INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus
Mécanique

Christian Licht and Thibaut Weller

Asymptotic analysis of plates in static and dynamic strain gradient elasticity

Volume 350 (2022), p. 325-342

Published online: 11 July 2022

https://doi.org/10.5802/crmeca.118

Les Comptes Rendus. Mécanique *sont membres du*
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 1873-7234

Comptes Rendus
Mécanique
2022, Vol. 350, p. 325-342
https://doi.org/10.5802/crmeca.118

Short Paper / *Note*

Asymptotic analysis of plates in static and dynamic strain gradient elasticity

Christian Licht [a, b] **and Thibaut Weller** [*, a]

[a] LMGC, Université de Montpellier, CNRS, Montpellier, France

[b] Department of Mathematics, Faculty of Science, Mahidol University, Bangkok 10400, Thailand

E-mails: christian.licht@umontpellier.fr (C. Licht), thibaut.weller@umontpellier.fr (T. Weller)

Abstract. We study the steady-state and transient responses of a second-order elastic plate by implementing an asymptotic analysis of the three-dimensional equations with respect to two geometric characteristics seen as *parameters*: the thickness of the plate and an inner material length. Depending on their ratio, four different models arise. Conditions under which Reissner–Mindlin kinematics may appear are discussed while the influence of crystalline symmetries is studied. The transient situation is solved through Trotter's theory of approximation of semi-groups of operators acting on variable spaces.

Keywords. Asymptotic analysis, Strain gradient elasticity, Plate models, Transient problems, m-dissipative operators, Approximation of semi-groups in the sense of Trotter.

Manuscript received 2 June 2022, accepted 3 June 2022.

As an abundance and variety of literature exhibits the interest in second-gradient approaches, here we propose an asymptotic mathematical modeling of thin strain gradient elastic plates. Even if "la Statique consiste simplement dans l'étude de problèmes de Dynamique particuliers", for the reader's convenience, the Sections 1 to 3 are dedicated to statics while Sections 4 and 5 deal with the unsteady equations. In accordance with some of our previous works, we have chosen to use notations that may seem daunting but have the advantage of conveying all the information necessary to express the complexity of the studied problem.

As usual we do not distinguish between the physical euclidean space and $\mathbb{R}^3$. For all $\xi = (\xi_1, \xi_2, \xi_3)$ of $\mathbb{R}^3$, $\hat{\xi}$ stands for (ξ_1, ξ_2). The Greek coordinate indices will run in $\{1,2\}$ whereas the Latin ones will run in $\{1,2,3\}$. Let $\mathbb{S}^n$ be the space of all $n \times n$ symmetric matrices, $\mathbb{T}^3$ the space of third-order matrices with entries symmetric with respect to the first two indices, and $\mathbb{K} := \mathbb{S}^3 \times \mathbb{T}^3$. These spaces are equipped with their usual inner product and norm denoted by $\cdot$ and $||$ (as well as in $\mathbb{R}^n$). The space of linear mappings from a space V into a space W is denoted by $\mathrm{Lin}(V, W)$ and when $W = V$ we simply write $\mathrm{Lin}(V)$. For any open subset G of $\mathbb{R}^n$, $H^1_\Gamma(G, \mathbb{R}^n)$ stands for the

[*] Corresponding author.

subspace of the Sobolev space $H^1(G,\mathbb{R}^n)$ composed of the elements which vanish on a smooth part Γ of the boundary ∂G of G while $H^2_\Gamma(G,\mathbb{R}^n)$ is made of the elements u which belong to $H^1_\Gamma(G,\mathbb{R}^n)$ such that the derivative $\partial_\nu u$ with respect to the unit outer normal ν also vanishes on Γ.

1. Setting the static problem

A reference configuration of a thin strain gradient linearly elastic plate is the closure of $\Omega^\varepsilon := \omega \times (-\varepsilon,\varepsilon)$. Here ε is a small positive number and ω a bounded domain of $\mathbb{R}^2$ with a Lipschitz-continuous boundary. We assume that the strain energy density function of the plate is $(1/2)m(x^\varepsilon)k^s(u)\cdot k^s(u)$ with

$$\begin{cases} k^s(u) := (e^\varepsilon(u), \ell g^\varepsilon(u)) \quad \forall u \in H^2(\Omega^\varepsilon,\mathbb{R}^3) \\ s := (\varepsilon,\ell) \end{cases} \tag{1.1}$$

with m^ε a symmetric element of $L^\infty(\Omega^\varepsilon,\mathrm{Lin}(\mathbb{K}))$ satisfying

$$\begin{cases} m^\varepsilon = \begin{bmatrix} a & b \\ b^T & c \end{bmatrix}, \quad a \in \mathrm{Lin}(\mathbb{S}^3), \quad b \in \mathrm{Lin}(\mathbb{S}^3,\mathbb{T}^3), \quad c \in \mathrm{Lin}(\mathbb{T}^3) \\ \exists \alpha_m > 0 \ s.t.\ m^\varepsilon(x^\varepsilon)k\cdot k \geq \alpha_m |k|^2, \quad \forall k \in \mathbb{K} \ a.e.\ x^\varepsilon \in \Omega^\varepsilon \end{cases} \tag{H_0}$$

