_4\mu^{(p+1)/(p-1)} + k_5}{k_1\mu + k_2}\right) \left(1 - e^{(k_1\mu + k_2)t}\right) \\ &= \frac{1}{(k_1\mu + k_2)} \left\{ [(k_1\mu + k_2) \|u_0\|_2^2 - k_4\mu^{(p+1)/(p-1)} - k_5] e^{(k_1\mu + k_2)t} + k_4\mu^{(p+1)/(p-1)} + k_5 \right\}. \end{aligned} \tag{2.19}$$

As in [14], we define

$$\begin{aligned} F(\mu) &= (k_1\mu + k_2) \|u_0\|_2^2 - k_4\mu^{(p+1)/(p-1)} - k_5 \\ &= \left[\lambda \left(2 - \frac{m}{p+1}\right) \mu + \frac{m-4}{2c^*} \right] \|u_0\|_2^2 - \mu^{(p+1)/(p-1)} \left(2 - \frac{m}{p+1}\right) \left(\frac{2}{p+1}\right)^{2/(p-1)} \left(\frac{p-1}{p+1}\right) \lambda |\Omega| - mJ_0. \end{aligned}$$

Thus, we can write (2.19) as follows:

$$\|u(t)\|_2^2 \geq \frac{1}{(k_1\mu + k_2)} \left\{ F(\mu) e^{(k_1\mu + k_2)t} + k_4\mu^{(p+1)/(p-1)} + k_5 \right\}. \tag{2.20}$$

Combining (2.20) with (2.5) yields

$$\frac{d}{dt} \|u(t)\|_2^2 \geq F(\mu) e^{(k_1\mu + k_2)t} + k_3 \|u(t)\|_2^{p+1}. \tag{2.21}$$

We now compute

$$F'(\mu) = \lambda \left(2 - \frac{m}{p+1}\right) \left[\|u_0\|_2^2 - |\Omega| \left(\frac{2}{p+1}\right)^{2/(p-1)} \mu^{2/(p-1)} \right].$$

By solving $F'(\mu) = 0$, we obtain

$$\mu_{\max} = \left(\frac{p+1}{2} \right) |\Omega|^{-(p-1)/2} \|u_0\|_2^{p-1},$$

where μ_{\max} is the absolutely maximum point of the function F . Thus, we have

$$k_1 \mu_{\max} + k_2 = k_1 \left(\frac{p+1}{2} \right) |\Omega|^{-(p-1)/2} \|u_0\|_2^{p-1} + k_2 = \frac{\lambda(2(p+1) - m)}{2} |\Omega|^{-(p-1)/2} \|u_0\|_2^{p-1} + k_2.$$

For convenience, we set $k_7 = (\lambda(2(p+1) - m))/2|\Omega|^{-(p-1)/2}$. Then we have

$$k_1 \mu_{\max} + k_2 = k_7 \|u_0\|_2^{p-1} + k_2.$$

We also have

$$\begin{aligned} F(\mu_{\max}) &= \left[\frac{\lambda(2(p+1) - m)}{2} |\Omega|^{-(p-1)/2} \|u_0\|_2^{p-1} + \frac{m-4}{2c^*} \right] \|u_0\|_2^2 \\ &\quad - \frac{\lambda(p-1)(2(p+1) - m)}{2(p+1)} |\Omega|^{-(p-1)/2} \|u_0\|_2^{p+1} - mJ_0 \\ &= \frac{\lambda(2(p+1) - m)}{(p+1)} |\Omega|^{-(p-1)/2} \|u_0\|_2^{p+1} + \frac{m-4}{2c^*} \|u_0\|_2^2 - mJ_0. \end{aligned}$$

Now by considering the values of k_2 and k_5 from (2.6) and setting $k_6 = (\lambda(2(p+1) - m))/(p+1)|\Omega|^{-(p-1)/2}$, we can write

$$F(\mu_{\max}) = k_6 \|u_0\|_2^{p+1} + k_2 \|u_0\|_2^2 - k_5.$$

By considering the preceding computations, we can write (2.20) and (2.21) as follows:

$$\|u(t)\|_2^2 \geq \frac{1}{(k_7 \|u_0\|_2^{p-1} + k_2)} \left\{ [k_6 \|u_0\|_2^{p+1} + k_2 \|u_0\|_2^2 - k_5] e^{(k_7 \|u_0\|_2^{p-1} + k_2)t} + k_4 (\mu_{\max})^{(p+1)/(p-1)} + k_5 \right\} \quad (2.22)$$

and

$$\frac{d}{dt} \|u(t)\|_2^2 \geq [k_6 \|u_0\|_2^{p+1} + k_2 \|u_0\|_2^2 - k_5] e^{(k_7 \|u_0\|_2^{p-1} + k_2)t} + k_3 \|u(t)\|_2^{p+1}. \quad (2.23)$$

