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Short paper / Note

# Probabilistic representation of helicity in viscous fluids

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**Abstract.** It is shown that the helicity of three-dimensional viscous incompressible flow can be identified with the overall linking of the fluid's initial vorticity to the expectation of a stochastic mean field limit. The relevant mean field limit is obtained by following the Lagrangian paths in the stochastic Hamiltonian interacting particle system of Hochgerner [S. Hochgerner, *Proc. R. Soc. A* **474** (2018) 20180178].

**Keywords.** Navier–Stokes equations, Stochastic fluid mechanics, Hamiltonian mechanics, Helicity, Interacting particle systems.

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## 1. Introduction

The evolution of the velocity field,  $u = u(t, x) = u_t(x)$ , of a three-dimensional incompressible fluid with constant mass density,  $\rho = 1$ , is given by

$$\frac{\partial}{\partial t} u = -\nabla_u u - \nabla p + \nu \Delta u, \quad \operatorname{div} u = 0, \quad u(0, \cdot) = u_0, \quad (1.1)$$

where  $t \in [0, T]$ ,  $x \in \mathbb{R}^3$ ,  $\nabla_u u = \langle u, \nabla \rangle u = \sum_{j=1}^3 u^j \partial_j u$  is the covariant derivative in  $\mathbb{R}^3$ ,  $p = p(t, x)$  is the pressure determined by  $\operatorname{div} u = 0$ , and the smooth divergence-free vector field  $u_0$  is an initial condition which decays sufficiently rapidly at infinity. If  $\nu > 0$  then (1.1) is the incompressible Navier–Stokes equation, and if  $\nu = 0$  it is the incompressible Euler equation in  $\mathbb{R}^3$ .

The helicity of the fluid is defined by

$$\mathcal{H}_t = \int_{\mathbb{R}^3} \langle u_t, \operatorname{curl} u_t \rangle dx, \quad (1.2)$$

where  $dx$  is the Euclidean volume element in  $\mathbb{R}^3$ . Helicity is a topological quantity measuring the overall degree of linking and knotting of vortex lines (Moffatt *et al.* [1–3]).

Let  $u$  be a solution to the Euler equation ( $\nu = 0$ ) and consider the Lagrangian flow  $g = g_t(x)$  generated by

$$\frac{\partial}{\partial t} g_t = u_t \circ g_t, \quad g_0 = e,$$

where  $e$  is the identity map in  $\mathbb{R}^3$ . Then  $g_t$  is a curve in the group of volume preserving diffeomorphisms  $\text{SDiff}(\mathbb{R}^3)$ . Arnold [4] has shown that the Euler equation has the structure of an infinite-dimensional Hamiltonian system with configuration space  $\text{SDiff}(\mathbb{R}^3)$ . Moreover, the Hamiltonian function,  $\int_{\mathbb{R}^3} \langle u, u \rangle dx / 2$ , is invariant under the relabeling symmetry, which is given by composition from the right in  $\text{SDiff}(\mathbb{R}^3)$ . Thus Noether's theorem applies, and yields

$$u_t = \text{Ad}(g_t^{-1})^\top u_0, \quad (1.3)$$

where the transpose adjoint action,  $\text{Ad}(\cdot)^\top$ , is defined as follows: for a vector field,  $X$ , let  $PX = X - \nabla \Delta^{-1} \text{div} X$  be the Leray–Hodge projection onto the divergence-free part. Then  $\text{Ad}(h)^\top v = P \cdot (Th)^\top \cdot (v \circ h)$  where  $h \in \text{SDiff}(\mathbb{R}^3)$ ,  $(Th)^\top$  is the transpose matrix, and  $v$  is a divergence-free vector field. In fact,  $\text{Ad}(h)^\top$  is the transpose with respect to the  $L^2$  inner product to the adjoint action (inverse vector field pullback)  $\text{Ad}(h) : v \mapsto Th \cdot (v \circ h^{-1})$ . We remark that  $Th = (\partial_i h^j)_{j,i}$  will throughout refer to differentiation in the space variable, while time differentiation will be denoted by  $\partial/\partial t$  (ordinary),  $\delta_t$  (Stratonovich), or  $d_t$  (Ito calculus).

The transport equation (1.3) and the identity  $\text{curl} \text{Ad}(g^{-1})^\top = \text{Ad}(g) \text{curl}$  yield

$$\mathcal{H}_t = \int_{\mathbb{R}^3} \langle \text{Ad}(g_t^{-1})^\top u_0, \text{curl} \text{Ad}(g_t^{-1})^\top u_0 \rangle dx = \int_{\mathbb{R}^3} \langle u_0, \text{Ad}(g_t^{-1}) \text{curl} \text{Ad}(g_t^{-1})^\top u_0 \rangle dx = \mathcal{H}_0.$$

Hence Euler flow conserves helicity [1, 5].

In the viscous case,  $\nu > 0$ , helicity is generally not conserved. The precise mechanism whereby helicity changes under Navier–Stokes flow is subject to ongoing investigation [6–9].

This note studies helicity in the viscous case from the point of view of stochastic Hamiltonian interacting particle systems (SHIPS) as in [10–12]. These systems can be viewed as a stochastic perturbation (along Hamiltonian vector fields) of ideal fluid mechanics. Ideal fluid mechanics (Euler flow) preserves energy and helicity. Both are quadratic invariants, but they have different geometric origins. Energy conservation follows because the Hamiltonian coincides with the energy functional, but such a quantity is generally not preserved under stochastic Hamiltonian perturbations. On the other hand, helicity is a Casimir function (constant on coadjoint orbits), and Casimirs are preserved by stochastic Hamiltonian perturbations. Hence the SHIPS approach can be expected to possess a helicity type invariant.

In [11] it is shown that solutions of incompressible Navier–Stokes equation can be obtained as the mean field limit of these interacting particle systems, and one may wonder how the helicity preserving stochastic Hamiltonian construction gives rise to a flow with non-constant helicity. The observation of this paper is that helicity in viscous fluids may be considered as an average over cross-helicities of stochastically perturbed ideal flows. Moreover, the group structure in  $\text{SDiff}(\mathbb{R}^3)$  allows to identify this as the cross-helicity of initial vorticity and an average over backward–forward transports of the initial velocity.

Concretely, and to describe the stochastic Hamiltonian equations in question, fix a (large) integer  $N$  and consider for,  $\alpha = 1, \dots, N$ , the IPS

$$(\delta_t g_t^\alpha) \circ (g_t^\alpha)^{-1} = \frac{1}{N} \sum_{\beta=1}^N u_t^\beta \delta_t t + \sqrt{2\nu} \delta_t W_t^\alpha, \quad g_0^\alpha = e, \quad u_t^\alpha = \text{Ad}((g_t^\alpha)^{-1})^\top u_0, \quad (1.4)$$

where  $\delta_t$  denotes Stratonovich differentiation and  $(W^\alpha)$  is a sequence of  $N$  mutually independent Brownian motions in  $\mathbb{R}^3$ . The process  $g_t^\alpha$  takes values in the group of volume preserving diffeomorphisms,  $\text{SDiff}(\mathbb{R}^3)$ , and  $e$  is the identity diffeomorphism. The system (1.4) is presented in Section 2 from the Lie–Poisson point of view.

The process  $u_t^\alpha$  takes values (by construction) in the space of divergence-free vector fields,  $\text{SVect}(\mathbb{R}^3)$ . Further, it depends on  $N$  and we can consider the mean field limit,  $u_t^\infty = \lim_{N \rightarrow \infty} u_t^\alpha$ , which is a limit in probability [13–15]. In fact, since all particles are identical, it suffices to consider the limit for  $\alpha = 1$ . Theorem 2.4, which is a summary of [11], shows that a given solution,

$u_t$ , to the Navier–Stokes equation can be represented as  $u_t = E[u_t^\infty] = \lim_{N \rightarrow \infty} \sum_{\alpha=1}^N u_t^\alpha / N$ . Therefore, and also by analogy to the ideal fluid case (1.3), it makes sense to call  $\text{Ad}((g_t^\alpha)^{-1})^\top u_0$  the forward transport of  $u_0$ . That is, the initial condition is transported forward in time along the stochastic Lagrangian path  $g_t^\alpha$ . The IPS (1.4) arises from a decomposition of each infinitesimally small blob of fluid (at each  $x \in \mathbb{R}^3$ ) into  $N$  identical sub-blobs, and insisting that the sub-blobs follow their common center of mass while at the same time each undergoing their own stochastic process. See Section 2.5.