and where $e^\varepsilon(u)$ is the linearized strain tensor associated with the field of displacement u, g^ε is the gradient of $e^\varepsilon(u)$ ($g^\varepsilon_{ijk} := \partial^\varepsilon_k e^\varepsilon_{ij}$) while ℓ is a positive number representing an "inner material length". The plate is clamped on $\Gamma^\varepsilon_D := \gamma_D \times (-\varepsilon,\varepsilon)$, where γ_D is a part with positive length of $\partial\omega$, and we assume that the work of the exterior loading on the plate stems from $\ell^2 L^\varepsilon$ where L^ε is an element of the strong dual $H^2_{\Gamma^\varepsilon_D}(\Omega^\varepsilon,\mathbb{R}^3)'$ of $H^2_{\Gamma^\varepsilon_D}(\Omega^\varepsilon,\mathbb{R}^3)$, which could be the classical loading through body and surface forces but also more special actions compatible with second-grade continuum theories such as couples, symmetric double forces, tangential surface couples, doubly normal and edge forces (see [1]). To simplify the presentation, as the problem of determining the equilibrium configuration of the plate is linear, we are free to choose a normalizing factor describing the magnitude of the loading, it is convenient to consider ℓ^2 (see the end of Remark 3.1). So the problem involves two main data $s := (\varepsilon,\ell)$ and the field of displacement u^s at equilibrium does satisfy:

$$(\mathscr{P}^s)\begin{cases} \text{Find } u^s \text{ in } H^2_{\Gamma^\varepsilon_D}(\Omega^\varepsilon,\mathbb{R}^3) \text{ such that} \\ \displaystyle\int_{\Omega^\varepsilon} m^\varepsilon(x^\varepsilon)k^s(u^s)\cdot k^s(u)\,\mathrm{d}x^\varepsilon = \ell^2 L^\varepsilon(u), \quad \forall u \in H^2_{\Gamma^\varepsilon_D}(\Omega^\varepsilon,\mathbb{R}^3). \end{cases} \tag{1.2}$$

We recall that if $\vartheta^s := (\sigma^s,\mu^s)$ denotes the couple made of the (classical) Cauchy stress tensor and the hyperstress tensor satisfying $\vartheta^s = m^\varepsilon k^s(u^s)$, the volumic balance equation implied by $(\mathscr{P}^s)$ reads as:

$$\partial_j \sigma^s_{ij} + \partial_{jk}\mu^s_{ijk} + f^s_i - C^s_{ij,j} - \phi^s_{ij,j} = 0, \tag{1.3}$$

where f^s, C^s and ϕ^s respectively denote densities of volumetric forces, volumetric couples and symmetric double forces.

Due to the above assumptions on the data, $(\mathscr{P}^s)$ appears as a fourth-order elliptic boundary value problem and has a unique solution, but the essential task is, due to various reasons—especially numerical ones—to propose a simplified but accurate enough model. This will be done by considering s as a couple *of parameters* taking values in a countable set of $(0,+\infty)^2$ with a unique cluster point $\bar{s} := (0,\bar{\ell})$ in $\{0\}\times[0,+\infty)$ and studying the asymptotic behavior of u^s when s goes to $\bar{s}$.

2. Asymptotic behavior for $(\mathscr{P}^s)$

Wishing to study various situations corresponding to the relative orders of magnitude of the thickness and the internal length, we make the following assumption:

$$\exists \bar{\bar{\ell}} \in [0,+\infty]; \quad \bar{\bar{\ell}} := \lim_{s\to\bar{s}} \frac{\ell}{\varepsilon} \tag{H_{stat_1}}$$

and consider four cases indexed by p:

$$\begin{cases} p=1: \bar{\ell}\in(0,+\infty) & \text{i.e. the internal length is of order 1;}\\ p=2: \bar{\ell}=0,\ \bar{\bar{\ell}}=+\infty & \text{i.e. the internal length is small but much larger than the thickness;}\\ p=3: \bar{\ell}=0,\ \bar{\bar{\ell}}\in(0,+\infty) & \text{i.e. both the internal length and the thickness are similarly small;}\\ p=4: \bar{\ell}=0,\ \bar{\bar{\ell}}=0 & \text{i.e. the thickness is small but the internal length is much smaller.}\end{cases} \tag{2.1}$$

Insofar as the internal length ℓ accounts for the microstructure inherent to a second-grade material, it goes without saying that the cases $p = 1,2$ seem unrealistic. However, we felt it was important to include them in the study in order to compare our results to the existing literature but also and more importantly to show that it is possible to extend existing studies to more interesting cases.

We also make assumption (H_{stat_2}) on the generalized stiffness and assumption (H_{stat_3}) on the loading which express that the previous quantities stem from quantities defined on $\Omega := \omega\times(-1,1)$, the image of Ω^ε by a change of coordinates (see [2])

$$x=(\hat{x},x_3)\in\bar{\Omega}\mapsto x^\varepsilon=\Pi^\varepsilon x:=(\hat{x},\varepsilon x_3)\in\Omega^\varepsilon \tag{2.2}$$

and, in the following, *systematically* x^ε *and* x *are connected by* $x^\varepsilon=\Pi^\varepsilon x$.

Assumption (H_{stat_2}) reads as:

$$\begin{cases}\text{There exists a symmetric element } m \text{ in } L^\infty(\Omega,\mathrm{Lin}(\mathbb{K})) \text{ such that}\\ \alpha_m|k|^2\le m(x)k\cdot k \quad a.e.\ x\in\Omega,\ \forall k\in\mathbb{K}\\ m^\varepsilon(x^\varepsilon)=m(x)\quad a.e.\ x\in\Omega.\end{cases} \tag{H_{stat_2}}$$

