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Volume 350 (2022), p. 269-282

Published online: 16 June 2022

https://doi.org/10.5802/crmeca.117

Les Comptes Rendus. Mécanique *sont membres du*
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 1873-7234

Comptes Rendus
Mécanique
2022, 350, p. 269-282
https://doi.org/10.5802/crmeca.117

Short paper / *Note*

A singular non-Newton filtration equation with logarithmic nonlinearity: global existence and blow-up

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Abstract. In this paper, we study the initial-boundary value problem of the singular non-Newton filtration equation with logarithmic nonlinearity. By using the concavity method, we obtain the existence of finite time blow-up solutions at initial energy $J(u_0) \leqslant d$. Furthermore, we discuss the asymptotic behavior of the weak solution and prove that the weak solution converges to the corresponding stationary solution as $t \to +\infty$. Finally, we give sufficient conditions for global existence and blow-up of solutions at initial energy $J(u_0) > d$.

Keywords. Non-Newton filtration equation, Singular potential, Global existence, Blow-up, Logarithmic nonlinearity.

Manuscript received 7th November 2021, revised 25th March 2022 and 23rd May 2022, accepted 24th May 2022.

1. Introduction

The main purpose of this paper is to consider global existence and blow-up of solutions for the following singular non-Newton filtration equation with logarithmic nonlinearity:

$$\begin{cases} |x|^{-s}u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = |u|^{q-2}u\ln(|u|), & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1}$$

where the initial value $u_0(x) \in W_0^{1,p}(\Omega)$, $\Omega \subset \mathbb{R}^n$ $(n \geqslant p)$ is a bounded domain including the origin 0 with smooth boundary $\partial\Omega$, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with $|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$, and the parameters satisfy

$$p \geqslant 2, \quad 0 \leqslant s \leqslant 2, \quad p < q < p^* = \frac{np}{n-p}. \tag{2}$$

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With regard to physical phenomenon, the movement of a fluid with the sauce in a rigid porous medium according to some assumptions is described in [1]. Through the principle of conservation

$$a(x)u_t - \operatorname{div}(\overrightarrow{V}u) = f(u), \tag{3}$$

where $a(x)$ is the void of medium, $u(x,t)$ is the density of fluid, $\overrightarrow{V}$ is the velocity of filtration of fluid and $f(u)$ is the source. For the non-Newton fluid, we have the following p-Laplace equation $a(x)u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(u)$. When $a(x) = |x|^{-s}$, we obtain the following singular non-Newton filtration equation

$$|x|^{-s}u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(u). \tag{4}$$

In the past years, many researchers have paid attention to the above problem (4) (see [2–9]). When the source $f(u)$ is a polynomial nonlinearity, Tan [2] investigated the following non-Newton filtration equation with special medium

$$\frac{u_t}{|x|^2} - \Delta_p u = u^q, \tag{5}$$

where p, q satisfies $2 < p < n$, $p-1 < q < (np/(n-p))-1$. The existence and asymptotic estimates of a global solution and the finite time blow-up of the local solution of problem were obtained (5). Subsequently, Zhou [4] considered the following multi-dimensional porous medium equation with special void

$$|x|^{-s}u_t - \Delta u^p = u^q, \tag{6}$$

where $0 \le s \le 2$, $1 < p < q \le ((n+2)p)/(n-2)$. A sufficient condition for the global existence of the solution and two sufficient conditions for the blow-up in finite time of the solution were given.

When the source $f(u)$ is a logarithmic nonlinearity, Deng and Zhou [9] investigated the following semilinear heat equation with singular potential and logarithmic nonlinearity

$$|x|^{-s}u_t - \Delta u = u\ln|u| \tag{7}$$

under some appropriate initial-boundary value conditions. They made use of the Sobolev logarithmic inequality in [10] to treat the difficulties caused by the nonlinear logarithmic term. By virtue of a family of potential wells, the global existence and infinite time blow-up of the solutions were obtained. In addition, the equations with logarithmic nonlinearity are not scaling invariant, this has attracted the attention of many researchers. For more non-scaling-invariant semilinear heat equations refer to [11–13].

As we know that the global well-posedness of solution to the evolution equation strongly relies on the initial data, especially the initial energy, the energy functional $J(u)$ and Nehari functional $I(u)$ will be given in (8) and (9) respectively. We aim to conduct a comprehensive study in this paper on the global well-posedness of solution at subcritical and critical initial energy $J(u_0) \le d$, where d is potential depth, and supercritical initial energy $J(u_0) > 0$. Fortunately, Liao *et al.* [14] recently considered for the first time the initial-boundary value problem of the singular non-Newton filtration equation with logarithmic nonlinearity for problem (1), and obtained a few good results at subcritical and critical initial energy $J(u_0) \le d$. Their main results are as follows:

(i) If $J(u_0) \leqslant d$ and $I(u_0) \geqslant 0$, then the solution exists globally;
(ii) if $J(u_0) < 0$, then the solution blows up at finite time;
(iii) if $J(u_0) < M$ and $I(u_0) < 0$, then the solution blows up at finite time, where M will be given in (10).

Their results are encouraging to us, but there are still some problems that seem to be resolved. We thought deeply about the following issues:

(QS1) What is the property of the solution under the conditions $M \leqslant J(u_0) \leqslant d$ and $I(u_0) < 0$?
(QS2) Whether the global solution of the problem (1) converges as $t \longrightarrow \infty$?
(QS3) What is the property of the solution at supercritical initial energy $J(u) > d$?

In this paper, we will try our best to address the three issues discussed above. Our paper is organized as follows:

In Section 2, we introduce some preliminaries and lemmas.

In Section 3, we demonstrate our main result.

(i) The solution $u(t)$ of problem (1) blows up in finite time and the estimation of the upper bound of blow-up time T is obtained at subcritical initial energy $J(u) < d$;
(ii) global solution $u(x,t)$ converges to the stationary solution of problem (1) as $t \to +\infty$;
(iii) sufficient conditions for the global existence and finite blow-up of solutions are obtained at supercritical initial energy $J(u) > d$.

2. Preliminaries and lemmas

Throughout this paper, we denote the norm of $L^p(\Omega)$ for $1 \leqslant p \leqslant \infty$ by $\|\cdot\|_p$ and the norm of $W_0^{1,p}(\Omega)$ by $\|\nabla(\cdot)\|_p$. For $u \in L^p(\Omega)$,

$$\|u\|_p = \begin{cases} \left(\int_\Omega |u(x)|^p \,\mathrm{d}x\right)^{1/p}, & \text{if } 1 \leqslant p < \infty; \\ \operatorname{esssup}_{x\in\Omega}|u(x)|, & \text{if } p = \infty. \end{cases}$$

And we denote by $(\cdot,\cdot)$ the inner product in $L^2(\Omega)$. In this paper, c is an arbitrary positive number which may be different from line to line.