Fix an index  $\alpha$  and consider the fluid collection that is made up of all  $\alpha$ -sub-blobs. In Section 3.1 it is observed that helicity is indeed preserved along the corresponding stochastic flow (1.4), that is

$$\int_{\mathbb{R}^3} \langle u_t^\alpha, \text{curl } u_t^\alpha \rangle dx = \mathcal{H}_0$$

for all  $\alpha = 1, \dots, N$ . This yields the representation

$$\mathcal{H}_t = \lim_{N \rightarrow \infty} \sum_{1 \leq \alpha \neq \beta \leq N} \int_{\mathbb{R}^3} \langle \text{Ad}(g_t^\alpha)^\top \text{Ad}((g_t^\beta)^{-1})^\top u_0, \text{curl } u_0 \rangle dx / N^2 \quad (\text{Theorem 3.1})$$

for the helicity (1.2) of Navier–Stokes flow (see also Remark 3.2). Hence the initial velocity,  $u_0$ , is transported forward along a stochastic Lagrangian path,  $g_t^\beta$ , and then backwards along another path,  $g_t^\alpha$ . The result,  $\mathcal{H}_t$ , is obtained by averaging over the  $L^2$  inner products of the initial vorticity and all such backward–forward transports with  $1 \leq \alpha \neq \beta \leq N$ , and letting  $N$  tend to infinity. Helicity at time  $t$  is thus the average over all possible linkings of integral curves, corresponding to  $\text{curl Ad}(g_t^\alpha)^\top \text{Ad}((g_t^\beta)^{-1})^\top u_0$ , and initial vortex lines, corresponding to  $\text{curl } u_0$ . See also Section 3.3.

If we set  $\nu = 0$ , the Lagrangian paths  $g_t^\alpha$  are deterministic and coincide with each other, whence it follows that  $\text{Ad}(g_t^\alpha)^\top \text{Ad}((g_t^\beta)^{-1})^\top u_0 = u_0$  and Theorem 3.1 reduces to  $\mathcal{H}_t = \int_{\mathbb{R}^3} \langle u_0, \text{curl } u_0 \rangle dx = \mathcal{H}_0$ , which is the conservation of helicity in the inviscid case.

Consider the SDE that follows from (1.4) in the mean field limit as  $N \rightarrow \infty$ . To obtain this limit we may fix  $\alpha = 1$  since all interacting particles (i.e., sub-blobs) are identical. The result is the stochastic mean field system

$$(\delta_t g_t^\infty) \circ (g_t^\infty)^{-1} = E[u_t^\infty] \delta_t t + \sqrt{2\nu} \delta_t W_t, \quad g_0^\infty = e, \quad u_t^\infty = \text{Ad}((g_t^\infty)^{-1})^\top u_0 \quad (1.5)$$

for processes  $g_t^\infty$  in  $\text{SDiff}(\mathbb{R}^3)$  and  $u_t^\infty$  in  $\text{SVect}(\mathbb{R}^3)$ , and where  $W$  is Brownian motion in  $\mathbb{R}^3$ . It follows that  $u_t = E[u_t^\infty]$  satisfies the Navier–Stokes equation (Theorem 2.4).

The IPS approach in [10–12] is a Hamiltonian analogue of the Constantin and Iyer [16] representation of solutions to the Navier–Stokes equation via the stochastic Weber formula. In fact, Equation (1.5) is equivalent to [11] via the stochastic Noether theorem (cf. Theorem 2.4), and coincides, up to notation, with the stochastic Weber formula of [16, Theorem 2.2]. In the present context the  $\text{Ad}(\cdot)^\top$  notation is kept because the group theoretic formulation is helpful in the helicity calculations.

The system (1.5) leads to an expression for the mean field limit in Theorem 3.1, which is

$$\mathcal{H}_t = E \left[ \int_{\mathbb{R}^3} \langle \text{Ad}(h_t)^\top u_0, \text{curl } u_0 \rangle dx \right], \quad (\text{Theorem 3.3})$$

where the process  $h_t$  in  $\text{SDiff}(\mathbb{R}^3)$  is the solution to the SDE with random coefficients

$$(\delta_t h_t) \circ h_t^{-1} = \sqrt{2\nu} (T g_t^\infty)^{-1} \cdot \delta_t (B_t - W_t), \quad h_0 = e$$

and where  $B$  is a Brownian motion in  $\mathbb{R}^3$  which is independent of  $W$ . Section 3.3 contains a physical interpretation of these equations.

These results depend on the existence of the mean field limits under consideration. The existence of these limits is assumed, at least for a short period of time  $[0, T]$ , but not proven (in this paper). Constantin and Iyer [16] have shown short time existence for (1.5).

## 2. Stochastic Hamiltonian interacting particle system (SHIPS) and mean-field limit

The stochastic Hamiltonian approach that is presented in this section has been developed in [10–12]. This approach is a Hamiltonian analogy to the mean field Weber formula theory of Constantin and Iyer [16], and it is also related to Holm's variational principle for stochastic fluid mechanics [17]. Section 2.4 contains the mean field evolution equation for stochastic vorticity, which is a straightforward consequence of [11] but has not been presented in this form elsewhere. A brief explanation of the physical picture underlying the mean field approach is given in Section 2.5.

### 2.1. Diffeomorphism groups

We fix  $s > 5/2$  and let  $\text{SDiff}(\mathbb{R}^3)$  denote the infinite-dimensional  $C^\infty$ -manifold of volume preserving  $H^s$ -diffeomorphisms on  $\mathbb{R}^3$ . This space is a topological group, but not a Lie group since left composition is only continuous but not smooth. Right composition is smooth. Let

$$\mathfrak{g} = \text{SVect}(\mathbb{R}^3)$$

be the space of divergence-free vector fields on  $\mathbb{R}^3$  of class  $H^s$ . The tangent space of  $\text{SDiff}(\mathbb{R}^3)$  at the identity  $e$  consists of divergence-free and compactly supported vector fields, denoted by

$$T_e \text{SDiff}(\mathbb{R}^3) = \mathfrak{g}_0 = \text{SVect}(\mathbb{R}^3)_{\text{cp}}.$$

We use right multiplication  $R^g : \text{SDiff}(\mathbb{R}^3) \rightarrow \text{SDiff}(\mathbb{R}^3)$ ,  $k \mapsto k \circ g = kg$  to trivialize the tangent bundle  $T\text{SDiff}(\mathbb{R}^3) \cong \text{SDiff}(\mathbb{R}^3) \times \mathfrak{g}_0$ ,  $v_g \mapsto (g, (TR^g)^{-1}v_g)$ .

The  $L^2$  scalar product  $\langle\langle \cdot, \cdot \rangle\rangle$  on  $\mathfrak{g}_0$  is defined by

$$\langle\langle v, w \rangle\rangle = \int_{\mathbb{R}^3} \langle v(x), w(x) \rangle dx$$

for  $v, w \in \mathfrak{g}_0$ , where  $dx$  is the standard volume element in  $\mathbb{R}^3$ , and  $\langle \cdot, \cdot \rangle$  is the Euclidean inner product. Via  $R^g$  this can be extended to a right invariant Riemannian metric on  $\text{SDiff}(\mathbb{R}^3)$ . See [18–21].