As

$$\frac{1}{\varepsilon\ell^2}\int_{\Omega^\varepsilon} m^\varepsilon(x^\varepsilon)(k^s(u^s)-\kappa^s)\cdot(k^s(u^s)-\kappa^s)\mathrm{d}x^\varepsilon=\int_\Omega m(x)(k_s(u_s)-\kappa_s)\cdot(k_s(u_s)-\kappa_s)\mathrm{d}x \tag{2.3}$$

it is *natural* and *convenient* to consider the field $u_s:=\mathscr{S}_\varepsilon u^s$ defined on Ω, where:

$$\begin{cases}(\mathscr{S}_\varepsilon u)(x)=\left(\frac{1}{\varepsilon}\hat{u}(x^\varepsilon),u_3(x^\varepsilon)\right)\ a.e.\ x\in\Omega\\ k_s(u):=\left(\frac{\varepsilon}{\ell}e(\varepsilon,u),g_\varepsilon(u)\right)\\ e_{\alpha\beta}(\varepsilon,u):=e_{\alpha\beta}(u):=\frac12(\partial_\alpha u_\beta+\partial_\beta u_\alpha)\\ \varepsilon e_{\alpha3}(\varepsilon,u)=\varepsilon e_{3\alpha}(\varepsilon,u):=e_{\alpha3}(u):=\frac12(\partial_\alpha u_3+\partial_3u_\alpha)\\ \varepsilon e_{33}(\varepsilon,u):=e_{33}(u):=\partial_3u_3\\ g_{\varepsilon,\alpha\beta\gamma}(u):=\varepsilon\partial_\gamma e_{\alpha\beta}(u),\quad g_{\varepsilon,\alpha\beta3}(u):=\partial_3e_{\alpha\beta}(u)\\ g_{\varepsilon,\alpha3\gamma}(u)=g_{\varepsilon,3\alpha\gamma}(u):=\partial_\gamma e_{\alpha3}(u),\quad g_{\varepsilon,\alpha33}(u)=g_{\varepsilon,3\alpha3}(u):=\frac{1}{\varepsilon}\partial_3e_{\alpha3}(u)\\ g_{\varepsilon,33\gamma}(u):=\frac{1}{\varepsilon}\partial^2_{\gamma3}u_3,\quad g_{\varepsilon,333}(u):=\frac{1}{\varepsilon^2}\partial^2_{33}u_3\\ \kappa^s(x^\varepsilon):=\ell\kappa_s(x)\ a.e.\ x\in\Omega,\quad\forall\kappa_s\in L^2(\Omega;\mathbb{K}).\end{cases} \tag{2.4}$$

Actually the obvious relation (2.3) expresses that the *relative* energetic gap between the two fields $k^s(u^s)$ and κ^s defined on the *physical* domain Ω^ε is of the same order of magnitude as the gap—measured on Ω—between their suitable images $k_s(u_s)$ and κ_s. It is by means of this tool that we will give a mechanical interpretation of our convergence results (in a rather classical sense) for the "scaled" problem $(\mathscr{P}_s)$ set on a *fixed* abstract domain Ω.

So if $\Gamma_D := (\Pi^\varepsilon)^{-1}(\Gamma_D^\varepsilon)$ and L^ε is such that

$$L^\varepsilon(v) := \varepsilon L_s(\mathscr{S}_\varepsilon v), \quad \forall v \in H^2_{\Gamma_D^\varepsilon}(\Omega, \mathbb{R}^3), \tag{2.5}$$

where L_s is an element of the strong dual $H^2_{\Gamma_D}(\Omega,\mathbb{R}^3)'$ of $H^2_{\Gamma_D}(\Omega,\mathbb{R}^3)$, the field u_s is the unique solution to

$$(\mathscr{P}_s)\begin{cases}\text{Find } u_s \text{ in } H^2_{\Gamma_D}(\Omega,\mathbb{R}^3) \text{ such that}\\ \displaystyle\int_\Omega m(x)k_s(u_s)\cdot k_s(u)\,\mathrm{d}x = L_s(u), \quad \forall u \in H^2_{\Gamma_D}(\Omega,\mathbb{R}^3).\end{cases}$$

The last assumption on the loading is:

$$\begin{cases}\text{There exists } (\hat{L}_s, L_{s_3}) \text{ in } H^2_{\Gamma_D}(\Omega,\mathbb{R}^3)' \text{ such that}\\ \bullet\ L_s(u) = \varepsilon\hat{L}_s(\hat{u}) + L_{s_3}(u_3), \quad \forall u = (\hat{u}, u_3) \in H^2_{\Gamma_D}(\Omega,\mathbb{R}^3)\\ \bullet\ (\hat{L}_s, L_{s_3}) \text{ converges strongly in } H^2_{\Gamma_D}(\Omega,\mathbb{R}^3)' \text{ toward } (\hat{L}, L_3).\end{cases} \tag{H_{stat_3}}$$

The following notations and notions make it possible to infer from convergence results for $(\mathscr{P}_s)$ the asymptotic behavior of the genuine physical problem $(\mathscr{P}^s)$ in a unified (they account for the various "limit" kinematics) and concise manner. Of course, a reader in a hurry or not interested in the details of rigorous mathematical developments can skip these however legitimate notations and notions and head for the remarks following the statement of the key Theorem 2.1.