Here we give some important definitions as follows: for $u_0 \in W_0^{1,p}(\Omega)$, we define the energy functional J and Nehari functional I as follows:

$$J(u) = \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{q}\int_\Omega |u|^q \ln|u| \,\mathrm{d}x + \frac{1}{q^2}\|u\|_q^q, \tag{8}$$

$$I(u) = \|\nabla u\|_p^p - \int_\Omega |u|^q \ln|u| \,\mathrm{d}x. \tag{9}$$

From (8) and (9), we obtain

$$J(u) = \frac{1}{q}I(u) + \frac{q-p}{pq}\|\nabla u\|_p^p + \frac{1}{q^2}\|u\|_q^q. \tag{10}$$

Furthermore, we define the potential depth by

$$d = \inf_{u\in\mathcal{N}} J(u),$$

and the Nehari manifold $\mathcal{N} := \{u \in W_0^{1,p}(\Omega)\backslash\{0\} \mid I(u) = 0\}$. By [14], we know

$$d \geq M := \frac{q-p}{pq} r_*^p, \tag{11}$$

where $r_* = \sup_{0<\sigma\leq(np/(n-p))-q}(\sigma/B_\sigma^{q+\sigma})^{1/(q+\sigma-p)}$ and B_σ is the optimal embedding constant of $W_0^{1,p}(\Omega) \hookrightarrow L^{p+\sigma}(\Omega)$.

The potential well $\mathcal{W}$ and its corresponding set $\mathcal{V}$ are defined by

$$\mathcal{W} := \{u \in W_0^{1,p}(\Omega) \mid I(u) > 0, J(u) < d\} \cup \{0\} \tag{12}$$

$$\mathcal{V} := \{u \in W_0^{1,p}(\Omega) \mid I(u) < 0, J(u) < d\}. \tag{13}$$

To consider the weak solution with high energy level, we need to introduce some new notations.

$$J^\alpha = \{u \in W_0^{1,p}(\Omega) \mid J(u) < \alpha\}, \tag{14}$$

$$\mathcal{N}_\alpha = \mathcal{N} \cap J^\alpha = \left\{u \in \mathcal{N} \mid \left(\frac{1}{p} - \frac{1}{q}\right)\|\nabla u\|_p^p + \frac{1}{q^2}\|u\|_q^q < \alpha\right\} \quad \text{for all } \alpha > d, \tag{15}$$

and

$$\lambda_\alpha = \inf\left\{\frac{1}{2}\int_\Omega |x|^{-s}|u|^2\,\mathrm{d}x \mid u\in\mathcal{N}_\alpha\right\},\quad \Lambda_\alpha = \sup\left\{\frac{1}{2}\int_\Omega |x|^{-s}|u|^2\,\mathrm{d}x \mid u\in\mathcal{N}_\alpha\right\}\quad \text{for all } \alpha > d, \tag{16}$$

where λ_α and Λ_α are well defined. Clearly, λ_α and Λ_α admit the following properties

$$\sigma \mapsto \lambda_\sigma \text{ is nonincreasing},\quad \sigma\mapsto \Lambda_\sigma \text{ is nondecreasing}. \tag{17}$$

Next we give the definitions of the weak solution and blow-up of the problem (1) as follows.

Definition 1 (Weak solution). $u = u(x,t) \in L^\infty([0,T], W_0^{1,p}(\Omega))$ with $|x|^{-s/2}u_t \in L^2([0,T], L^2(\Omega))$, is said to be a weak solution of problem (1) on $\Omega\times[0,T)$, if it satisfies the initial condition $u(x,0) = u_0(x)$, and

$$(|x|^{-s}u_t,\phi) + (|\nabla u|^{p-2}\nabla u,\nabla\phi) = (|u|^{q-2}u\ln|u|,\phi) \tag{18}$$

for any $\phi\in W_0^{1,p}(\Omega)$. Moreover,

$$\int_0^t \||x|^{-s/2}u_\tau\|_2^2\,\mathrm{d}\tau + J(u(x,t)) = J(u_0). \tag{19}$$

Remark 2. For the global weak solution $u(t) = u(x,t)$ of problem (1), we define the ω-limit set of u_0 by

$$\omega(u_0) = \bigcap_{t\geqslant 0}\overline{\{u(s): s\geqslant t\}}.$$

Definition 3 (Maximal existence time). Let $u(t)$ be a weak solution of problem (1). We define the maximal existence time T of $u(t)$ as follows

(i) If $u(t)$ exists for $0\le t<\infty$, then $T = +\infty$;
(ii) if there exists a $t_0\in(0,\infty)$ such that $u(t)$ exists for $0\leqslant t<t_0$, but does not exist at $t = t_0$, then $T = t_0$.

Definition 4 (Finite time blow-up). Let $u(x,t)$ be a weak solution of problem (1). We say $u(x,t)$ blows up in finite time if the maximal existence time T is finite and $\lim_{t\to T}\|u\|^2_{H_0^1(\Omega)} = +\infty$.

The following lemmas will be used for our main goals.

Lemma 5. *Let σ be a positive number, then the following inequality holds*

$$\log x \leqslant \frac{e^{-1}}{\sigma}x^\sigma$$

for all $x\in(0,+\infty)$.

Lemma 6 ([15]). (i) *For any function $u\in W_0^{1,p}(\Omega)$, we have the inequality*

$$\|u\|_q \leqslant B_{p,q}\|\nabla u\|_p$$

for all $q\in[1,\infty)$ if $n\leqslant p$, and $1\leqslant q\leqslant np/(n-p)$ if $n>p$. The best constant $B_{q,p}$ depends only on Ω, n, p and q. We will denote the constant $B_{q,p}$ by B_q.

(ii) *Let $2\leqslant p<q<p^*$. For any $u\in W_0^{1,p}(\Omega)$ we have*

$$\|u\|_q\leqslant c\|\nabla u\|_p^\alpha\||u||_2^{1-\alpha},$$

where c is a positive constant and $\alpha = ((1/2)-(1/q))((1/n)-(1/p)+(1/2))^{-1}$.

Lemma 7. *For any $\alpha>d$, λ_α and Λ_α defined in (16) satisfy $0<\lambda_\alpha\leqslant\Lambda_\alpha<+\infty$.*

Proof. For any $u\in\mathcal{N}_\alpha$, using Hardy–Sobolev inequality and Hölder inequality, we obtain

$$\frac{1}{2}\int_\Omega |x|^{-s}|u|^2\,\mathrm{d}x \leqslant \frac{c}{2}\left(\int_\Omega |\nabla u|^{2n/(n+2-s)}\,\mathrm{d}x\right)^{(n+2-s)/n}. \tag{20}$$

Taking that $2n/(n+2-s) \leqslant p$ and $p \geqslant 2$, then

$$\frac{c}{2}\left(\int_\Omega |\nabla u|^{2n/(n+2-s)}\,\mathrm{d}x\right)^{(n+2-s)/n} \leqslant \frac{c}{2}|\Omega|^{(2n/(n+2-s)-(2/p))}\|\nabla u\|_p^2.$$

From (15), we get

$$\frac{c}{2}\left(\int_\Omega |\nabla u|^{2n/(n+2-s)}\,\mathrm{d}x\right)^{(n+2-s)/n} \leqslant \frac{c}{2}|\Omega|^{(2n/(n+2-s)-(2/p))}\left(\frac{\alpha pq}{q-p}\right)^{2/p} < +\infty. \tag{21}$$

From (20) and (21), we have $\Lambda_\alpha = \sup_{u\in N_\alpha}(1/2)\int_\Omega |x|^{-s}|u|^2\,\mathrm{d}x < +\infty$.