### 2.2. Phase space

The configuration space of incompressible fluid mechanics on  $\mathbb{R}^3$  is  $\text{SDiff}(\mathbb{R}^3)$ . The corresponding phase space is trivialized via right multiplication as

$$T^* \text{SDiff}(\mathbb{R}^3) \cong \text{SDiff}(\mathbb{R}^3) \times \mathfrak{g}^*,$$

where  $\mathfrak{g}^*$  is defined as  $\mathfrak{g}^* = \Omega^1(\mathbb{R}^3)/d\mathcal{F}(\mathbb{R}^3)$ . Here  $\Omega^k(\mathbb{R}^3)$  are  $k$ -forms (of class  $H^s$ ),  $\mathcal{F}(\mathbb{R}^3)$  are functions (of class  $H^{s+1}$ ), and  $\Omega^1(\mathbb{R}^3)/d\mathcal{F}(\mathbb{R}^3)$  is the space of equivalence classes modulo exact one-forms. Elements in  $\mathfrak{g}^*$  will thus be denoted by  $[\xi]_{\mathfrak{g}^*}$  where  $\xi \in \Omega^1(\mathbb{R}^3)$  is a representative of the class in  $\mathfrak{g}^* = \Omega^1(\mathbb{R}^3)/d\mathcal{F}(\mathbb{R}^3)$ .

Let  $\flat : \text{Vect}(\mathbb{R}^3) \rightarrow \Omega^1(\mathbb{R}^3)$ ,  $X \mapsto X^\flat$  be the metric (musical) isomorphism with inverse  $\flat^{-1} = \sharp$ . Let  $P : \text{Vect}(\mathbb{R}^3) \rightarrow \text{SVect}(\mathbb{R}^3)$ ,  $X \mapsto X - \nabla \Delta^{-1} \text{div}(X)$  be the Hodge projection onto divergence-free vector fields. Then we obtain an isomorphism  $\mu : \mathfrak{g} \rightarrow \mathfrak{g}^*$ ,  $X \mapsto [X^\flat]_{\mathfrak{g}^*}$  with inverse  $\mu^{-1} : \mathfrak{g}^* \rightarrow \mathfrak{g}$ ,  $[\xi]_{\mathfrak{g}^*} \mapsto P\xi^\sharp$ .

**Remark 2.1.** The restriction of  $\mu$  to  $\mathfrak{g}_0$  does not induce an isomorphism of  $T\text{SDiff}(\mathbb{R}^3) \cong \text{SDiff}(\mathbb{R}^3) \times \mathfrak{g}_0$  and  $T^* \text{SDiff}(\mathbb{R}^3) \cong \text{SDiff}(\mathbb{R}^3) \times \mathfrak{g}^*$  since  $\mu^{-1}([\xi]_{\mathfrak{g}^*})$  need not be compactly supported.

### 2.3. SHIPS

In [11] a stochastic Hamiltonian interacting particle system is constructed which yields, in the mean field limit, the solution to the incompressible Navier–Stokes equation. Let  $N$  be the number

of interacting particles (or interacting blobs of fluid). The approach is Hamiltonian and the relevant phase space is

$$\mathcal{P}^N = (\text{SDiff}(\mathbb{R}^3) \times \mathfrak{g}^*)^N.$$

This space is equipped with the direct product symplectic structure obtained from the canonical symplectic form on each copy  $\text{SDiff}(\mathbb{R}^3) \times \mathfrak{g}^*$ . The direct product group  $\text{SDiff}(\mathbb{R}^3)^N$  acts on  $\mathcal{P}^N$  through the product action and there is a corresponding momentum map

$$J: \mathcal{P}^N \rightarrow (\mathfrak{g}^*)^N, \quad (g^\alpha, [\xi^\alpha]_{\mathfrak{g}^*})_{\alpha=1}^N \mapsto (\text{Ad}(g^\alpha)^* [\xi^\alpha]_{\mathfrak{g}^*})_{\alpha=1}^N, \quad (2.6)$$

where the coadjoint representation is determined by  $\langle \text{Ad}(g)^* [\xi]_{\mathfrak{g}^*}, X \rangle = \int_{\mathbb{R}^3} \langle \xi, \text{Ad}(g)X \rangle dx$  and  $\text{Ad}(g)X = Tg \cdot (X \circ g^{-1}) = (g^{-1})^* X$  for  $g \in \text{SDiff}(\mathbb{R}^3)$ ,  $[\xi]_{\mathfrak{g}^*} \in \mathfrak{g}^*$  and  $X \in \mathfrak{g}_0$ . That is,

$$\text{Ad}(g^\alpha)^* [\xi^\alpha]_{\mathfrak{g}_0^*} = [(g^\alpha)^* \xi^\alpha]_{\mathfrak{g}_0^*} = [(\xi^\alpha \circ g^\alpha) \cdot Tg^\alpha]_{\mathfrak{g}_0^*}.$$

The infinitesimal adjoint representation is given by  $\text{ad}(X).Y = [X, Y] = -L_X Y$  where  $L_X Y = \nabla_X Y - \nabla_Y X$  is the Lie derivative. The corresponding coadjoint representation  $\text{ad}(X)^*: \mathfrak{g}^* \rightarrow \mathfrak{g}^*$  is characterized by  $\langle \text{ad}(X)^* [\xi]_{\mathfrak{g}^*}, Y \rangle = \langle \langle P\xi^\sharp, [X, Y] \rangle \rangle$ . Thus,  $\text{ad}(X)^* [\xi]_{\mathfrak{g}^*} = [L_X \xi]_{\mathfrak{g}^*}$  where  $L_X \xi$  is the Lie derivative of a one-form.

Let  $e_1, e_2, e_3$  be the standard basis vectors in  $\mathbb{R}^3$ . In the following, the vectors  $e_j$  will be viewed as constant vector fields on  $\mathbb{R}^3$ . Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$  be a filtered probability space satisfying the usual assumptions as specified in [22]. All stochastic processes shall be understood to be adapted to this filtration. For  $\alpha = 1, \dots, N$  consider a sequence of mutually independent Brownian motions  $W^\alpha = \sum W^{j,\alpha} e_j$  in  $\mathbb{R}^3$ . The Stratonovich differential will be denoted by  $\delta_t$ .

The equations of motion for a path  $(g_t^\alpha, [\xi_t^\alpha]_{\mathfrak{g}^*})_{\alpha=1}^N$  in  $\mathcal{P}^N$  are given by the system of Stratonovich SDEs (see [11, Equations (2.11)–(2.12)]):

$$\delta_t g_t^\alpha = TR^{g_t^\alpha} \left( \frac{1}{N} \sum_{\beta=1}^N P(\xi_t^\beta)^\sharp \delta_t t + \varepsilon \sum_{j=1}^3 e_j \delta_t W^{j,\alpha} \right), \quad g_0^\alpha = e \quad (2.7)$$

$$\delta_t [\xi_t^\alpha]_{\mathfrak{g}_0^*} = -\text{ad} \left( \frac{1}{N} \sum_{\beta=1}^N P(\xi_t^\beta)^\sharp \right)^* [\xi_t^\alpha]_{\mathfrak{g}_0^*} \delta_t t - \varepsilon \sum_{j=1}^3 \text{ad}(e_j)^* [\xi_t^\alpha]_{\mathfrak{g}_0^*} \delta_t W^{j,\alpha}, \quad \xi_0^\alpha = u_0^b, \quad (2.8)$$

where  $\delta_t$  indicates Stratonovich differentiation,  $\varepsilon > 0$  is a constant,  $e$  is the identity diffeomorphism and  $u_0 \in \mathfrak{g}$  is a smooth deterministic and divergence-free vector field. This system depends on the empirical average  $(1/N) \sum_{\beta=1}^N P(\xi_t^\beta)^\sharp$  and is therefore an interacting particle system. In the following we assume that the mean field limit of this IPS exists such that the limit in probability,  $\lim_{N \rightarrow \infty} (1/N) \sum_{\beta=1}^N P(\xi_t^\beta)^\sharp$ , is a deterministic time-dependent vector field and satisfies the desired initial condition. Moreover, for each  $\alpha$ , the process  $[\xi_t^\alpha]_{\mathfrak{g}^*}$  converges, as  $N \rightarrow \infty$ , to a stochastic process  $[\xi_t]_{\mathfrak{g}^*}$ . Since the particles (i.e., fluid blobs) are identical, it suffices to consider  $[\xi_t^1]_{\mathfrak{g}^*}$ , that is  $[\xi_t]_{\mathfrak{g}^*} = \lim_{N \rightarrow \infty} [\xi_t^1]_{\mathfrak{g}^*}$ . It follows that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{\beta=1}^N P(\xi_t^\beta)^\sharp = PE[\xi_t]^\sharp =: u_t. \quad (2.9)$$

**Remark 2.2.** The process  $(g_t^\alpha, [\xi_t^\alpha]_{\mathfrak{g}^*})$  depends on  $N$ . It would thus be more concise to write

$$(g_t^{\alpha,N}, [\xi_t^{\alpha,N}]_{\mathfrak{g}^*})$$

such that  $[\xi_t]_{\mathfrak{g}^*} = \lim_{N \rightarrow \infty} [\xi_t^{1,N}]_{\mathfrak{g}^*}$ . However, to make the notation more readable the superscript  $N$  is omitted, but it is always tacitly implied. A solution to (2.7)–(2.8) will mean a strong solution on an interval  $[0, T]$  independent of  $N$  and such that the mean field limit exists. See [13–15, 23] for background on mean field SDEs.