$$\begin{cases}H^1_{\partial_3}(\Omega,\mathbb{R}^n) := \{u \in L^2(\Omega,\mathbb{R}^n); \partial_3 u \in L^2(\Omega,\mathbb{R}^n)\}\\ {}^1H := H^2_{\Gamma_D}(\Omega,\mathbb{R}^2)\times H^1_{\Gamma_D}(\Omega)\times H^2_{\Gamma_D}(\Omega)\times H^1_{\Gamma_D}(\Omega,\mathbb{R}^2)\\ {}^2H := H^1_{\Gamma_D}(\Omega,\mathbb{R}^2)\times H^1_{\partial_3}(\Omega)\times H^2_{\Gamma_D}(\Omega)\times H^1_{\partial_3}(\Omega,\mathbb{R}^2)\\ {}^3H := H^1_{\Gamma_D}(\Omega,\mathbb{R}^2)\times H^1_{\partial_3}(\Omega)\times H^2_{\Gamma_D}(\Omega)\times H^1_{\partial_3}(\Omega,\mathbb{R}^2)\\ {}^p\mathscr{U} := \left\{ U = (u^M, z, u^F, y) \in \left\{\begin{array}{ll} H^2_{\gamma_D}(\omega,\mathbb{R}^2)\times H^1_{\gamma_D}(\omega)\times H^2_{\gamma_D}(\omega)\times H^1_{\gamma_D}(\omega,\mathbb{R}^2) & \text{if } p=1\\ H^1_{\gamma_D}(\omega,\mathbb{R}^2)\times L^2(\omega)\times H^2_{\gamma_D}(\omega)\times L^2(\omega,\mathbb{R}^2) & \text{if } p=2\\ {}^3H;\ \partial_3 u^F = 0,\ u^M + \frac{x_3}{\bar{\bar{\ell}}}\hat{\nabla}u^F \in H^1_{\gamma_D}(\omega,\mathbb{R}^2) & \text{if } p=3\end{array}\right.\right\}\\ \left.\begin{array}{l}{}^pe_{\alpha\beta}(U) := \frac{1}{2}(\partial_\alpha u^M_\beta + \partial_\beta u^M_\alpha)\\ {}^pe_{\alpha 3}(U) := y_\alpha,\ {}^pe_{33}(U) := z\\ {}^pL(U) := \bar{\ell}\hat{L}(u^M) + L_3(u^F)\end{array}\right\} \quad \forall U = (u^M, z, u^F, y) \in {}^p\mathscr{U}, \quad \forall p \in \{1,2,3\}.\end{cases} \tag{2.6}$$

Of course we will consider that any element of $H^m(\Omega,\mathbb{R}^n)$ which does not depend on x_3 is assimilated to an element of $H^m(\omega,\mathbb{R}^n)$. Let

$$\begin{cases}{}^pI^- := \big\{(i,j,k) \in \{1,2,3\}^3;\ k \in \emptyset \text{ if } (i,j) \in \{1,2\}^2,\\ \qquad\qquad k = 3 \text{ if } (i,j) \in (\{1,2,3\}\times 3)\cup(\{3\}\times\{1,2,3\})\big\}, \quad p = 1,2, \quad {}^3I^- = \emptyset\\ {}^1I^0 = \emptyset, \quad {}^pI^0 := \{(i,j,k) \in \{1,2,3\}^3;\ k \neq 3\}, \quad p = 2,3\\ {}^pI^+ := \{1,2,3\}^3 \setminus ({}^pI^- \cup {}^pI^0), \quad \forall p \in \{1,2,3\}\\ \left.\begin{array}{l}\left.\begin{array}{l}({}^pg^\diamond)_{ijk} := g_{ijk},\ \forall (i,j,k) \in {}^pI^\diamond,\ \forall g \in \mathbb{T}^3\\ {}^p\mathbb{T}^\diamond := \{{}^pg^\diamond;\ g \in \mathbb{T}^3\},\ {}^p\mathbb{K}^\diamond := \mathbb{S}^3 \times {}^p\mathbb{T}^\diamond\end{array}\right\} \forall\diamond \in \{-,0,+\}\\ {}^pk^+ := (e, {}^pg^+),\ {}^pk^\diamond := (0, {}^pg^\diamond),\ \forall\diamond \in \{-,0\},\ \forall k = (e,g) \in \mathbb{K}\end{array}\right\} \quad \forall p \in \{1,2,3\}.\end{cases} \tag{2.7}$$

For all p in $\{1,2,3\}$, let $({}^p\widetilde{m}, {}^pK, {}^pK^s)$ in $\mathrm{Lin}({}^p\mathbb{K}^+) \times \mathrm{Lin}(L^2(\Omega, {}^p\mathbb{K}^+), L^2(\Omega,\mathbb{K})) \times \mathrm{Lin}(L^2(\Omega, {}^p\mathbb{K}^+), L^2(\Omega^\varepsilon,\mathbb{K}))$ defined by

$$\begin{cases} {}^p\widetilde{m}\, q\cdot q := \mathrm{Min}\,\{m\,k\cdot k\,;\ k\in\mathbb{K},\ {}^pk^+ = q,\ {}^2k^0 = 0,\ {}^3k^0 = 0\}, \quad \forall q\in {}^p\mathbb{K}^+ \\ m\,{}^pK(q)\cdot {}^pK(q) = {}^p\widetilde{m}\, q\cdot q, \quad \forall q\in L^2(\Omega, {}^p\mathbb{K}^+) \\ {}^pK^s(q)(x^\varepsilon) = \ell\, {}^pK(q)(x), \quad \forall q\in L^2(\Omega, {}^p\mathbb{K}^+), \quad a.e.\ x^\varepsilon\in\Omega^\varepsilon \end{cases} \tag{2.8}$$

and finally for all $U = (u^M, z, u^F, y)$ in ${}^p\mathscr{U}$, let ${}^pg(U)$ in ${}^p\mathbb{T}^+$ defined by

$$\begin{cases} {}^1g_{\alpha\beta\gamma}(U) := \bar\ell\partial_\gamma e_{\alpha\beta}(u^M), \quad {}^1g_{\alpha\beta3}(U) := 2\bar\ell e_{\alpha\beta}(y) - \partial^2_{\alpha\beta}u^F \\ {}^1g_{\alpha3\gamma}(U) := \bar\ell\partial_\gamma y_\alpha, \quad {}^1g_{33\gamma}(U) := \bar\ell\partial_\gamma z \\ {}^2g_{\alpha\beta3}(U) := -\partial^2_{\alpha\beta}u^F \\ {}^3g_{\alpha\beta3}(U) := -\partial^2_{\alpha\beta}u^F, \quad {}^3g_{\alpha33}(U) := \bar{\bar\ell}\partial_3 y_\alpha, \quad {}^3g_{333}(U) := \bar{\bar\ell}\partial_3 z \\ {}^pk(U) := ({}^pe(U), {}^pg(U)). \end{cases} \tag{2.9}$$