On the other hand, since Ω is a bounded domain in $\mathbb{R}^n$, there exists a positive constant ρ such that

$$|x| = \sqrt{x_1^2+x_2^2+\cdots+x_n^2} \leqslant \rho, \quad \forall x \in \overline{\Omega}, \tag{22}$$

then we have

$$\int_\Omega |x|^{-s}|u|^2\,\mathrm{d}x \geqslant \rho^{-s}\int_\Omega |u|^2\,\mathrm{d}x = \rho^{-s}\|u\|_2^2. \tag{23}$$

We apply Lemmas 5 and 6 again to show that

$$\int_\Omega |u|^q \ln|u|\,\mathrm{d}x \leqslant c\|u\|_{q+\sigma}^{q+\sigma} \leqslant c\|\nabla u\|_p^{\alpha(q+\sigma)}\|u\|_2^{(1-\alpha)(q+\sigma)}, \tag{24}$$

where $\sigma > 0$ is suitably small such that $q+\sigma < p^* = np/(n-p)$ and $p-\alpha(q+\sigma) > 0$, $\alpha = ((1/2)-(1/(q+\sigma)))((1/n)-(1/p)+(1/2)) \in (0,1)$.

Therefore, for any $u \in \mathcal{N}_\alpha (\alpha > d)$, we obtain from (15) that

$$\|\nabla u\|_p^p = \int_\Omega u^q \ln|u|\,\mathrm{d}x \leqslant c\|\nabla u\|_p^{\alpha(q+\sigma)}\|u\|_2^{(1-\alpha)(q+\sigma)},$$

which yields that

$$\|\nabla u\|_p^{p-\alpha(q+\sigma)} \leqslant c\|u\|_2^{(1-\alpha)(q+\sigma)}. \tag{25}$$

From Lemmas 5 and 6, we have

$$\int_\Omega |u|^q \ln|u|\,\mathrm{d}x \leqslant c\|u\|_{q+\sigma}^{q+\sigma} \leqslant c\|\nabla u\|_p^{(q+\sigma)}. \tag{26}$$

For any $u \in \mathcal{N}_\alpha$ and $q > p$, then $\|\nabla u\|_p^p = \int_\Omega u^q \ln|u|\,\mathrm{d}x \leqslant c\|\nabla u\|_p^{q+\sigma}$, i.e., $\|\nabla u\|_p \geqslant c$. By virtue of (23) and (25), $\int_\Omega |x|^{-s}|u|^2\,\mathrm{d}x > 0$. Then we have $\lambda_\alpha = \inf_{u\in N\alpha}(1/2)\int_\Omega |x|^{-s}|u|^2\,\mathrm{d}x > 0$. Finally, by the definition of λ_α and Λ_α, it is easy to see that $\lambda_\alpha \leqslant \Lambda_\alpha$, so Lemma 7 is proved. □

Lemma 8. *For any $u \in \mathcal{N}_+$, we have $J(u_0) > 0$. Furthermore, for any $\alpha > 0$ and $u \in J^\alpha \bigcap \mathcal{N}_+$, it holds that*

$$\|\nabla u\|_p \leqslant \left(\frac{pq}{q-p}\alpha\right)^{1/p}.$$

Proof. By the definition of $\mathcal{N}_+$, we have $I(u) > 0$, i.e., $\|\nabla u\|_p^p > \int_\Omega |u|^q \ln|u|\,\mathrm{d}x$. Since $p < q$, we get $1/p\|\nabla u\|_p^p > (1/q)\int_\Omega |u|^q \ln|u|\,\mathrm{d}x$. Then it follows from the definition of $J(u)$ that $J(u) > 0$.

On the other hand, for any $u \in J^\alpha \bigcap \mathcal{N}_+$, i.e. $J(u) < \alpha$ and $I(u) > 0$, we have $\alpha > J(u) = (1/q)I(u) + (q-p)/pq\|u\|_p^p + 1/q^2\|u\|_q^q > (q-p)/pq\|\nabla u\|_p^p$, i.e., $\|\nabla u\|_p \leqslant ((pq/(q-p))\alpha)^{1/p}$. □

Lemma 9 ([14]). *Let (2) hold and $u_0(x) \in W_0^{1,p}(\Omega)$. Assume that u is a weak solution of problem (1) in $\Omega \times [0,T)$.*

(i) *If $J(u_0) < d$ and $I(u_0) > 0$, then $u(t) \in \mathcal{W}$ for $0 \leqslant t < T$;*
(ii) *if $J(u_0) < d$ and $I(u_0) < 0$, then $u(t) \in \mathcal{V}$ for $0 \leqslant t < T$.*

Lemma 10 ([14]). *Let u be a weak solution of problem (1). Then for all $t \in [0,T)$,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\||x|^{-s/2}u\|_2^2 = -2I(u).$$

Lemma 11 ([16]). *Let (2) hold and $u \in W_0^{1,p}(\Omega)$ satisfy $I(u) < 0$, then there exists a $\lambda^* \in (0,1)$ such that $I(\lambda^* u) = 0$.*

Lemma 12 ([17]). *Suppose that $0 < T \leqslant \infty$ and suppose a non-negative function $F(t) \in C[0,T)$ satisfy*

$$F''(t)F(t) - (1+\gamma)(F'(t))^2 \geq 0$$

for some constant $\gamma > 0$. If $F(0) > 0$, $F'(0) > 0$, then

$$T \leq \frac{F(0)}{\gamma F'(0)} < \infty$$

and $F(t) \to \infty$ as $t \to T$.

3. Main results

Theorem 13. *Let (2) hold. If $J(u_0) \leqslant d$, and u is a weak solution to problem (1), then u blows up at finite time T with*

$$T \leqslant \frac{4(q-1)\||x|^{-s/2}u_0\|_2^2}{q(d-J(u_0))(q-2)^2}.$$

Proof. Step 1: Blow-up in finite time

For $J(u_0) \leqslant d$, we are going to discuss two cases.

Case 1. $J(u_0) < d, I(u_0) < 0$. By contradiction, we supposed that u is global weak solution of problem (1) with $I(u_0) < 0, J(u_0) < d$, then $T_{\max} = +\infty$. First, we define

$$G(t) = \int_0^t \||x|^{-s/2}u(\tau)\|_2^2 \,\mathrm{d}\tau, \quad \text{for all } t \geqslant 0. \tag{27}$$

Through a direct calculation, we have

$$G'(t) - G'(0) = \||x|^{-s/2}u\|_2^2 - \||x|^{-s/2}u_0\|_2^2 = 2\int_0^t (|x|^{-s/2}u_\tau, |x|^{-s/2}u)\,\mathrm{d}\tau \tag{28}$$

and

$$G''(t) = (|x|^{-s/2}u_t, |x|^{-s/2}u) = -2I(u). \tag{29}$$

It follows from (10) and (19) that

$$\begin{aligned} G''(t) = -2I(u) &= -2qJ(u) + \frac{2}{q}\|u\|_q^q + \left(\frac{2q}{p} - 2\right)\|\nabla u\|_p^p \\ &\geq -2qJ(u_0) + 2q\int_0^t \||x|^{-s/2}u_\tau\|_2^2\,\mathrm{d}\tau + \frac{2}{q}\|u\|_q^q + \left(\frac{2q}{p} - 2\right)\|\nabla u\|_p^p. \end{aligned} \tag{30}$$