This completes the proof. \square

Lemma 2.5. Assume that u is the weak solution of (1.1). Moreover, assume that $p > 1$, $4 < m < 2(p+1)$ and $0 < \lambda < 1$. If the following condition holds:

$$k_6 \|u_0\|_2^{p+1} + k_2 \|u_0\|_2^2 - k_5 \geq 0, \quad (2.24)$$

then the solution of problem (1.1) blows up at finite time t_* and $t_* \leq T$ with

$$T = \frac{2\|u_0\|_2^{1-p}}{(p-1)k_3},$$

where k_2, k_3, k_5 and k_6 are defined in (2.6) and (2.18).

Proof. By considering the condition (2.24), the inequality (2.17) implies that

$$\frac{d}{dt} \|u(t)\|_2^2 \geq k_3 \|u(t)\|_2^{p+1}.$$

This inequality yields

$$(\|u(t)\|_2^2)^{-(p+1)/2} \frac{d}{dt} \|u(t)\|_2^2 \geq k_3. \quad (2.25)$$

Integrating (2.25) from 0 to t gives

$$\|u(t)\|_2^{-(p-1)} \leq \|u_0\|_2^{1-p} - \frac{(p-1)}{2} k_3 t. \tag{2.26}$$

This inequality cannot hold for all $t > 0$. In fact we conclude from (2.26) that $\lim_{t \rightarrow t_*^-} \|u(t)\|_2^2 = +\infty$, where t_* is bounded from above by

$$t_* \leq T = \frac{2\|u_0\|_2^{1-p}}{(p-1)k_3}.$$

This completes the proof. □

Remark 2.6. In the case of $J_0 \leq 0$, the inequality (2.12) implies that

$$\frac{d}{dt} \|u(t)\|_2^2 \geq \left(2 - \frac{m}{p+1}\right) \|u(t)\|_{p+1}^{p+1}.$$

Making use of the inequality (2.14), the preceding inequality becomes

$$\frac{d}{dt} \|u(t)\|_2^2 \geq \left(2 - \frac{m}{p+1}\right) |\Omega|^{-(p-1)/2} \|u\|_2^{p+1}. \tag{2.27}$$

Similar to Lemma 2.5, we can conclude from the inequality (2.27) that $\lim_{t \rightarrow t_*^-} \|u(t)\|_2^2 = +\infty$, and:

$$t_* \leq T = \frac{2(p+1)\|u_0\|_2^{1-p}}{(p-1)(2(p+1)-m)} |\Omega|^{(p-1)/2}.$$

Remark 2.7. By considering Lemma 2.5 and Remark 2.6, we know that if $J_0 > 0$ and the condition (2.24) holds or $J_0 \leq 0$, then $\lim_{t \rightarrow t_*^-} \|u(t)\|_2^2 = +\infty$, thus the inequality (2.14) implies that $\lim_{t \rightarrow t_*^-} \|u(t)\|_{p+1}^{p+1} = +\infty$. Finally, the Sobolev inequality $\|u(t)\|_{p+1} \leq c_s \|u_{xx}\|_2$ deduces that $\lim_{t \rightarrow t_*^-} \|u_{xx}\|_2^2 = +\infty$. Hence, we can write

$$\lim_{t \rightarrow t_*^-} (\|u_{xx}(t)\|_2^2 + \|u(t)\|_{p+1}^{p+1}) = +\infty.$$

In the following lemma, we use the technique used in [15] and obtain a lower bound for the blow-up time, when the blow-up occurs.

Lemma 2.8. Assume that u is the weak solution of (1.1). Also, assume that $p > 1$. If $J_0 > 0$ and the condition (2.24) holds or $J_0 \leq 0$, then the blow-up occurs at finite time t_* and t_* is bounded from below by

$$t_* \geq \frac{1}{2^p c_s^{2p} (p-1) \psi^{p-1}(0)},$$

where c_s is the best constant in the Sobolev inequality and:

$$\psi(0) = \frac{1}{2} \|u_{0xx}\|_2^2 + \frac{1}{p+1} \|u_0\|_{p+1}^{p+1}.$$

Proof. In view of Lemma 2.5 and Remark 2.6, we know that if $J_0 > 0$ and the condition (2.24) holds or $J_0 \leq 0$, then the blow-up occurs at finite time t_* . Thus, we find a lower bound for the blow-up time. In order to find a lower bound for the blow-up time, we define the function ψ as:

$$\psi(t) = \frac{1}{2} \|u_{xx}(t)\|_2^2 + \frac{1}{p+1} \|u(t)\|_{p+1}^{p+1}.$$