Thus the asymptotic behavior of u^s is determined through the

Theorem 2.1. *Under assumptions $(H_{stat_i})_{1\le i\le 3}$, there holds:*

- *When $p\in\{1,2,3\}$*

$$\lim_{s\to\bar s}\left(\frac{1}{\varepsilon\ell^2}\int_{\Omega^\varepsilon} m^\varepsilon(k^s(u^s) - {}^pk^s({}^pU))\cdot(k^s(u^s) - {}^pk^s({}^pU))\right)\mathrm{d}x^\varepsilon = 0, \tag{2.10}$$

  where ${}^pU$ is the unique solution to:

$$({}^p\mathscr{P})\begin{cases} \text{Find } {}^pU \text{ in } {}^p\mathscr{U} \text{ such that} \\ \displaystyle\int_\Omega {}^p\widetilde{m}\,{}^pk({}^pU)\cdot{}^pk(U)\,\mathrm{d}x = {}^pL(U), \quad \forall U\in{}^p\mathscr{U}, \end{cases}$$

  where we simply denote ${}^pK^s({}^pk({}^pU))$ by ${}^pk^s({}^pU)$.
- *When $p = 4$, one has:*

$$\lim_{s\to\bar s}\frac{1}{\varepsilon\ell^2}\int_{\Omega^\varepsilon} m^\varepsilon(x^\varepsilon)\,k^s(u^s)\cdot k^s(u^s)\,\mathrm{d}x^\varepsilon = 0. \tag{2.11}$$

Proof. As usual C denotes a constant independent of s which may vary from line to line. The fundamental identity:

$$\partial_{ij}u_k = \partial_j e_{ik}(u) + \partial_i e_{jk}(u) - \partial_k e_{ij}(u) \tag{2.12}$$

implies

$$|D^2(\varepsilon\hat u_s, u_{s_3})|_{L^2(\Omega,\mathbb{T}^3)} \le C|g_\varepsilon(u_s)|_{L^2(\Omega,\mathbb{T}^3)} \tag{2.13}$$

so that taking $u = u_s$ in the formulation of $(\mathscr{P}_s)$ successively yields:

$$|g_\varepsilon(u_s)|_{L^2(\Omega,\mathbb{T}^3)} \le C, \quad \frac{\varepsilon}{\ell}|e(\varepsilon, u_s)|_{L^2(\Omega,\mathbb{T}^3)} \le C. \tag{2.14}$$

Hence, when $p = 4$, $e(u_s)$ and $e(\varepsilon, u_s)$ converge strongly toward 0 in $L^2(\Omega,\mathbb{S}^3)$. Hence (H_{stat_3}) implies that $g_\varepsilon(u_s)$ converges strongly toward 0 in $L^2(\Omega,\mathbb{T}^3)$ and, by (2.3), that (2.11) is true.

When p belongs to $\{1,2,3\}$, we go straightforwardly through the three usual steps:

• Step 1: *First results on the asymptotic behavior of u_s.*

Proposition 2.1. *For all p in $\{1,2,3\}$, there exists a not relabeled subsequence and ${}^pU = ({}^pu^M, {}^pz, {}^pu^F, {}^py)$ in ${}^p\mathscr{U}$ such that $(((\varepsilon/\ell)\hat u_s, (1/\ell\varepsilon)\partial_3 u_{s_3}, u_{s_3}, (1/\ell)e_{\alpha3}(u_s)), k_s(u_s))$ weakly converges in ${}^pH\times L^2(\Omega,\mathbb{K})$ toward $({}^pU, {}^p\bar k)$ with*

$${}^p({}^p\bar k)^+ = {}^pk({}^pU) = ({}^pe({}^pU), {}^pg({}^pU)). \tag{2.15}$$

Proof. The weak convergence in ${}^pH \times L^2(\Omega,\mathbb{K})$ is an immediate consequence of (2.13) and (2.14). That ${}^pU$ belongs to ${}^p\mathscr{U}$ is classical when $p = 3$ (cf. [2]). If p belongs to $\{1,2\}$, $(2.14)_1$ implies $\partial_3{}^py = 0$, $\partial_3{}^pz = 0$ while $(2.14)_2$ yields $\partial_3{}^pu^F = 0$ and $(\varepsilon/l)\,u_s$ weakly converges in $H^1(\Omega,\mathbb{R}^3)$ toward some ${}^pu^\star$ such that $\widehat{{}^pu^\star}(x) = {}^pu^M(\hat{x}) - x_3\hat{\nabla}\,{}^pu^{\star F}(\hat{x})$, a.e. $x \in \Omega$, with ${}^pu^{\star F}$ the weak limit of $(\varepsilon/l)\,u_{s_3}$ which is 0! The very definition of $g_\varepsilon(u_s)$ and the identity

$$\partial_3 e_{\alpha\beta}(u_s) = \partial_\alpha e_{3\beta}(u_s) + \partial_\beta e_{\alpha 3}(u_s) - \partial^2_{\alpha\beta} u_{s_3} \tag{2.16}$$

imply that $\partial_3 e_{\alpha\beta}(u_s)$ weakly converges in $L^2(\Omega)$ toward $e_{\alpha\beta}({}^py) - \partial^2_{\alpha\beta}({}^pu^F)$. Eventually when $p = 2$, ℓ goes to 0 yields the expression (2.9) of ${}^2g({}^2U)$. □

- Step 2: *Identification of ${}^pU$ which describes the asymptotic behavior of u_s.*

Proposition 2.2. *The whole sequences indexed by s converge and ${}^pU$ is the unique solution to $({}^p\mathscr{P})$.*