**Remark 2.3.** The canonical symplectic form on  $\text{SDiff}(\mathbb{R}^3) \times \mathfrak{g}^*$  is only weakly symplectic. This follows from Remark 2.1 and implies that the induced homomorphism  $T(\text{SDiff}(\mathbb{R}^3) \times \mathfrak{g}^*) \rightarrow T^*(\text{SDiff}(\mathbb{R}^3) \times \mathfrak{g}^*)$  is only injective but not surjective. Hence the Hamiltonian vector field does

not exist for all functions on  $\text{SDiff}(\mathbb{R}^3) \times \mathfrak{g}^*$ . In fact, Equations (2.7)–(2.8) do not arise from a Hamiltonian vector field since neither  $(1/N) \sum_{\beta=1}^N P(\xi_t^\beta)^\sharp$  nor  $e_j$  are compactly supported. Not even the initial condition,  $u_0$ , is assumed to have compact support. In [11] this problem was circumvented by taking the torus (which is compact) as the fluid's domain. Therefore, in the present context, the Hamiltonian approach can be used only as a guiding principle. This means that, if  $(g_t^\alpha, [\xi_t^\alpha]_{\mathfrak{g}^*})$  is a solution to (2.7)–(2.8), the (Hamiltonian) conclusion  $J(g_t^\alpha, [\xi_t^\alpha]_{\mathfrak{g}^*}) = J(g_0^\alpha, [\xi_0^\alpha]_{\mathfrak{g}^*})$  has to be proved directly. Equation (2.8) is a stochastic Euler equation, and so restricting  $P(\xi_t^\alpha)^\sharp$  to be of compact support does not seem to be reasonable since solutions of the (deterministic) Euler equation are generally not expected to be compactly supported [24].

**Theorem 2.4 ([11]).** *Consider a solution  $(g_t^\alpha, [\xi_t^\alpha]_{\mathfrak{g}^*})$  of (2.7)–(2.8). Then:*

- (1) *The equations for the stochastic mean field limit of the interacting particle system (2.7)–(2.8) are*

$$\delta_t g_t = TR^{g_t} \left( u_t \delta_t t + \varepsilon \sum_{j=1}^3 e_j \delta_t W^j \right), \quad g_0 = e \quad (2.10)$$

$$\delta_t [\xi_t]_{\mathfrak{g}_0^*} = -\text{ad} \left( u_t \delta_t t + \varepsilon \sum_{j=1}^3 e_j \delta_t W_t^j \right)^* [\xi_t]_{\mathfrak{g}_0^*} \quad \xi_0 = u_0^\flat, \quad (2.11)$$

where  $(g_t, \xi_t) = \lim_{N \rightarrow \infty} (g_t^1, \xi_t^1)$  and  $u_t = E[P\xi_t^\sharp]$ .

- (2)  $u_t = \lim_{N \rightarrow \infty} (1/N) \sum_{\beta=1}^N P(\xi_t^\beta)^\sharp = E[P\xi_t^\sharp]$  solves the incompressible Navier–Stokes equation

$$\frac{\partial}{\partial t} u = -\nabla_u u - \nabla p + \nu \Delta u, \quad \text{div } u = 0, \quad (2.12)$$

where  $p$  is the pressure and  $\nu = \varepsilon^2/2$ . Conversely, if  $u_t$  satisfies (2.12) and  $g_t$  is defined by (2.10), then  $[\xi_t]_{\mathfrak{g}^*} = \text{Ad}(g_t^{-1})^* [u_0^\flat]_{\mathfrak{g}^*}$  is a solution to (2.11) and  $E[P\xi_t^\sharp] = u_t$ .

- (3) Assume  $(g_t^\alpha)$  satisfies (2.7). Then (2.8) holds if, and only if,  $\text{Ad}(g_t^\alpha)^* [\xi_t^\alpha]_{\mathfrak{g}^*} = [(g_t^\alpha)^* \xi_t^\alpha]_{\mathfrak{g}^*} = [u_0^\flat]_{\mathfrak{g}^*}$  for all  $\alpha = 1, \dots, N$ . In the limit,  $N \rightarrow \infty$ , this implies  $\text{Ad}(g_t)^* [\xi_t]_{\mathfrak{g}^*} = [u_0^\flat]_{\mathfrak{g}^*}$ .

**Proof.** These assertions are shown in [11] where item (3) follows because (2.7)–(2.8) constitute a right invariant Hamiltonian system whence the momentum map (2.6) is constant along solutions. In the present context (Remark 2.3) item (3) is shown directly:

For an arbitrary  $k$ -form  $\sigma$  we have the identity  $\delta_t (g_t^\alpha)^* \sigma = (g_t^\alpha)^* L_{(TR^{g_t^\alpha})^{-1} \delta_t g_t^\alpha} \sigma$ . Now,  $\text{Ad}(g_t^\alpha)^* [\xi_t^\alpha]_{\mathfrak{g}^*} = [(g_t^\alpha)^* \xi_t^\alpha]_{\mathfrak{g}^*} = [u_0^\flat]_{\mathfrak{g}^*}$  for all  $\alpha = 1, \dots, N$  holds if, and only if,

$$\delta_t \int_{\mathbb{R}^3} \langle (g_t^\alpha)^* \xi_t^\alpha, X \rangle dx = \int_{\mathbb{R}^3} \langle (g_t^\alpha)^* L_{(TR^{g_t^\alpha})^{-1} \delta_t g_t^\alpha} \xi_t^\alpha, X \rangle dx + \int_{\mathbb{R}^3} \langle (g_t^\alpha)^* \delta_t \xi_t^\alpha, X \rangle dx = 0$$

for all  $X \in \mathfrak{g}_0$ . Because of (2.7) the assertion follows.  $\square$

## 2.4. Vorticity formulation

The system (2.7)–(2.8) is a stochastic version of ideal incompressible flow. Consequently, the corresponding vorticity may be expected to be transported along the stochastic flow. In this section it is shown that this is indeed the case. The vorticity,  $\omega = \omega([\xi]_{\mathfrak{g}^*})$ , associated to an element  $[\xi]_{\mathfrak{g}^*} \in \mathfrak{g}^*$  is defined as

$$\omega = d\xi \in \mathcal{C}^2 \subset \Omega^2(\mathbb{R}^3),$$

where  $\mathcal{C}^2$  denotes the space of closed two-forms. If  $X = \mu^{-1}[\xi]_{\mathfrak{g}^*} = P\xi^\sharp$  and  $*$  is the Hodge star operator, then we have  $(*\omega)^\sharp = \nabla \times X$ , which is the expression of the vorticity when considered as a vector field. We thus obtain an isomorphism,  $[\xi]_{\mathfrak{g}^*} \mapsto \omega([\xi]_{\mathfrak{g}^*})$ , from  $\mathfrak{g}^*$  to  $\mathcal{C}^2$ . The induced coadjoint action on  $\mathcal{C}^2$  is given by pullback, that is  $\text{Ad}(g)^* \omega = g^* \omega = (\omega \circ g) \cdot \Lambda^2 Tg$ . The infinitesimal coadjoint action is given by the Lie derivative,  $\text{ad}(X)^* \omega = L_X \omega$ .