We now compute

$$\begin{aligned} \psi'(t) &= \int_{\Omega} u_{xx}(t) u_{xxt}(t) \, dx + \int_{\Omega} |u(t)|^{p-1} u(t) u_t(t) \, dx \\ &= \int_{\Omega} (u_{xxx}(t) + |u(t)|^{p-1} u(t)) u_t(t) \, dx \\ &= \int_{\Omega} \left(2|u(t)|^{p-1} u(t) - u_t(t) - |\Omega|^{-1} \int_{\Omega} |u(t)|^{p-1} u(t) \, dx \right) u_t(t) \, dx \end{aligned}$$

$$\begin{aligned}
 &= \int_{\Omega} 2|u(t)|^{p-1}u(t)u_t(t) \, dx - \int_{\Omega} u_t(t)^2 \, dx - \left(|\Omega|^{-1} \int_{\Omega} |u(t)|^{p-1}u(t) \, dx \right) \left(\frac{d}{dt} \int_{\Omega} u(t) \, dx \right) \\
 &= \int_{\Omega} 2|u(t)|^{p-1}u(t)u_t(t) \, dx - \int_{\Omega} u_t(t)^2 \, dx, \tag{2.28}
 \end{aligned}$$

where we have used from (2.3). We apply the Young inequality to the first term on the right hand side of (2.28) and use the Sobolev inequality $\|u(t)\|_{2p} \leq c_s \|u_{xx}\|_2$ to obtain

$$\int_{\Omega} 2|u(t)|^{p-1}u(t)u_t(t) \, dx \leq \|u(t)\|_{2p}^{2p} + \|u_t(t)\|_2^2 \leq c_s^{2p} \|u_{xx}(t)\|_2^{2p} + \|u_t(t)\|_2^2 \leq 2^p c_s^{2p} (\psi(t))^p + \|u_t(t)\|_2^2.$$

Here, c_s is the best constant in the Sobolev inequality. This inequality along with (2.28) yields

$$\psi'(t) \leq 2^p c_s^{2p} (\psi(t))^p.$$

Thus,

$$\psi^{-p} d\psi \leq 2^p c_s^{2p} dt.$$

Integration of the preceding inequality, we deduce

$$-\frac{1}{(p-1)\psi^{p-1}(t)} + \frac{1}{(p-1)\psi^{p-1}(0)} \leq 2^p c_s^{2p} t.$$

From Remark 2.7, $\lim_{t \rightarrow t_*^-} \psi(t) = +\infty$, thus we obtain

$$t_* \geq \frac{1}{2^p c_s^{2p} (p-1)\psi^{p-1}(0)}.$$

Thus, we obtain the desired result. □

By considering the previous lemmas and their conditions, we can obtain the exact blow-up time. In order to do this, we equate the upper and lower bounds obtained for the blow-up time and obtain a condition. Under this condition, the blow-up time is equal to the common value of the upper and lower bounds.

3. Exponential growth and non-extinction of solutions

In this section, we prove the exponential growth and non-extinction of solutions.

Lemma 3.1. *Assume that $J_0 > 0$ and u is the weak solution of (1.1). Also, assume that condition (2.24) holds in the strict sense. Then for every $q \geq 2$, $\|u(t)\|_q$ grows exponentially for all $0 < t < t_*$.*

Proof. By considering $J_0 > 0$, the inequality (2.22) yields

$$\|u(t)\|_2^2 \geq \frac{1}{(k_7 \|u_0\|_2^{p-1} + k_2)} [k_6 \|u_0\|_2^{p+1} + k_2 \|u_0\|_2^2 - k_5] e^{(k_7 \|u_0\|_2^{p-1} + k_2)t}. \tag{3.1}$$

Making use of Hölder’s inequality, for every $q > 2$, we can write

$$|\Omega|^{(q-2)/q} \|u(t)\|_q^2 \geq \|u(t)\|_2^2. \tag{3.2}$$

This inequality along with (3.1) gives

$$\|u(t)\|_q^2 \geq \frac{1}{(k_7 \|u_0\|_2^{p-1} + k_2)} [k_6 \|u_0\|_2^{p+1} + k_2 \|u_0\|_2^2 - k_5] |\Omega|^{-(q-2)/q} e^{(k_7 \|u_0\|_2^{p-1} + k_2)t}.$$

In view of the condition (2.24), we see that the inequality (3.1) and the last inequality deduce that for every $q \geq 2$, $\|u(t)\|_q$ grows exponentially for all $0 < t < t_*$. □