From the Lemmas 9 and 11, we get $I(u(t)) < 0$, $t \geqslant 0$, then there exists a $\lambda^* \in (0,1)$ such that $I(\lambda^* u) = 0$. Therefore, by the definition of d, it follows that

$$\left(\frac{1}{p} - \frac{1}{q}\right)\|\nabla u\|_p^p + \frac{1}{q^2}\|u\|_q^q \geq \left(\frac{1}{p} - \frac{1}{q}\right)\lambda_*{}^p\|\nabla u\|_p^p + \frac{1}{q^2}\lambda_*{}^q\|u\|_q^q = J(\lambda_* u) \geq d. \tag{31}$$

Combining (30) and (31), we have

$$G''(t) \geq 2q\int_0^t \||x|^{-s/2}u_\tau\|_2^2\,\mathrm{d}\tau + 2q(d - J(u_0)). \tag{32}$$

By (29) and $I(u) < 0$, then $G''(t) = -2I(u) > 0$, so

$$G'(t) > G'(0) = \||x|^{-s/2}u_0\|_2^2 > 0, \quad \text{for all } t > 0. \tag{33}$$

From (28) and Hölder's inequality, we obtain

$$\frac{1}{4}(G'(t) - G'(0))^2 \leqslant \int_0^t \||x|^{-s/2}u\|_2^2\,\mathrm{d}\tau \int_0^t \||x|^{-s/2}u_\tau\|_2^2\,\mathrm{d}\tau. \tag{34}$$

Combining (27), (32) and (34), we get

$$
\begin{aligned}
G(t)G''(t) &\geqslant 2q\int_0^t \||x|^{-s/2}u\|_2^2\,\mathrm{d}\tau \int_0^t \||x|^{-s/2}u_\tau\|_2^2\,\mathrm{d}\tau + 2q(d-J(u_0))G(t) \\
&\geqslant \frac{q}{2}(G'(t)-G'(0))^2 + 2q(d-J(u_0))G(t). \qquad (35)
\end{aligned}
$$

By (33), we get

$$
G(t) \geqslant G(t_0) \geqslant \||x|^{-s/2}u_0\|_2^2 t_0 > 0, \quad \text{for all } t \geqslant t_0. \qquad (36)
$$

Furthermore, combining (35), (36) and $J(u_0) < d$, we have

$$
G(t)G''(t) - \frac{q}{2}(G'(t)-G'(0))^2 \geqslant 2q(d-J(u_0))\||x|^{-s/2}u_0\|_2^2 t_0 > 0, \quad \text{for all } t \geqslant t_0. \qquad (37)
$$

Next, we define $F(t) = G(t) + (T-t)\||x|^{-s/2}u_0\|_2^2$, for all $t \in [0,T]$, then

$$
F(t) \geqslant G(t) > 0, \quad F'(t) = G'(t) - G'(0) \quad \text{and} \quad F''(t) = G''(t) > 0, \quad \text{for all } t \in [0,T]. \qquad (38)
$$

By (37) and (38), we get

$$
F(t)F''(t) - \frac{q}{2}(F'(t))^2 \geqslant 2q(d-J(u_0))\||x|^{-s/2}u_0\|_2^2 t_0 > 0, \quad \text{for all } t \in [t_0,T]. \qquad (39)
$$

Let $y(t) = F(t)^{-(q-2)/2}$, then

$$
y'(t) = -\frac{q-2}{2}(F(t))^{-q/2}F'(t) \quad \text{and} \quad y''(t) = -\frac{q-2}{2}F^{-(q+2)/2}\Big(F(t)F''(t) - \frac{q}{2}(F'(t))^2\Big). \qquad (40)
$$

By (33), (38) and (39), we get $y''(t) < 0$, $t \in [t_0,T]$. Since $y(t_0) > 0$, $y'(t_0) < 0$, then $T_* \in [0,T)$ exists such that $\lim_{t\to T_*^-} y(t) = 0$ if we choose T sufficiently large. Consequently, we obtain $\lim_{t\to T_*^-} \||x|^{-s/2}u\|_2^2 = +\infty$.

Case 2. $J(u_0) = d, I(u_0) < 0$. From continuities of $J(u)$ and $I(u)$ with respect to t, we know that there exists a sufficiently small $t_1 \in (0,+\infty)$ such that $J(u(t)) > 0$ and $I(u(t)) < 0$ for $t \in [0,t_1]$. By $(|x|^{-s}u_t, u) = -I(u)$, we have $(|x|^{-s}u_t, u) > 0$ and $\||x|^{-s/2}u_t\|_2^2 > 0$ for $t \in [0,t_1]$. From (19), we have $0 < J(u(t_1)) \leqslant d - \int_0^{t_1} \||x|^{-s/2}u_\tau\|_2^2\,\mathrm{d}\tau < d$. Thus, we take t_1 as the initial time, then the remaining proof is similar to the proof of Case 1.

Step 2: Upper bound estimation of the blow-up time.

We next give an upper bound estimation of T. Suppose $u(t)$ be a solution of problem (1) with initial value u_0 satisfying $I(u_0) < 0$ and $J(u_0) < d$. From the Lemma 9, we get $u(t) \in \mathcal{V}$, $\forall\, t \in [0,T)$, i.e., $I(u(t)) < 0$, $t \in [0,T)$. We define a functional as follows:

$$
H(t) = \int_0^t \||x|^{-s/2}u\|_2^2\,\mathrm{d}t + (T-t)\||x|^{-s/2}u_0\|_2^2 + \beta(t+\gamma)^2, \quad \text{for all } t \in [0,T). \qquad (41)
$$

By $\mathrm{d}/\mathrm{d}t\||x|^{-s/2}u\|_2^2 = -2I(u(t)) < 0$, for all $t \in [0,T)$, we get

$$
\begin{aligned}
H'(t) &= \||x|^{-s/2}u\|_2^2 - \||x|^{-s/2}u_0\|_2^2 + 2\beta(t+\gamma) \\
&\geqslant 2\beta(t+\gamma) > 0, \quad \text{for all } t \in [0,T) \qquad (42)
\end{aligned}
$$

and

$$
H(t) \geqslant H(0) = T\||x|^{-s/2}u_0\|_2^2 + \beta\gamma^2, \quad \text{for all } t \in [0,T). \qquad (43)
$$

Combining (32) and Lemma 10, we have

$$
\begin{aligned}
H''(t) &= -2I(u(t)) + 2\beta > 2q(d-J(u(t))) + 2\beta \\
&= 2q(d-J(u_0)) + 2q\int_0^t \||x|^{-s/2}u_\tau\|_2^2\,\mathrm{d}\tau + 2\beta, \quad \text{for all } t \in [0,T). \qquad (44)
\end{aligned}
$$