Let  $[\xi_t^\alpha]_{\mathfrak{g}^*}$ , for  $\alpha = 1, \dots, N$ , be a solution to (2.8). The Maurer–Cartan formula,  $L_X = di_X + i_X d$ , then implies that the vorticity,  $\omega_t^\alpha = d\xi_t^\alpha$ , satisfies

$$\delta_t \omega^\alpha = -\text{ad} \left( u^{(N)} \delta_t t + \varepsilon \sum_{j=1}^3 e_j \delta_t W^{j,\alpha} \right)^* \omega^\alpha = -L_{u^{(N)}} \omega^\alpha \delta_t t - \varepsilon \sum_{j=1}^3 L_{e_j} \omega^\alpha \delta_t W^{j,\alpha}, \quad (2.13)$$

where

$$u^{(N)} = \sum_{\beta=1}^N P(\xi^\beta)^\sharp / N = \sum_{\beta=1}^N BS \left( (*\omega^\beta)^\sharp \right) / N,$$

and  $(*\omega^\alpha)^\sharp$  is the divergence-free vector field associated to  $\omega^\alpha$  and  $BS$  is the Biot–Savart operator. The latter is defined as

$$BS(w)(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{w(y) \times (x - y)}{(x - y)^3} dy \quad (2.14)$$

for  $w \in \mathfrak{g} = \text{SVect}(\mathbb{R}^3)$ . Closed two-forms and divergence-free vector fields are in one-to-one correspondence via  $\omega \mapsto (*\omega)^\sharp$ . This can be used to define  $BS^* : \mathcal{C}^2 \rightarrow \Omega^1(\mathbb{R}^3)$ ,  $\omega \mapsto (BS((*\omega)^\sharp))^\flat$ , which satisfies  $BS^* d\xi = \xi$ .

Applying  $BS^*$  to (2.13) yields (2.8). Hence these equations are equivalent, and the former is the vorticity formulation of the latter. Furthermore, the constancy of the momentum map (2.6), i.e. Theorem 2.4(3), along solutions implies that the vorticity is transported along the stochastic flow:

$$\text{Ad}(g_t^\alpha)^* \omega_t^\alpha = \omega_0 \quad (2.15)$$

for all  $\alpha = 1, \dots, N$ . Note that the initial conditions are assumed to be independent of  $\alpha$ ,  $\omega_0^\alpha = \omega_0 = d\mu_0^\flat$  and  $g_0^\alpha = e$ .

Under the assumption that the mean field limit exists, we consider  $\omega_t = \lim_{N \rightarrow \infty} \omega_t^1$ . It follows that

$$\begin{aligned} \delta_t \omega_t &= -\text{ad} \left( u_t \delta_t t + \varepsilon \sum_{j=1}^3 e_j \delta_t W^j \right)^* \omega_t \\ u_t &= (BS^*(E[\omega_t]))^\sharp = \lim_{N \rightarrow \infty} \left( BS^* \left( \sum_{\alpha=1}^N \omega_t^\alpha / N \right) \right)^\sharp \end{aligned} \quad (2.16)$$

which is a mean field SDE because the drift depends on the expectation. If  $g_t = \lim g_t^1$  with  $g_0 = e$ , this can be restated as  $\omega_t = \text{Ad}(g_t^{-1})^* \omega_0$ . Moreover, since  $[BS^* \omega_t]_{\mathfrak{g}^*}$  satisfies (2.11), Theorem 2.4 implies that  $u$  is a solution to the incompressible Navier–Stokes equation (2.12).

## 2.5. Physical interpretation of SHIPS

The picture underlying the IPS (2.7)–(2.8) is that each infinitesimal blob of fluid is divided into  $N$  identical sub-blobs. These sub-blobs interact to follow their common center of mass and, at the same time, each undergo their own Brownian motion. The barycentric component of the motion is due to the  $\sum_{\beta=1}^N P(\xi_t^\beta)^\sharp / N$  part of the equation, while the stochastic perturbation is encoded in  $\varepsilon \sum e_j \delta_t W_t^{j,\alpha}$ .

In [11] the equations (2.7)–(2.8) are given the structure of a stochastic Hamiltonian system. In fact, [11] treats the case where the fluid's domain is a torus and the perturbation vectors are given by a certain infinite sequence of divergence-free vector fields. These are then interpreted as the velocities of molecules which impart their momenta on the sub-blobs. In the present context, because helicity is usually defined on simply-connected domains, the domain is chosen to be  $\mathbb{R}^3$ . Thus the stochastic perturbation,  $\varepsilon \sum e_j \delta_t W_t^{j,\alpha}$ , is interpreted as a model for the combined effect of molecules of different momenta hitting the sub-blob indexed by  $\alpha$ . Due to the non-compactness of  $\mathbb{R}^3$ , the Hamiltonian interpretation encounters structural



difficulties (Remark 2.3). Nevertheless, the central (and only) conclusion from the Hamiltonian approach, namely that the momentum map (2.6) is constant along solutions, still holds. In fact, Theorem 2.4(3) shows that (2.8) is equivalent to the preservation of the momentum map.

Each infinitesimal element,  $dx$ , is subdivided into a partition of  $N$  identical  $dx^\alpha$ , and the initial conditions in each  $dx^\alpha$  are given by  $(g_0^\alpha, [\xi_0^\alpha]_{\mathfrak{g}^*}) = (e, [u_0^b]_{\mathfrak{g}^*})$ , independently of  $\alpha$ . As this subdivision is carried out simultaneously for all infinitesimal elements  $dx$  in the domain, this may also be seen as  $N$  copies of the domain at each point in time. Thus for each  $t \in [0, T]$  we have a system of  $N$  interacting stochastic fluid states  $(g_t^\alpha, [\xi_t^\alpha]_{\mathfrak{g}^*})$ , and, as  $N$  becomes large, their average velocity converges to the macroscopically observed state  $u_t$ , which is the solution to the Navier–Stokes equation (2.12).

The construction is Hamiltonian (with the caveat mentioned in Remark 2.3) and therefore the state of each copy,  $(g_t^\alpha, \xi_t^\alpha)$ , evolves according to a stochastic Hamiltonian system. This does not, in general, mean that energy is conserved (and it is not for the case at hand as discussed in [11]). However, coadjoint orbits are conserved by stochastic Hamiltonian mechanics when the phase space is the dual of a Lie algebra. Via  $d : \mathfrak{g}^* \rightarrow \mathcal{C}^2$  the coadjoint orbits in  $\mathfrak{g}^*$  are isomorphic to sets of the form  $\{\text{Ad}(g)^* \omega : g \in \text{SDiff}(\mathbb{R}^3)\}$ . Hence, for each  $\alpha$ , vorticity is transported along flow lines, i.e. (2.15) holds.

### 3. Helicity representation

Let  $u = u(t, x)$ , with  $t \in [0, T]$  and  $x \in \mathbb{R}^3$ , be a solution to the incompressible Navier–Stokes equation. The helicity at time  $t \in [0, T]$  is

$$\mathcal{H}_t = \int_{\mathbb{R}^3} \langle u_t, \text{curl } u_t \rangle dx = \int_{\mathbb{R}^3} u_t^b \wedge \omega_t = \int_{\mathbb{R}^3} BS^*(\omega_t) \wedge \omega_t, \quad (3.17)$$

where the vorticity,  $\omega_t = du_t^b$ , and Biot–Savart operator  $BS^*$  have been defined in Section 2.4. The physical meaning of the following calculations is discussed in Section 3.3.