Remark 3.2. In the case of $J_0 \leq 0$, the inequality (2.12) implies that

$$\frac{d}{dt} \|u(t)\|_2^2 \geq \frac{1}{c^*} \left(-2 + \frac{m}{2}\right) \|u(t)\|_2^2 + \left(2 - \frac{m}{p+1}\right) \|u(t)\|_{p+1}^{p+1},$$

with $4 < m < 2(p+1)$. Making use of the inequality (2.16) with $\mu = ((p+1)/2)|\Omega|^{-(p-1)/2} \|u_0\|_2^{p-1}$, the preceding inequality becomes

$$\frac{d}{dt} \|u(t)\|_2^2 \geq k_9 \|u\|_2^2 - k_{10} \tag{3.3}$$

with $k_9 = ((2(p+1) - m))/2|\Omega|^{-(p-1)/2} \|u_0\|_2^{p-1} + k_2$ and $k_{10} = ((p-1)(2(p+1) - m))/2(p+1)|\Omega|^{-(p-1)/2} \|u_0\|_2^{p+1}$. This inequality yields

$$\begin{aligned} \|u(t)\|_2^2 &\geq \|u_0\|_2^2 e^{k_9 t} + \frac{k_{10}}{k_9} (1 - e^{k_9 t}) \\ &= \frac{1}{k_9} [k_9 \|u_0\|_2^2 - k_{10}] e^{k_9 t} + \frac{k_{10}}{k_9} \\ &= \frac{1}{k_9} \left[\frac{(2(p+1) - m)}{p+1} |\Omega|^{-(p-1)/2} \|u_0\|_2^{p+1} + k_2 \|u_0\|_2^2 \right] e^{k_9 t} + \frac{k_{10}}{k_9}. \end{aligned}$$

For every $q > 2$, this inequality along with (3.2) yields

$$\|u(t)\|_q^2 \geq \frac{1}{k_9} \left[\frac{(2(p+1) - m)}{p+1} |\Omega|^{-(p-1)/2} \|u_0\|_2^{p+1} + k_2 \|u_0\|_2^2 \right] |\Omega|^{-(q-2)/q} e^{k_9 t}, \quad 0 < t < t_*.$$

Thus, we conclude that if $J_0 \leq 0$ and $\|u_0\|_2 > 0$, then for every $q \geq 2$, $\|u(t)\|_q$ grows exponentially for all $0 < t < t_*$.

Lemma 3.3. Assume that u is the weak solution of (1.1) and $u_0 \in H^2(\Omega)$ with $\|u_0\|_2 > 0$. Moreover, assume that $J_0 > 0$ and the condition (2.24) holds. Then, the solution does not extinct in finite time.

Proof. By considering the condition (2.24) in the strict sense, we can write the inequality (2.22) as follows:

$$\begin{aligned} \|u(t)\|_2^2 &\geq \frac{1}{(k_7 \|u_0\|_2^{p-1} + k_2)} \left[k_6 \|u_0\|_2^{p+1} + k_2 \|u_0\|_2^2 + k_4 (\mu_{\max})^{(p+1)/(p-1)} \right] \\ &= \frac{1}{(k_7 \|u_0\|_2^{p-1} + k_2)} \left[k_7 \|u_0\|_2^{p-1} + k_2 \right] \|u_0\|_2^2 \\ &= \|u_0\|_2^2 > 0, \quad t \geq 0. \end{aligned}$$

Here, we have used from the facts that $e^{(k_7 \|u_0\|_2^{p-1} + k_2)t} \geq 1$ and:

$$k_6 \|u_0\|_2^{p+1} + k_4 (\mu_{\max})^{(p+1)/(p-1)} = k_7 \|u_0\|_2^{p+1}.$$

Also, when the condition (2.24) holds in the equal sense, the inequality (2.22) implies that

$$\|u(t)\|_2^2 \geq \frac{k_4 (\mu_{\max})^{(p+1)/(p-1)} + k_5}{k_7 \|u_0\|_2^{p-1} + k_2} > 0, \quad t \geq 0.$$

We now apply the Hölder inequality for every $q > 1$ to obtain

$$0 < \|u(t)\|_2^2 \leq \|u(t)\|_q^2 \|u(t)\|_{\frac{q}{q-1}}^2, \quad t \geq 0.$$

Thus, there does not exist $\bar{t} > 0$ such that

$$\lim_{t \rightarrow \bar{t}} \|u(t)\|_q = 0. \quad \square$$

Remark 3.4. We see that in the case of $J_0 \leq 0$, the condition (2.24) is a trivial inequality. Thus in this case under the condition $\|u_0\|_2 > 0$, the result of Lemma 3.3 holds.

Conflicts of interest

Authors have no conflict of interest to declare.

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