By Hölder's inequality,

$$\begin{aligned}\frac{1}{2}\int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau}\||x|^{-s/2}u(\tau)\|_2^2\,\mathrm{d}\tau &= \int_0^t (|x|^{-s/2}u_\tau, |x|^{-s/2}u)\,\mathrm{d}\tau \\ &\leqslant \int_0^t \||x|^{-s}u_\tau\|_2\||x|^{-s}u\|_2\,\mathrm{d}\tau \\ &\leqslant \left(\int_0^t \||x|^{-s}u_\tau\|_2^2\,\mathrm{d}\tau\right)^{1/2}\left(\int_0^t \||x|^{-s}u\|_2^2\,\mathrm{d}\tau\right)^{1/2}, \quad \text{for all } t\in[0,T). \quad (45)\end{aligned}$$

Furthermore,

$$\begin{aligned}&(H(t)-(T-t)\||x|^{-s/2}u_0\|_2^2)\left(\int_0^t \||x|^{-s/2}u_\tau\|_2^2\,\mathrm{d}\tau+\beta\right)\\ &=\left(\int_0^t \||x|^{-s/2}u\|_2^2\,\mathrm{d}\tau+\beta(t+\gamma)^2\right)\left(\int_0^t \||x|^{-s/2}u_\tau\|_2^2\,\mathrm{d}\tau+\beta\right)\\ &=\int_0^t \||x|^{-s/2}u\|_2^2\,\mathrm{d}\tau\int_0^t \||x|^{-s/2}u_\tau\|_2^2\,\mathrm{d}\tau\\ &\quad+\beta\int_0^t \||x|^{-s/2}u\|_2^2\,\mathrm{d}\tau+\beta(t+r)^2\int_0^t \||x|^{-s/2}u_\tau\|_2^2\,\mathrm{d}\tau+\beta^2(t+\gamma)^2\\ &\geqslant\int_0^t \||x|^{-s/2}u\|_2^2\,\mathrm{d}\tau\int_0^t \||x|^{-s/2}u_\tau\|_2^2\,\mathrm{d}\tau\\ &\quad+2\beta(t+r)\left(\int_0^t \||x|^{-s/2}u\|_2^2\,\mathrm{d}\tau\right)^{1/2}\left(\int_0^t \||x|^{-s/2}u_\tau\|_2^2\,\mathrm{d}\tau\right)^{1/2}+\beta^2(t+\gamma)^2\\ &=\left[\left(\int_0^t \||x|^{-s/2}u\|_2^2\,\mathrm{d}\tau\int_0^t \||x|^{-s/2}u_\tau\|_2^2\,\mathrm{d}\tau\right)^{1/2}+\beta(t+\gamma)\right]^2\\ &\geqslant\left[\frac{1}{2}\int_0^t \||x|^{-s/2}u\|_2^2\,\mathrm{d}\tau+\beta(t+\gamma)\right]^2, \quad \text{for all } t\in[0,T). \qquad (46)\end{aligned}$$

By (42) and (46), we get

$$\begin{aligned}(H'(t))^2 &= 4\left(\frac{1}{2}\int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau}\||x|^{-s/2}u(\tau)\|_2^2\,\mathrm{d}\tau+\beta(t+r)\right)^2\\ &\leqslant 4H(t)\left(\int_0^t \||x|^{-s/2}u_\tau\|_2^2\,\mathrm{d}\tau+\beta\right), \quad t\in[0,T). \qquad (47)\end{aligned}$$

Combining (43), (44) and (47), we have

$$\begin{aligned}&H(t)H''(t)-\frac{q}{2}(H'(t))^2\\ &> H(t)\left[2q(d-J(u_0))+2q\int_0^t \||x|^{-s/2}u_\tau\|\,\mathrm{d}\tau+2\beta-2q\int_0^t \||x|^{-s/2}u_\tau\|\,\mathrm{d}\tau-2q\beta\right]\\ &= H(t)[2q(d-J(\alpha_0))-2(q-1)\beta]=H(t)[2q(d-J(u_0))-2(q-1)\beta]. \qquad (48)\end{aligned}$$

Restricting β to satisfy

$$0<\beta\leqslant\frac{q(d-J(u_0))}{q-1}, \qquad (49)$$

then $H(t)H''(t)-(q/2)(H'(t))^2>0$, $t\in[0,T)$. From Lemma 11, we get

$$T\leqslant\frac{H(0)}{\left(\frac{q}{2}-1\right)H'(0)}=\frac{T\||x|^{-s/2}u_0\|_2^2+\beta\gamma^2}{\left(\frac{q}{2}-1\right)2\beta\gamma}=\frac{1}{q-2}\left(\gamma+\frac{\||x|^{-s/2}u_0\|_2^2}{\beta\gamma}T\right). \qquad (50)$$

Then

$$T\leqslant\frac{\beta\gamma^2}{(q-2)\beta\gamma-\||x|^{-s/2}u_0\|_2^2}, \quad \gamma\in\left(\frac{\||x^{-s/2}u_0\|_2^2}{(q-2)\beta},+\infty\right). \qquad (51)$$

Let

$$\omega(\beta,\gamma)=\frac{\beta\gamma^2}{(q-2)\beta\gamma-\||x|^{-s/2}u_0\|_2^2}, \tag{52}$$

then

$$T\leqslant\min_{(\beta,\gamma)\in\Theta}w(\beta,\gamma),\quad \Theta=\{(\beta,\gamma):\beta,\gamma \text{ satisfy (49) and (51), respectively}\}. \tag{53}$$

Let $\alpha=\gamma\beta$, we have

$$\alpha>\frac{\||x|^{-s/2}u_0\|_2^2}{(q-2)},\quad \gamma\geqslant\frac{(q-1)\alpha}{q(d-J(u_0))}\quad\text{and}\quad w(\alpha,\gamma)=\frac{\alpha\gamma}{(q-2)\alpha-\||x|^{-s/2}u_0\|_2^2}. \tag{54}$$

It is easy to find that $\omega(\alpha,\gamma)$ is increasing with γ, then

$$\begin{aligned}T&\leqslant\inf_{\alpha>(\||x|^{-s/2}u_0\|_2^2)/(q-2)}\omega\left(\alpha,\frac{(q-1)\alpha}{q(d-J(u_0))}\right)\\&=\inf_{\alpha>(\||x|^{-s/2}u_0\|_2^2)/(q-2)}\frac{(q-1)\alpha^2}{q(d-J(u_0))[(q-2)\alpha-\||x|^{-s/2}u_0\|_2^2]}\\&=\frac{(q-1)\alpha^2}{q(d-J(u_0))[(q-2)\alpha-\||x|^{-s/2}u_0\|_2^2]}\Bigg|_{\alpha=(2\||x|^{-s/2}u_0\|_2^2)/(q-2)}\\&=\frac{4(q-1)\||x|^{-s/2}u_0\|_2^2}{q(d-J(u_0))(q-2)^2}.\end{aligned} \tag{55}$$

The proof of Theorem 13 is complete. □

Theorem 14 (Stationary solution). *If the global solution $u(x,t)$ of problem (1) is uniformly bounded with respect to time in $W_0^{1,p}(\Omega)$, then $u(x,t)$ converges to the stationary solution of problem (1) as $t\to+\infty$.*