#### 3.1. SHIPS representation

Note that  $d : \mathfrak{g}^* \rightarrow \mathcal{C}$  has the equivariance property  $d \circ \text{Ad}(g)^* = \text{Ad}(g)^* \circ d$ , for all  $g \in \text{SDiff}(\mathbb{R}^3)$ . Define the map

$$I : \mathcal{C}^2 \rightarrow \mathbb{R}, \quad \omega \mapsto \int_{\mathbb{R}^3} BS^*(\omega) \wedge \omega.$$

It follows that  $I(\text{Ad}(g)^* \omega) = I(g^* \omega) = I(\omega)$  for all  $g \in \text{SDiff}(\mathbb{R}^3)$ . Let  $(g_t^\alpha, \omega_t^\alpha)_{\alpha=1}^N$  be a solution to the system (2.7), (2.13). Equation (2.15), which expresses the observation that the vorticity  $\omega_t^\alpha$  is transported along the stochastic flow  $g_t^\alpha$ , implies that

$$\sum_{\alpha=1}^N I(\text{Ad}(g_t^\alpha)^* \omega_t^\alpha) = \sum_{\alpha=1}^N I(\omega_0) = N \mathcal{H}_0. \quad (3.18)$$

Consider the mean field limit (2.16) of the system (2.7), (2.13). Hence the (time-dependent and deterministic) vector field  $u$ , defined in (2.16), satisfies the incompressible Navier–Stokes equation. Conversely, every solution  $u$  can be represented in this manner (Theorem 2.4).

Define

$$\lambda_t = \lim_{N \rightarrow \infty} \sum_{1 \leq \alpha \neq \beta \leq N} \text{Ad}\left((g_t^\alpha)^{-1} g_t^\beta\right)^* \omega_0 / N^2, \quad (3.19)$$

where the limit is taken in probability.

**Theorem 3.1.** *The helicity (3.17) of  $u$  satisfies*

$$\mathcal{H}_t = \int_{\mathbb{R}^3} BS^*(\lambda_t) \wedge \omega_0.$$

**Proof.** Indeed, the mean field vorticity formulation (2.16) yields

$$\begin{aligned}\mathcal{H}_t &= \int_{\mathbb{R}^3} BS^*(du_t^b) \wedge du_t^b = \int_{\mathbb{R}^3} BS^* \left( \lim_{N \rightarrow \infty} \sum_{\alpha} \omega_t^{\alpha} / N \right) \wedge \lim_{N \rightarrow \infty} \sum_{\beta} \omega_t^{\beta} / N \\ &= \lim_{N \rightarrow \infty} \left( \underbrace{\sum_{\alpha} \int_{\mathbb{R}^3} BS^*(\omega_t^{\alpha}) \wedge \omega_t^{\alpha} / N^2}_{\mathcal{H}_0 / N \rightarrow 0} + \sum_{\alpha \neq \beta} \int_{\mathbb{R}^3} BS^*(\omega_t^{\alpha}) \wedge \omega_t^{\beta} / N^2 \right) \\ &= \lim_{N \rightarrow \infty} \sum_{\alpha \neq \beta} \int_{\mathbb{R}^3} (g_t^{\beta})^* (BS^*(\omega_t^{\alpha}) \wedge \omega_t^{\beta}) / N^2 \\ &= \int_{\mathbb{R}^3} BS^*(\lambda_t) \wedge \omega_0,\end{aligned}$$

where  $\alpha$  and  $\beta$  range from 1 to  $N$ , and we have used the invariance property of  $I$  together with (2.15), which implies

$$(g_t^{\beta})^* (BS^*(\omega_t^{\alpha}) \wedge \omega_t^{\beta}) = ((g_t^{\beta})^* BS^*(\omega_t^{\alpha})) \wedge ((g_t^{\beta})^* \omega_t^{\beta}) = (Ad(g_t^{\beta})^* BS^*(Ad((g_t^{\alpha})^{-1})^* \omega_0)) \wedge \omega_0$$

and  $\int_{\mathbb{R}^3} (Ad(g_t^{\beta})^* BS^*(Ad((g_t^{\alpha})^{-1})^* \omega_0)) \wedge \omega_0 = \int_{\mathbb{R}^3} (BS^*(Ad(g_t^{\beta})^* Ad((g_t^{\alpha})^{-1})^* \omega_0)) \wedge \omega_0$ ; the last equality holds because  $d(Ad(g_t^{\beta})^* BS^* \omega_t^{\alpha} - BS^* Ad(g_t^{\beta})^* \omega_t^{\alpha}) = 0$ .  $\square$

**Remark 3.2.** The initial vorticity,  $\omega_0$ , is related to the curl of the initial velocity,  $u_0$ , as  $(*\omega_0)^{\sharp} = (*du_0^b)^{\sharp} = \text{curl } u_0$ . Further, we have  $\int_{\mathbb{R}^3} \langle (*\omega_0)^{\sharp}, u_0 \rangle dx = \int_{\mathbb{R}^3} \omega_0 \wedge u_0^b = \int_{\mathbb{R}^3} \langle \text{curl } u_0, u_0 \rangle dx$ . If  $X$  is a divergence-free vector field its pullback,  $h^*X = Th^{-1} \cdot (X \circ h) = Ad(h^{-1})X$ , by  $h \in \text{SDiff}(\mathbb{R}^3)$  is again divergence free. Hence Theorem 3.1 can be reformulated as

$$\begin{aligned}\mathcal{H}_t &= \int_{\mathbb{R}^3} (BS(*\lambda_t)^{\sharp}) \wedge u_0^b = \int_{\mathbb{R}^3} \langle BS(*\lambda_t)^{\sharp}, \text{curl } u_0 \rangle dx = \int_{\mathbb{R}^3} (* * \lambda_t) \wedge u_0^b \\ &= \lim_{N \rightarrow \infty} \sum_{1 \leq \alpha \neq \beta \leq N} \int_{\mathbb{R}^3} Ad(g_t^{\beta})^* Ad((g_t^{\alpha})^{-1})^* \omega_0 \wedge u_0^b / N^2 \\ &= \lim_{N \rightarrow \infty} \sum_{1 \leq \alpha \neq \beta \leq N} \int_{\mathbb{R}^3} \langle Ad((g_t^{\beta})^{-1}) Ad(g_t^{\alpha}) \text{curl } u_0, u_0 \rangle dx / N^2 \\ &= \lim_{N \rightarrow \infty} \sum_{1 \leq \alpha \neq \beta \leq N} \int_{\mathbb{R}^3} \langle Ad(g_t^{\alpha})^{\top} Ad((g_t^{\beta})^{-1})^{\top} u_0, \text{curl } u_0 \rangle dx / N^2.\end{aligned}$$

### 3.2. Mean field limit

To find the evolution equation characterizing the limit  $\lambda_t$  defined in (3.19), set  $h_t^{\beta, \alpha} = (g_t^{\beta})^{-1} g_t^{\alpha}$  and

$$\eta_t^{\beta, \alpha} = Ad((g_t^{\alpha})^{-1} g_t^{\beta})^* \omega_0 = Ad((h_t^{\beta, \alpha})^{-1})^* \omega_0. \quad (3.20)$$

The product formula for Stratonovich equations implies that the process  $h_t^{\beta, \alpha}$  in  $\text{SDiff}(\mathbb{R}^3)$  is the solution to

$$\begin{aligned}\delta_t h_t^{\beta, \alpha} &= TR^{h_t^{\beta, \alpha}} \cdot \left( - (Tg_t^{\beta})^{-1} \underbrace{(\delta_t g_t^{\beta})(g_t^{\beta})^{-1}}_{(2.7)} \underbrace{g_t^{\alpha}(h_t^{\beta, \alpha})^{-1}}_{g_t^{\beta}} + (Tg_t^{\beta})^{-1} \underbrace{(\delta_t g_t^{\alpha})(g_t^{\alpha})^{-1}}_{(2.7)} g_t^{\beta} \right) \\ &= \varepsilon TR^{h_t^{\beta, \alpha}} \cdot (Ad(g_t^{\beta})^{-1} \delta_t \hat{W}^{\alpha, \beta}) \\ h_0^{\beta, \alpha} &= e,\end{aligned} \quad (3.21)$$

where  $\hat{W}^{\alpha,\beta} = \sum e_j(W^{j,\alpha} - W^{j,\beta})$  is a difference of two independent Brownian motions. It is therefore, up to a multiplicative factor, again a Brownian motion with quadratic variation  $[\hat{W}^{\alpha,\beta}, \hat{W}^{\alpha,\beta}]_t = [W^\alpha, W^\alpha]_t + [W^\beta, W^\beta]_t = 2t$ .