Proof. When p belongs to $\{1,2\}$, by going to the limit in the formulation of $(\mathscr{P}_s)$ with u such that first

$$u = \varepsilon(\hat{\eta}, \varepsilon\eta_3), \quad \eta(x) = \int_{-1}^{x_3}\int_{-1}^{t} w(\hat{x},\tau)\,\mathrm{d}\tau\,\mathrm{d}t,\ a.e.\ x \in \Omega,\ w \text{ arbitrary in } C^\infty(\bar{\Omega},\mathbb{R}^3) \tag{2.17}$$

and next

$$u = \ell\left(\frac{1}{\varepsilon}u^M + x_3\left(2y - \frac{1}{\ell}\hat{\nabla}u^F\right), \frac{1}{\ell}u^F + \varepsilon x_3 z\right), \quad (u^M, z, u^F, y) \text{ arbitrary in } {}^p\mathscr{U} \tag{2.18}$$

one successively deduces ${}^p(m\bar{k})^- = 0$ and ${}^pU$ is the unique solution to $({}^p\mathscr{P})$. When $p = 3$, it is right to use u defined by:

$$u = \ell\left(\frac{1}{\varepsilon}u^M + 2x_3 y, \frac{1}{\ell}u^F + \varepsilon x_3 z\right) + \varepsilon\int_{-1}^{x_3}\left(2y(\cdot,t) - \int_{-1}^{t}\hat{\nabla}z(\cdot,\tau)\,\mathrm{d}\tau, \varepsilon z(\cdot,t)\right)\mathrm{d}t,$$
$$(u^M, z, u^F, y) \text{ arbitrary in } {}^p\mathscr{U}. \tag{2.19}$$

□

- Step 3: *Strong convergence in $L^2(\Omega,\mathbb{K})$ of $k_s(u_s)$ toward ${}^pK({}^pk({}^pU))$.*

By taking $u = u_s$ in the formulation of $(\mathscr{P}_s)$, Proposition 2.1, the very definitions of ${}^p\widetilde{m}$ and ${}^pK$ infer

$$\lim_{s\to\bar{s}}\int_\Omega m\,k_s(u_s)\cdot k_s(u_s)\,\mathrm{d}x = \int_\Omega m\,{}^pK({}^pk({}^pU))\cdot{}^pK({}^pk({}^pU))\,\mathrm{d}x \tag{2.20}$$

which establishes the strong convergence of $k_s(u_s)$ toward ${}^pK({}^pk({}^pU))$.

Thus (2.10) stems from the identity (2.3) and the definition (2.8) of ${}^pK^s$! □

3. Some remarks

Remark 3.1. As $\int_{\Omega^\varepsilon} m^\varepsilon\,{}^pk^s({}^pU)\cdot{}^pk^s({}^pU)\,\mathrm{d}x^\varepsilon = \varepsilon\ell^2\int_\Omega m\,{}^pk({}^pU)\cdot{}^pk({}^pU)\,\mathrm{d}x$, Theorem 2.1 supplies a "simplified and accurate" equivalent of the generalized *physical* strain $k^s(u^s)$ by a convergence of *relative energetic gaps* measured on the real *physical* plate (the only one which has a meaning because the strain energies are going to zero! ...). Note also that (H_0) and (H_{stat_2}) imply that

$$\lim_{s\to\bar{s}}\frac{1}{\varepsilon\ell^2}\int_{\Omega^\varepsilon}(m^\varepsilon)^{-1}(\vartheta^s - {}^p\Upsilon^s)\cdot(\vartheta^s - {}^p\Upsilon^s)\,\mathrm{d}x^\varepsilon = 0, \tag{3.1}$$

where ${}^p\Upsilon^s := m^\varepsilon\,{}^pk^s({}^pU)$, which provides an equivalent to the couple ϑ^s made of the Cauchy stress and hyperstress tensors (see (1.2) and (1.3)). This equivalent can be viewed as simplified because the field ${}^pk^s({}^pU)$ is obtained through the formula ${}^pk^s({}^pU) = {}^pK^s({}^pk({}^pU))$ by solving a linear problem $({}^p\mathscr{P})$ set on the abstract fixed domain Ω.

But even if $({}^p\mathscr{P})$ is posed on an abstract domain and involves abstract fields, it is interesting to interpret it in mechanical terms as say a model for thick plates. The state variable U of this limit system can be considered as a displacement field with in-plane component u^M and transversal one u^F but with:

- a kind of "strain tensor" ${}^pe(U)$ such that
$$ {}^pe_{\alpha\beta}(U) = \frac{1}{2}(\partial_\alpha u^M_\beta + \partial_\beta u^M_\alpha), \quad {}^pe_{\alpha 3}(U) = y_\alpha, \quad {}^pe_{33}(U) = z \tag{3.2} $$
thus introducing additional state variables y and z.
- a kind of "strain gradient tensor" ${}^pg(U)$ in ${}^p\mathbb{T}^+$ determined by the components ${}^pg_{ijk}(U)$ with indexes i, j, k in a suitable set ${}^pI^+$ corresponding to some appropriate first partial derivatives of ${}^pe(U)$ and second partial derivatives of u^F.

As to the (kind of) internal forces, they can be represented by ${}^p\vartheta := ({}^p\sigma, {}^p\mu)$ in ${}^p\mathbb{K}^+ := \mathbb{S}^3 \times {}^p\mathbb{T}^+$ satisfying
$$ {}^p\vartheta = {}^p\widetilde{m}\,({}^pe({}^pU), {}^pg({}^pU)). \tag{3.3} $$
Note that unlike the Kirchhoff–Love model, all the components of the Cauchy stress tensor are involved here.