Proof. We choose a monotone increasing sequence $\{t_n\}_{n=1}^{+\infty}$ such that $t_n\to+\infty$ $(n\to+\infty)$, and let $u_n=u(t_n)$. Since the sequence $\{u(t_n)\}_{n=1}^{+\infty}$ is uniformly bounded in $W_0^{1,p}(\Omega)$, there exists a subsequence of $\{u(t_n)\}_{n=1}^{+\infty}$ which is still denoted by $\{u(t_n)\}_{n=1}^{+\infty}$ and a function ω such that

$$u_n\longrightarrow w \text{ weakly in } W_0^{1,p}(\Omega)\quad\text{and}\quad u_n\longrightarrow w \text{ a.e. in } \Omega. \tag{56}$$

Next we will introduce some suitable test functions. For $\widetilde{T}<+\infty$, we take two functions φ and ρ fulfilling

$$\varphi\in W_0^{1,p}(\Omega),\quad \rho\in C_0(0,\widetilde{T}),\quad \rho\geqslant 0,\quad \int_0^{\widetilde{T}}p(s)\,\mathrm{d}s=1,$$

and let

$$\phi(x,t):=\begin{cases}\rho(t-t_n)\varphi(x), & (x,t)\in\bar{\Omega}\times(t_n,+\infty)\\ 0, & (x,t)\in\bar{\Omega}\times[0,t_n].\end{cases} \tag{57}$$

In (18), integrating it over $(t_n,t_n+\widetilde{T})$ with respect to t, we get

$$\int_{t_n}^{\widetilde{T}+t_n}\int_\Omega|x|^{-s}u_t\phi\,\mathrm{d}x\,\mathrm{d}t+\int_{t_n}^{\widetilde{T}+t_n}\int_\Omega|\nabla u|^{p-2}\nabla u\nabla\phi\,\mathrm{d}x\,\mathrm{d}t=\int_{t_n}^{\widetilde{T}+t_n}\int_\Omega|u|^{q-2}u\ln|u|\phi\,\mathrm{d}x\,\mathrm{d}t. \tag{58}$$

By integrating the first term on the left in (30) by parts and $\rho \in C_0^1(0,\widetilde{T})$, we get

$$\begin{aligned}\int_{t_n}^{\widetilde{T}+t_n}\int_\Omega |x|^{-s}u_t\phi\,\mathrm{d}x\,\mathrm{d}t &= \int_\Omega\int_{t_n}^{\widetilde{T}+t_n}|x|^{-s}u_t\rho(t-t_n)\cdot\varphi\,\mathrm{d}t\,\mathrm{d}x\\ &= \int_\Omega |x|^{-s}u\rho(t-t_n)\varphi\Big|_{t_n}^{\widetilde{T}+t_n}\mathrm{d}x-\int_\Omega\int_{t_n}^{t_n+\widetilde{T}}|x|^{-s}u\rho'(t-t_n)\varphi\,\mathrm{d}t\,\mathrm{d}x\\ &= \int_\Omega |x|^{-s}u\rho(\widetilde{T})\varphi-|x|^{-s}u\rho(0)\varphi\,\mathrm{d}x-\int_{t_n}^{t_n+\widetilde{T}}\int_\Omega|x|^{-s}u\rho'(t-t_n)\varphi\,\mathrm{d}x\,\mathrm{d}t.\\ &= -\int_{t_n}^{t_n+\widetilde{T}}\int_\Omega |x|^{-s}u\rho'(t-t_n)\varphi\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}\tag{59}$$

Hence, by (58) and (59)

$$\begin{aligned}&\int_{t_n}^{t_n+\widetilde{T}}\int_\Omega |x|^{-s}u\rho'(t-t_n)\varphi\,\mathrm{d}x\,\mathrm{d}t-\int_{t_n}^{t_n+\widetilde{T}}\int_\Omega|\nabla u|^{p-2}\nabla u\rho(t-t_n)\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\int_{tn}^{t_n+\widetilde{T}}\int_\Omega |u|^{q-2}u\ln|u|\rho(t-t_n)\varphi\,\mathrm{d}x\,\mathrm{d}t=0.\end{aligned}\tag{60}$$

In (60), taking $t=t_n+\widetilde{s}$, then we get

$$\begin{aligned}&\int_0^{\widetilde{T}}\int_\Omega |x|^{-s}u(t_n+\widetilde{s})\rho'(\widetilde{s})\varphi\,\mathrm{d}x\,\mathrm{d}\widetilde{s}-\int_0^{\widetilde{T}}\int_\Omega|\nabla u(t_n+\widetilde{s})|^{p-2}\nabla u(t_n+\widetilde{s})\rho(\widetilde{s})\nabla\varphi\,\mathrm{d}x\,\mathrm{d}\widetilde{s}\\ &\quad+\int_0^{\widetilde{T}}\int_\Omega |u(t_n+\widetilde{s})|^{q-2}u(t_n+\widetilde{s})\ln|u(t_n+\widetilde{s})|\rho(\widetilde{s})\varphi\,\mathrm{d}x\,\mathrm{d}\widetilde{s}=0.\end{aligned}\tag{61}$$

By virtue of (2), we know the embedding $W_0^{1,p}(\Omega)\hookrightarrow L^2(\Omega)$ is compact and that $\{u(t_n+\widetilde{s})\}_{n=1}^{+\infty}$ is uniformly bounded in $W_0^{1,p}(\Omega)$, hence there exists a subsequence of $\{u(t_n+\widetilde{s})\}_{n=1}^{+\infty}$ which is still denoted by $\{u(t_n+\widetilde{s})\}_{n=1}^{+\infty}$ and $\overline{\omega}\in L^2(\Omega)$ such that

$$u(t_n+\widetilde{s})\to\overline{w}\quad\text{in } L^2(\Omega),\quad u(t_n)\to w\quad\text{in } L^2(\Omega).\tag{62}$$

Next, we claim that $\overline{\omega}=\omega$ a.e. in Ω. In fact, we know that the solution is global, by Lemma 8 we know that $J(u(t))>0$ for all $t\in[0,\infty)$, then by (19) we have

$$\int_0^t \||x|^{-s/2}u_\tau\|_2^2\,\mathrm{d}\tau\leqslant J(u_0)<+\infty,$$

which implies

$$\int_{t_n}^{t_n+\widetilde{T}}\||x|^{-s/2}u_\tau\|_2^2\,\mathrm{d}\tau\to 0\quad\text{as } n\to+\infty.\tag{63}$$

By Hölder's inequality and (63), we obtain

$$\begin{aligned}\rho^{-s}\int_\Omega|u(t_n+\widetilde{s})-u(t_n)|^2\,\mathrm{d}x&\leqslant\int_n|x|^{-s}|u(t_n+\widetilde{s})-u(t_n)|^2\,\mathrm{d}x\\ &=\int_\Omega|x|^{-s}\left(\int_{t_n}^{t_n+\widetilde{s}}(u(x,t))_t\,\mathrm{d}t\right)^2\mathrm{d}x\leqslant\widetilde{s}\int_{t_n}^{t_n+\widetilde{s}}\int_\Omega|x|^{-s}[(u(x,t))_t]^2\,\mathrm{d}x\,\mathrm{d}t\\ &\leqslant\widetilde{T}\int_{t_n}^{t_n+\widetilde{T}}\||x|^{-s/2}u_\tau\|_2^2\,\mathrm{d}\tau\to0\quad(n\to\infty).\end{aligned}\tag{64}$$

Thus $\tilde{w}=w$ a.e. in Ω for any fixed $\widetilde{T}<\infty$ and $\tilde{s}\in[0,\widetilde{T}]$.