Equation (3.21) implies that  $\eta_t^{\beta,\alpha}$  satisfies

$$\delta_t \eta_t^{\beta,\alpha} = -\varepsilon \operatorname{ad} \left( \operatorname{Ad}(g_t^\beta)^{-1} \delta_t \hat{W}^{\alpha,\beta} \right)^* \eta_t^{\beta,\alpha}, \quad \eta_0^{\beta,\alpha} = \omega_0. \quad (3.22)$$

Equations (3.21) and (3.22) are SDEs with random coefficients, since there is a dependence on realizations of  $g_t^\beta$ . But neither depend on  $g_t^\alpha$  for  $\alpha \neq \beta$ . Therefore, for all  $N$  and all  $\beta \leq N$ , the sequence  $\eta_t^{\beta,\alpha}$  with  $\alpha = 1, \dots, \hat{\beta}, \dots, N$  ( $\beta$  omitted) is i.i.d., and we have that

$$\sum_{\alpha \neq \beta, \alpha=1}^N \eta_t^{\beta,\alpha} \sim (N-1) \eta_t^{\beta,\beta+1}, \quad (3.23)$$

where  $\sim$  means equivalence in distribution. (For  $\beta = N$  the expression  $\eta_t^{\beta,\beta+1}$  does not make sense, thus one should write, e.g.,  $\sum_{\alpha=1}^{N-1} \eta_t^{N,\alpha} \sim (N-1) \eta_t^{N,N-1}$  for this case. However, below we will only need the case  $\beta = 1$  and so the inconsistency at  $\beta = N$  will be ignored from now on.)

Recall from Remark 2.2 that  $g_t^\beta$  depends on  $N$ . Let  $N$  go to infinity and assume  $g_t = \lim_{N \rightarrow \infty} g_t^\beta$  is the stochastic mean field limit (for an arbitrarily fixed  $\beta$ , e.g.  $\beta = 1$ ) of the IPS (2.7), given by the mean field SDE (2.10) and where the driving Brownian motion is denoted by  $W$ .

Equations (3.21) and (3.22) imply, respectively, that  $h_t = \lim_{N \rightarrow \infty} h_t^{\beta,\beta+1}$  satisfies

$$\delta_t h_t = \varepsilon T R^{h_t} \cdot (\operatorname{Ad}(g_t)^{-1} \delta_t \hat{W}), \quad h_0 = e \quad (3.24)$$

and that  $\eta_t = \operatorname{Ad}(h_t^{-1})^* \omega_0 = \lim_{N \rightarrow \infty} \eta_t^{\beta,\beta+1}$  satisfies

$$\delta_t \eta_t = -\varepsilon \operatorname{ad} \left( \operatorname{Ad}(g_t)^{-1} \delta_t \hat{W} \right)^* \eta_t, \quad \eta_0 = \omega_0, \quad (3.25)$$

where  $\hat{W} = B - W$  and  $B$  is a Brownian motion in  $\mathbb{R}^3$  independent of  $W$ .

Now, to find the evolution equation for (3.19), note that (3.23) yields

$$\lambda_t = \lim_{N \rightarrow \infty} \sum_{\beta=1}^N \sum_{\alpha \neq \beta, \alpha=1}^N \eta_t^{\beta,\alpha} / N^2 = \lim_{N \rightarrow \infty} \sum_{\beta=1}^N (N-1) \eta_t^{\beta,\beta+1} / N^2 = \lim_{N \rightarrow \infty} \sum_{\beta=1}^N \eta_t^{\beta,\beta+1} / N = E[\eta_t]. \quad (3.26)$$

Here we use the propagation of chaos property of mean field limits [13, 15] which implies independence of  $g_t^\beta$  in the limit as  $N \rightarrow \infty$ , such that the  $\eta_t^{\beta,\beta+1}$  are asymptotically i.i.d.

However, because (3.25) is an SDE with random coefficients, it does not have the Markov property and one cannot expect the evolution of  $\lambda_t = E[\eta_t]$  to be given by a deterministic PDE.

**Theorem 3.3.** *The helicity (3.17) satisfies*

$$\mathcal{H}_t = \int_{\mathbb{R}^3} BS^*(E[\eta_t]) \wedge \omega_0 = E \left[ \int_{\mathbb{R}^3} (\operatorname{Ad}(h_t^{-1})^* \omega_0) \wedge u_0^b \right] = E \left[ \int_{\mathbb{R}^3} \langle \operatorname{Ad}(h_t)^\top u_0, \operatorname{curl} u_0 \rangle dx \right].$$

Moreover,

$$\frac{\partial}{\partial t} E[\eta_t] = \varepsilon^2 E \left[ \operatorname{Ad}(g_t)^* \Delta \operatorname{Ad}(g_t^{-1})^* \eta_t \right], \quad (3.27)$$

where  $\Delta = (d + *d*)^2$  is the Laplacian.

**Proof.** The first part follows from (3.26) and Theorem 3.1, and because  $\int_{\mathbb{R}^3} (\operatorname{Ad}(h_t^{-1})^* \omega_0) \wedge u_0^b = \int_{\mathbb{R}^3} *(\operatorname{curl} u_0)^b \wedge \operatorname{Ad}(h_t)^* u_0^b = \int_{\mathbb{R}^3} \langle \operatorname{Ad}(h_t) \operatorname{curl} u_0, u_0 \rangle dx$ . For the second statement, it remains to transform (3.25) into Ito form. By definition of the Stratonovich integral (see [22]) it follows that

$$\begin{aligned} \eta_t - \eta_0 &= -\varepsilon \sum_{j=1}^3 \int_0^t \operatorname{ad}(\operatorname{Ad}(g_s)^{-1} e_j)^* \eta_s \delta_s \hat{W}_s^j \\ &= -\varepsilon \sum_{j=1}^3 \int_0^t \operatorname{ad}(\operatorname{Ad}(g_s)^{-1} e_j)^* \eta_s d_s \hat{W}_s^j - \frac{\varepsilon}{2} \sum_{j=1}^3 \left[ \operatorname{ad}(\operatorname{Ad}(g_s)^{-1} e_j)^* \eta_s, \hat{W}_s^j \right]_t, \end{aligned}$$

where  $d_t$  indicates Ito differentiation and  $[\cdot, \cdot]_t$  is the quadratic variation process. The product formula for Stratonovich SDEs applied to (2.7) and (3.25) implies

$$\delta_t (\text{ad}(g_t^{-1})^* \eta_t) = -\text{ad}(u \delta_t t + \varepsilon \delta_t B_t)^* \text{Ad}(g_t^{-1})^* \eta_t \quad (3.28)$$

whence

$$\begin{aligned} \delta_t (\text{ad}(\text{Ad}(g_t^{-1}) e_j)^* \eta_t) &= \delta_t (\text{Ad}(g_t)^* \text{ad}(e_j)^* \text{Ad}(g_t^{-1})^* \eta_t) \\ &= (\dots) \delta_t t + \varepsilon \sum_{k=1}^3 \text{Ad}(g_t)^* \text{ad}(e_k)^* \text{ad}(e_j)^* \text{Ad}(g_t^{-1})^* \eta_t \delta_t W^k \\ &\quad - \varepsilon \sum_{l=1}^3 \text{Ad}(g_t)^* \text{ad}(e_j)^* \text{ad}(e_l)^* \text{Ad}(g_t^{-1})^* \eta_t \delta_t B^l. \end{aligned}$$