When p belongs to $\{1,2\}$, $({}^p\mathscr{P})$ reduces to a linear boundary value problem set on ω with "coefficients" involving the averages with respect to x_3 of the entries of ${}^p\widetilde{m}$. Hence ${}^pk({}^pU)$ depends only on $\hat{x}$, but if the entries of ${}^p\widetilde{m}$ depend on x_3, ${}^pK^s({}^pk({}^pU))$ will depend on x_3 so that ${}^pk^s$ does depend on x_3^ε which is not the case when m does not depend on x_3. In the following Remark 3.3, conditions implying a decoupling between (u^M, z) and (u^F, y) will be given. As to $({}^3y(\hat{x},\cdot), {}^3z(\hat{x},\cdot))$, it satisfies a Sturm–Liouville problem on $(-1,1)$ with data like $e(u^{M0})(\hat{x})f(x) + \partial^2_{\alpha\beta}u^F(\hat{x})g(x)$ where $u^{M0} = u^M + (x_3/2)\widehat{\nabla}u^F$ and f, g are algebraic functions of a, b, c and of their partial derivatives with respect to x_3, so that $({}^3\mathscr{P})$ reduces to a standard problem set on ω like the one encountered in the static theory of Kirchhoff–Love plates.

Eventually it is possible to introduce an additional parameter r accounting for the magnitude of the loading. The framework of the study being linear, the solution of the corresponding problem $(\mathscr{P}^{s,r})$ would be in this case $(r/\ell^2)u^s$ and the results of convergence of energetic gap would be the same by replacing ${}^pk({}^pU)$ by $(r/\ell^2)\,{}^pk({}^pU)$!

Remark 3.2. This study improves [3] which confines to the case $p = 1$ with *isotropic* strain gradient elasticity (where $b = 0$ and ${}^1\widetilde{m}$ yields a decoupling between (u^M, z) and (u^F, y)), and a loading such that $\hat{L}_\varepsilon(\hat{u}) = l\varepsilon\int_\Omega \hat{f}(x)\cdot\hat{u}(x)\,\mathrm{d}x$, $L_{\varepsilon_3}(u_3) = \int_\Omega f_3(x)u_3(x)\,\mathrm{d}x$ which imply vanishing in-plane effects!

Remark 3.3. It is worthwhile to highlight some properties of the operators ${}^p\widetilde{m}$ supplying the constitutive equations of the second-grade elastic plate when p belongs to $\{1,2,3\}$. In this direction it is advantageous to define ${}^p\mathscr{U}_M := \{U \in {}^p\mathscr{U}; u^F = 0, y = 0\}$ and ${}^p\mathscr{U}_F := \{U \in {}^p\mathscr{U}; u^M = 0, z = 0\}$ so that ${}^p\mathscr{U} = {}^p\mathscr{U}_M \oplus {}^p\mathscr{U}_F$. The spaces ${}^p\mathscr{U}_M$ and ${}^p\mathscr{U}_F$ are said to be ${}^p\widetilde{m}$-polar when ${}^p\widetilde{m}\;{}^pk(U)\cdot{}^pk(V) = 0$ for all (U,V) in ${}^p\mathscr{U}_M \times {}^p\mathscr{U}_F$, $p \in \{1,2\}$.

Considering the influence of crystalline symmetries of the material constituting the thin plate as well as the possible symmetry classes of $\widetilde{m}^\varepsilon$ (see the very valuable works of [4] and the notations therein), we deduce:

- the operators ${}^p\widetilde{m}$ are symmetric elements of $L^\infty(\Omega, \mathrm{Lin}({}^p\mathbb{K}^+))$ and, similarly to (H_0) and (H_{stat_2}), we can write:
$$ {}^p\widetilde{m} = \begin{bmatrix} {}^p\widetilde{a} & {}^p\widetilde{b} \\ {}^p\widetilde{b}^T & {}^p\widetilde{c} \end{bmatrix}, \quad {}^p\widetilde{a} \in \mathrm{Lin}(\mathbb{S}^3), \quad {}^p\widetilde{b} \in \mathrm{Lin}(\mathbb{S}^3, {}^p\mathbb{T}^+), \quad {}^p\widetilde{c} \in \mathrm{Lin}({}^p\mathbb{T}^+), $$
- the very definitions of ${}^3\widetilde{m}$, ${}^3k^0$ and ${}^3k^-$ imply that the coefficients never mix and that the genuine elastic tensor a is left unchanged through the asymptotic process when $p = 3$

which means that ${}^3\widetilde{a} = a$ while the sub-operators ${}^3\widetilde{b}$ and ${}^3\widetilde{c}$ are only composed of elastic coupling and second-order elastic coefficients, respectively,
- the limit elastic tensors ${}^1\widetilde{a}$ and ${}^2\widetilde{a}$ are always identical: ${}^1\widetilde{a} = {}^2\widetilde{a}$,
- of course, for any centro-symmetric class material (which obviously implies isotropy) the genuine coupling elastic tensor b vanishes. This property is not altered in the limit model and ${}^p\widetilde{b} = 0$ for $p = 1,2,3$ in these situations. Perhaps more surprisingly, the decoupling between the first and second-order elasticities also occur when $p = 2,3$ for materials whose symmetry class is D_{10}^h!
- for materials with Z_8^-, D_8^h and D_{10}^h symmetry classes, we have ${}^1\widetilde{a} = {}^2\widetilde{a} = {}^3\widetilde{a} = a$. This is also true for materials with centro-symmetric microstructure. In all other situations, ${}^1\widetilde{a}$ and ${}^2\widetilde{a}$ involve a mixture of elastic, elastic coupling and second-order elastic coefficients,
- for materials with trigonal or hexagonal microstructures (and only for them), the elastic tensors a and ${}^p\widetilde{a}$ do not share the same symmetry class, $p = 1,2$. More precisely, ${}^p\widetilde{a}_{66} \neq 1/2({}^p\widetilde{a}_{11} - {}^p\widetilde{a}_{12})$ in these cases,
- from the very definitions of ${}^p\widetilde{m}$ and ${}^pk^\diamond$, ${}^1\widetilde{m}$-polarity implies ${}^2\widetilde{m}$-polarity. Accordingly it can be shown that the spaces ${}^p\mathcal{U}_M$ and ${}^p\mathcal{U}_F$ are ${}^p\widetilde{m}$-polar for Z_2^-, Z_6^-, D_6, D_6^h and D_{10}^h symmetry class materials when $p = 1,2$. Moreover, the same result holds for all centro-symmetric classes that are not trigonal ($p = 2$), or all symmetry classes that are neither trigonal nor pentagonal ($p = 1$). This illustrates a generalization to strain gradient elasticity of the decoupling between flexural and membrane displacements which is well known in some cases of first-order elastic plates,
- when $p = 3$, it should be noted that u^M and u^F are not independent of each other and that the field (u^M, y, z) depends on x_3 (see (2.19)). Therefore the notion of ${}^3\widetilde{m}$-polarity is not relevant.