By (61), let $n\to\infty$, it follows from the dominated convergence theorem, (56) and (62) that

$$\int_0^{\widetilde{T}}\int_\Omega|x|^{-s}\omega\rho'(\widetilde{s})\varphi\,\mathrm{d}x\,\mathrm{d}\widetilde{s}-\int_0^{\widetilde{T}}\int_\Omega|\nabla\omega|^{p-2}\nabla\omega\rho(\widetilde{s})\nabla\varphi\,\mathrm{d}x\,\mathrm{d}\widetilde{s}+\int_0^{\widetilde{T}}\int_\Omega|\omega|^{q-2}\omega\ln|\omega|\rho(\widetilde{s})\varphi\,\mathrm{d}x\,\mathrm{d}\widetilde{s}=0.\tag{65}$$

By integrating the first term on the left by parts and $\rho \in C_0^1(0, \widetilde{T})$, we get

$$\int_0^{\widetilde{T}} \int_\Omega |x|^{-s} \omega \rho'(\tilde{s}) \varphi \,\mathrm{d}x \,\mathrm{d}\tilde{s} = \int_\Omega |x|^{-s} \omega [\rho(\widetilde{T}) - \rho(0)] \varphi \,\mathrm{d}x = 0. \tag{66}$$

Then, we have

$$\int_0^{\widetilde{T}} \int_\Omega |\nabla \omega|^{p-2} \nabla \omega \rho(\tilde{s}) \nabla \varphi \,\mathrm{d}x \,\mathrm{d}\tilde{s} - \int_0^{\widetilde{T}} \int_\Omega |\omega|^{q-2} \omega \ln|\omega| \rho(\tilde{s}) \varphi \,\mathrm{d}x \,\mathrm{d}\tilde{s} = 0.$$

By $\int_0^{\widetilde{T}} \rho(\tilde{s}) \,\mathrm{d}\tilde{s} = 1$, then

$$\begin{aligned}
&\int_\Omega |w|^{q-2} \omega \ln(|w|) \varphi - |\nabla w|^{p-2} \nabla w \nabla \varphi \,\mathrm{d}x \\
&\quad = \int_0^{\widetilde{T}} \rho(\tilde{s}) \,\mathrm{d}\tilde{s} \int_\Omega |w|^{q-2} \omega \ln(|w|) \varphi - |\nabla w|^{p-2} \nabla w \nabla \varphi \,\mathrm{d}x \\
&\quad = \int_0^{\widetilde{T}} \int_\Omega \rho(\tilde{s}) |w|^{q-2} \omega \ln(|w|) \varphi - \rho(\tilde{s}) |\nabla w|^{p-2} \nabla w \nabla \varphi \,\mathrm{d}x \,\mathrm{d}\tilde{s} \\
&\quad = 0,
\end{aligned}$$

which implies ω is a stationary solution of problem (1).

The proof of Theorem 14 is complete. □

Theorem 15. *For any $\alpha \in (d, +\infty)$, the following conclusions hold.*

(i) *If $u_0 \in \Phi_\alpha$, then the solution of problem (1) exists globally and $u(t) \longrightarrow 0$, as $t \longrightarrow \infty$;*

(ii) *if $u_0 \in \Psi_\alpha$, then the solution of problem (1) blows up in finite or infinite time, where*

$$\Phi_\alpha = \mathcal{N}_+ \cap \left\{ u(t) \in W_0^{1,p}(\Omega) \,\middle|\, \frac{1}{2} \int_\Omega |x|^{-s} |u(t)|^2 \,\mathrm{d}x < \lambda_\alpha, d < J(u(t)) \leqslant \alpha \right\},$$

$$\Psi_\alpha = \mathcal{N}_- \cap \left\{ u(t) \in W_0^{1,p}(\Omega) \,\middle|\, \frac{1}{2} \int_\Omega |x|^{-s} |u(t)|^2 \,\mathrm{d}x > \Lambda_\alpha, d < J(u(t)) \leqslant \alpha \right\}.$$

λ_α and Λ_α are two constants defined.

Proof. (i) Assume that $u_0 \in \Phi_\alpha$, then by the definition of Φ_α and the monotonicity property of λ_α, we have $d < J(u_0) \leqslant \alpha$, $u_0 \in \mathcal{N}_+$ and

$$\frac{1}{2} \int_\Omega |x|^{-s} |u_0|^2 \,\mathrm{d}x < \lambda_\alpha \leqslant \lambda_{J(u_0)}. \tag{67}$$

We first claim that $u(t) \in \mathcal{N}_+$ for all $t \in [0, T)$. If not, there would exist a $t_0 \in (0, T)$ such that $u(t) \in \mathcal{N}_+$ for $t \in [0, t_0)$ and $u(t_0) \in \mathcal{N}$. By Lemma 10, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2} \int_\Omega |x|^{-s} |u(t)|^2 \,\mathrm{d}x \right) = -I(u(t)). \tag{68}$$

Then by the definition of $\mathcal{N}_+$ and (68), we know that $\int_\Omega |x|^{-s} |u(t)|^2 \,\mathrm{d}x$ is strictly decreasing on $[0, t_0)$. On the other hand, from (19), we know that $J(u(t))$ is non-increasing with respect to t. Thus, we get

$$J(u(t)) \leqslant J(u_0) \quad \text{for all } t \in [0, T). \tag{69}$$

From (67), we have

$$\frac{1}{2} \int_\Omega |x|^{-s} |u(t_0)|^2 \,\mathrm{d}x < \frac{1}{2} \int_\Omega |x|^{-s} |u_0|^2 \,\mathrm{d}x < \lambda_{J(u_0)}. \tag{70}$$

By $u(t_0) \in \mathcal{N}$ and (69), we get $u(t_0) \in \mathcal{N}_{J(u_0)}$. According to the definition of $\lambda_{J(u_0)}$, we have

$$\lambda_{J(u_0)} = \inf_{u \in \mathcal{N}_J(u_0)} \frac{1}{2} \int_\Omega |x|^{-s} |u|^2 \,\mathrm{d}x \leqslant \frac{1}{2} \int_\Omega |x|^{-s} |u(t_0)|^2 \,\mathrm{d}x,$$

which contradicts (70) and the claim is proved. Therefore, we have $u \in \mathcal{N}_+$ for all $t \in [0, T)$ and $u(t) \in J^{J(u_0)}$, i.e., $u \in J^{J(u_0)} \bigcap \mathcal{N}_+$ for all $t \in [0, T)$. By Lemma 8, we get

$$\|\nabla u\|_p < \left(\frac{pq}{q-p} J(u_0)\right)^{1/p}, \quad \forall t \in [0, T). \tag{71}$$

Since the right-hand of (71) is dependent on T, then we can extend the solution to infinity, i.e., $T = +\infty$. It indicates that $u(t)$ is bounded uniformly in $W_0^{1,p}(\Omega)$. Hence, ω-limit set is not an empty set.