Using [22, Ch. 2, Thm. 29],

$$\begin{aligned} \left[ \text{ad}(\text{Ad}(g_t^{-1}) e_j)^* \eta_t, \hat{W}_t^j \right]_t &= \varepsilon \sum_{k=1}^3 \left[ \int_0^t \text{Ad}(g_s)^* \text{ad}(e_k)^* \text{ad}(e_j)^* \text{Ad}(g_s^{-1})^* \eta_s d_t W_s^k, -\hat{W}_t^j \right]_t \\ &\quad - \varepsilon \sum_{l=1}^3 \left[ \int_0^t \text{Ad}(g_s)^* \text{ad}(e_j)^* \text{ad}(e_l)^* \text{Ad}(g_s^{-1})^* \eta_s d_t B_s^l, \hat{W}_t^j \right]_t \\ &= -2\varepsilon \int_0^t \text{Ad}(g_s)^* \text{ad}(e_j)^* \text{ad}(e_j)^* \text{Ad}(g_s^{-1})^* \eta_s ds. \end{aligned}$$

Therefore,

$$d_t \eta_t = \varepsilon^2 \text{Ad}(g_t)^* \Delta \text{Ad}(g_t^{-1})^* \eta_t d_t t - \varepsilon \text{ad}(\text{Ad}(g_t)^{-1} d_t \hat{W}_t)^* \eta_t \quad (3.29)$$

and the claim follows since  $E[\int_0^t \text{ad}(\text{Ad}(g_s)^{-1} d_t \hat{W}_s)^* \eta_s] = 0$ .  $\square$

**Remark 3.4.** Equation (3.28) has the same structure and initial condition as (2.16). But the driving Brownian motions are different, thus these equations do not imply path-wise equality of  $\text{Ad}(g_t^{-1})^* \eta_t$  and  $\omega_t$ .

The transpose of the adjoint operator,  $\text{ad}(\cdot)^\top : \mathfrak{g} \rightarrow \mathfrak{g}$ , is characterized by  $\langle \langle \text{ad}(X)^\top Y, Z \rangle \rangle = \langle \langle Y, \text{ad}(X)Z \rangle \rangle$  for  $X, Y \in \mathfrak{g}$  and  $Z \in \mathfrak{g}_0$ , and given by

$$\text{ad}(X)^\top Y = P(\nabla_X Y + (\nabla^\top X)Y),$$

where  $(\nabla^\top X)Y = \sum (\partial_i X^j) Y^j e_i$ . Consider now a solution,  $[\xi_t]_{\mathfrak{g}^*}$ , to the mean field equation (2.11) and let the corresponding vector field valued process be defined by  $u_t^W = P\xi_t^\sharp$ . Then  $u_t^W$  satisfies

$$\delta_t u_t^W = -\text{ad}(u_t \delta_t t + \varepsilon \delta_t W_t)^\top u_t^W = -P(\nabla_{u_t} u_t^W + (\nabla^\top u_t) u_t^W) \delta_t t - \sum_{k=1}^3 P \nabla_{e_k} u_t^W \delta_t W_t^k$$

with  $u_t = E[u_t^W]$  and where  $W$  is the same Brownian motion as in (2.10). Consider furthermore  $u_t^B$  defined by

$$\delta_t u_t^B = -\text{ad}(u_t \delta_t t + \varepsilon \delta_t B_t)^\top u_t^B = -P(\nabla_{u_t} u_t^B + (\nabla^\top u_t) u_t^B) \delta_t t - \sum_{k=1}^3 P \nabla_{e_k} u_t^B \delta_t B_t^k$$

with  $u_t = E[u_t^B] = E[u_t^W]$  and where  $B$  is the same as in (3.25). Comparing (2.16) and (3.28) then implies  $\text{curl } u_t^B = (*\text{Ad}(g_t^{-1})^* \eta_t)^\sharp$ . Since  $\varepsilon^2 = 2\nu$  Theorem 3.3 implies in particular that

$$\begin{aligned} \frac{\partial}{\partial t} \mathcal{H}_t &= \varepsilon^2 E \left[ \int_{\mathbb{R}^3} (\text{Ad}(g_t)^* \Delta \text{Ad}(g_t^{-1})^* \eta_t) \wedge B S^* \omega_0 \right] = \varepsilon^2 E \left[ \int_{\mathbb{R}^3} \underbrace{(\Delta \text{Ad}(g_t^{-1})^* \eta_t)}_{*(\text{curl } u_t^B)^\flat} \wedge \underbrace{\text{Ad}(g_t^{-1})^* \xi_0}_{(u_t^W)^\flat} \right] \\ &= \varepsilon^2 E \left[ \int_{\mathbb{R}^3} \langle \text{curl } \Delta u_t^B, u_t^W \rangle dx \right] = \varepsilon^2 \int_{\mathbb{R}^3} \langle \text{curl } \Delta E[u_t^B], E[u_t^W] \rangle dx = 2\nu \int_{\mathbb{R}^3} \langle \text{curl } \Delta u_t, u_t \rangle dx, \end{aligned} \quad (3.30)$$

where we use  $*\Delta = \Delta*$  and that  $u^W$  and  $u^B$  are independent. This coincides, of course, with the result of using the Navier–Stokes equation to obtain  $(\partial/\partial t)\mathcal{H}_t$  from the definition (3.17).

### 3.3. Physical interpretation

The proof of Theorem 3.1 offers two representations for the helicity  $\mathcal{H}_t$ :

The first is

$$\mathcal{H}_t = \lim_{N \rightarrow \infty} \sum_{1 \leq \alpha \neq \beta \leq N} \int_{\mathbb{R}^3} BS^*(\omega_t^\alpha) \wedge \omega_t^\beta / N^2. \quad (3.31)$$

Each integral of the form  $\int_{\mathbb{R}^3} BS^*(\omega_t^\alpha) \wedge \omega_t^\beta$  represents the overall linking of  $(*\omega_t^\alpha)^\sharp$ -lines to  $(*\omega_t^\beta)^\sharp$ -lines [1]. Let us refer to this quantity as simply the linking of  $\omega_t^\alpha$  and  $\omega_t^\beta$ . Then (3.31) says that helicity can be interpreted as the average linking of  $\omega_t^\alpha$  to  $\omega_t^\beta$  (over all pairs  $\alpha, \beta$  with  $\alpha \neq \beta$ ). Because each vorticity,  $\omega_t^\alpha$ , is transported along its own stochastic Lagrangian path,  $g_t^\alpha$ , the self-linking of  $\omega_t^\alpha$  remains constant. This is a consequence of (2.15), which implies that  $\int_{\mathbb{R}^3} BS^*(\omega_t^\alpha) \wedge \omega_t^\alpha = \mathcal{H}_0$  for all  $\alpha$ . However, as  $\omega_t^\alpha$  and  $\omega_t^\beta$  are driven by different Brownian motions for  $\alpha \neq \beta$ , the linking of their respective vortex lines can change as time progresses.

The second representation of helicity (compare also with Remark 3.2), is

$$\mathcal{H}_t = \int_{\mathbb{R}^3} BS^*(\lambda_t) \wedge \omega_0 = \lim_{N \rightarrow \infty} \sum_{1 \leq \alpha \neq \beta \leq N} \int_{\mathbb{R}^3} \left\langle \text{Ad}(g_t^\alpha)^\top \text{Ad}((g_t^\beta)^{-1})^\top u_0, \text{curl } u_0 \right\rangle dx / N^2. \quad (3.32)$$

This arises because of the transport property (2.15). Indeed, the stochastic flow,  $g_t^\beta$ , lies in the group of volume preserving diffeomorphisms, and this allows to pull-back  $\omega_t^\alpha$  along  $g_t^\beta$ . The result is (3.32), which is now the average overall linking of the initial vorticity,  $\omega_0$ , to the vorticities associated to backward–forward transports of  $u_0$ .

The second interpretation leads to the mean field limit  $\lambda_t = E[\eta_t]$ . This shows that  $\mathcal{H}_t$  equals the expectation of the cross-helicity of the stochastically transported initial condition,  $\text{Ad}(h_t)^\top u_0$ , and  $\text{curl } u_0$ .

### Conflicts of interest

The author has no conflict of interest to declare.

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