Next, for any $\omega \in \omega(u_0)$, by the above discussions, we get

$$J(w) \leqslant J(u_0) \quad \text{and} \quad \frac{1}{2}\int_\Omega |x|^{-s}|w|^2\,\mathrm{d}x < \lambda J(u_0).$$

According to the first inequality, this shows $\omega \in J^{J(u_0)}$. According to the second inequality and the definition of $\lambda_{J(u_0)}$, we know that $\omega \notin \mathcal{N}_{J(u_0)}$. Since $\mathcal{N}_{J(u_0)} = \mathcal{N} \bigcap J^{J(u_0)}$, we obtain $\omega \notin \mathcal{N}$.

Finally, we prove that

$$\omega(u_0) = \{0\}. \tag{72}$$

With $u(t) \in \mathcal{N}_+$, we know $J(u_0) > 0$ for all $t \in [0,\infty)$. So $J(u(t))$ is bounded below, and there exists a non-negative constant c such that $\lim_{t\to+\infty} J(u(t)) = c$. Now, selecting any $\omega \in \omega(u_0)$, we have $J(u_\omega(t)) = c$ for all $t \geqslant 0$, where $u_\omega(t)$ is the solution of problem (1) with initial value ω. Choosing $u = u_\omega$ into (19), we obtain $\int_0^t \||x|^{-s/2} u_\tau\|_2^2 \mathrm{d}\tau = 0$, $0 \leqslant t < +\infty$. It implies that $u_\omega(t) \equiv \omega$. From (68), we have $(\mathrm{d}/\mathrm{d}t)((1/2)\int_\Omega |x|^{-s}|u_\omega|^2\,\mathrm{d}x) = -I(u_\omega(t)) = 0$, then

$$I(\omega) = 0, \quad \forall \omega \in \omega(u_0). \tag{73}$$

Combining (73), $w \notin \mathcal{N}$ and the definition of $\mathcal{N}$, we know $\omega = 0$. Thus, (72) holds and implies that the solution $u(t) \to 0$ as $t \to +\infty$.

(ii) If $u_0 \in \Psi_\alpha$, by the definition of Ψ_α, it is clear that $u_0 \in \mathcal{N}_-$ and $d < J(u_0) \leqslant \alpha$. This is combined with the monotonicity of Λ_α, then

$$\frac{1}{2}\int_\Omega |x|^{-s}|u_0|^2\,\mathrm{d}x > \Lambda_\alpha \geqslant \Lambda_{J(u_0)}. \tag{74}$$

We claim that $u(t) \in \mathcal{N}_-$ for all $t \in [0, T)$. If not, there would exist a $t_1 \in (0, T)$ such that $u(t) \in \mathcal{N}_-$ for $0 \leqslant t < t_1$ and $u(t_1) \in \mathcal{N}$. By Lemma 10, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_\Omega |x|^{-s}|u(t)|^2\,\mathrm{d}x\right) = -I(u(t)). \tag{75}$$

Thus $(\mathrm{d}/\mathrm{d}t)((1/2)\int_\Omega |x|^{-s}|u(t)|^2\,\mathrm{d}x) = -I(u(t)) > 0$ for $0 \leqslant t < t_1$. Then by the definition of $\mathcal{N}_-$, we deduce that $(1/2)\int_\Omega |x|^{-s}|u(t)|^2\,\mathrm{d}x$ is strictly increasing on $[0, t_1)$. This along with (75) yields

$$\frac{1}{2}\int_\Omega |x|^{-s}|u(t_1)|^2\,\mathrm{d}x > \frac{1}{2}\int_\Omega |x|^{-s}|u_0|^2\,\mathrm{d}x > \Lambda_{J(u_0)}, \quad J(u(t_1)) \leqslant J(u_0). \tag{76}$$

By (15), $u(t_1) \in \mathcal{N}_{J(u_0)}$. Thus, it follows from the definition of $\Lambda_{J(u_0)}$ that

$$\Lambda_{J(u_0)} = \sup_{u\in\mathcal{N}_{J(u_0)}} \frac{1}{2}\int_\Omega |x|^{-s}|u|^2\,\mathrm{d}x \geqslant \frac{1}{2}\int_\Omega |x|^{-s}|u(t_1)|^2\,\mathrm{d}x,$$

which is incompatible with (76), so we get $u(t) \in \mathcal{N}_-$.

Next, we assume that $u(t)$ exists globally, i.e., $T = +\infty$, then $u(t) \in J^{J(u_0)} \bigcap \mathcal{N}_-$, $\forall t \in [0, +\infty)$ and $(1/2)\int_\Omega |x|^{-s}|u(t)|^2\,\mathrm{d}x$ is strictly increasing on $[0, +\infty)$. Furthermore more, we claim that $u(t)$ is uniformly bounded in $W_0^{1,p}(\Omega)$, i.e., there exists a positive constant M such that

$$\|\nabla u(t)\|\|_p^p \leqslant M, \quad \forall t \in [0, +\infty).$$

Indeed, if this is false, then there must exist a monotone increasing sequence $\{t_n\}_{n=1}^{+\infty}$ such that

$$\|\nabla u(t)\|_p^p > n, \quad n = 1, 2, \ldots. \tag{77}$$

Then it follows from the monotone property of $\{t_n\}_{n=1}^{+\infty}$, that $t_n \to +\infty (n \to +\infty)$ or there exists $t^* \in (0,+\infty)$ such that $t_n \to t^*$. If the first case happens, then by (77), we know $u(t)$ blows up at infinite time. On the other hand, we know $u(t)$ exists globally, which contradicts the conclusion that blowup exists at infinite time. If the last case happens, then by (77), we know $u(t_0)$ blows up at finite time. It contradicts the assumption that $u(t)$ exists globally. Thus $u(t)$ is uniformly bounded in $W_0^{1,p}(\Omega)$, i.e., $w(u_0)$ is not an empty set.

For any $\omega \in \omega(u_0)$, by using the above claim and since $J(u(t))$ is non-increasing with respect to t, we have the following two cases:

$$\lim_{t\to+\infty} J(u) = -\infty \quad \text{or} \quad \lim_{t\to+\infty} J(u) = c,$$

where c is a constant.

Next we will prove that both the above cases contradict $T = +\infty$. If the first case happens, there must exist a $t^* \in [0,+\infty)$ such that $J(u(t^*)) < 0$. By [14, Theorem 2.6], the solution of problem (1) will blow up at finite time, which contradicts our hypothesis. If the second case happens, then we obtain $\omega(u_0) = \{0\}$ by the similar way as proof (i). However, by [14, Lemma 3.1(2)], we have $\|u\|_p > r(\alpha)$, i.e.,

$$\operatorname{dist}(0,\mathcal{N}_-) = \inf_{u\in\mathcal{N}_-} \operatorname{dist}(0,u) = \inf_{u\in\mathcal{N}_-} \|u\|_{W_0^{1,p}(\Omega)} = \inf_{u\in\mathcal{N}_-} \|\nabla u\|_p > 0.$$

Hence, $0 \notin \omega(u_0)$. We get a contradiction, then $T \neq +\infty$.

The proof of Theorem 15 is complete. □

Conflicts of interest

The authors declare no conflict of interest.

Acknowledgments

This research was supported by the project of Guizhou province science and technology plan under (No. Qiankehe foundation-ZK[2021]YIBAN317), and by the project of Guizhou Minzu University under (No. GZMU[2019]YB04), and central leading local science and technology development special foundation for Sichuan province, PR China under (No. 2021ZYD0020).

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