We dwelve into more specific polarity properties in appendix A.

Remark 3.4. As written previously, the strain $e^\varepsilon(u^s)$ in the real plate is equivalent to ${}^pe^s({}^pU) = \ell\, {}^pe({}^pU)(\Pi_\varepsilon^{-1} x^\varepsilon)$ which *in general is not of Reissner–Mindlin type*. Indeed, when $p = 3$, ${}^pe_{i3}({}^pU)$ depends on x_3 in general while, when p belongs to $\{1,2\}$, ${}^pz = {}^pe_{33}({}^pU)$ does not vanish and ${}^pe_{\alpha\beta}({}^pU)$ does not depend on x_3 so that it is not an affine function of x_3. A remedy to that last point is to add to $\ell({}^pu^M, {}^pu^F)$ the field $(x_3^\varepsilon(2\,{}^py - \widehat{\nabla}\,{}^pu^F), 0)$ whilst ${}^pz = 0$ is obtained when both ${}^p\hat{L} = 0$ (only for $p = 1$) and ${}^p\widetilde{b} = 0$. That highlights the connection between the microstructure and Reissner–Mindlin plate model (see [5]).

Remark 3.5. As in polycrystalline plates one expects to have only one grain in the thickness and many grains in the surface with an internal length related to the grain size, it is interesting to consider the case of an internal length anisotropy. For this purpose it suffices to replace the scalar ℓ in the definition (1.1) of the generalized strain k^s by an operator in $\mathrm{Lin}(\mathbb{T}^3)$ and in (1.2) to replace ℓ^2 by the square of the norm of this new operator which will be assumed to converge to some limit in $\mathrm{Lin}(\mathbb{T}^3)$ with various relative behavior with respect to ε. This will generate new definitions of ${}^pg(U)$!

Remark 3.6. One can also introduce another parameter which accounts for the coupling between strain and strain gradient as a scalar factor λ in front of b in (H_0). Because of (2.14) one easily deduces that when (s,λ) goes to $(\bar{s},0)$ the coupling disappears in the sense that $({}^p\mathcal{P})$ involves ${}^p\widetilde{m}_0$ built from $m_0 := \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}$.

Remark 3.7. As there are no kinematic links between the fields of displacement and the electric and magnetic fields, it is obvious to derive an asymptotic modeling of electromagnetoelastic plates with gradients of strain, electric and magnetic fields by simply superposing the present analysis and the one done in [6, 7].

4. Setting the dynamic problem

We denote the field of velocities in the plate by v^s and introduce

$$h^s(v^s) := (v^s, \ell\nabla^\varepsilon v^s). \tag{4.1}$$

Referring to [8] we consider that the kinetic energy of the plate reads as $\int_{\Omega^\varepsilon} 1/\ell\, j^\varepsilon(x^\varepsilon)\, h^s(v^s)\cdot h^s(v^s)\,\mathrm{d}x^\varepsilon$ where

$$\begin{cases} j^\varepsilon(x^\varepsilon) = j(x), \quad a.e.\ x\in\Omega \\ j \text{ belongs to } L^\infty(\Omega, \mathrm{Lin}(\mathbb{R}^3\times\mathbb{M}^3)) \text{ and is such that} \\ \quad \text{(i) } j = \begin{bmatrix} \rho I & d \\ d^T & J \end{bmatrix} \\ \quad \text{with } I \text{ the identity of } \mathrm{Lin}(\mathbb{R}^3), (d,J)\in \mathrm{Lin}(\mathbb{M}^3,\mathbb{R}^3)\times\mathrm{Lin}(\mathbb{M}^3,\mathbb{M}^3), \\ \qquad \mathbb{M}^3 \text{ the space of } 3\times 3 \text{ matrices,} \\ \quad \text{(ii) } j(x)q\cdot q \geq \alpha_m |q|^2, \quad \forall q\in\mathbb{R}^3\times\mathbb{M}^3,\ a.e.\ x\in\Omega. \end{cases} \tag{H_{dyn_1}}$$

The space $\mathbb{R}^3\times\mathbb{M}^3$ is equipped with its usual inner product and norm still denoted by $\cdot$ and $|\,|$.

Now we assume that the previous loading L^s is smoothly time dependent, namely:

$$L^s \in C^{1,1}([0,T], H^2_{\Gamma^\varepsilon_D}(\Omega^\varepsilon,\mathbb{R}^3)') \tag{H_{dyn_2}}$$

so that the problem (Q^s) of determining the evolution on the interval of time $[0,T]$ of the plate in an initial state $X^{s0} := (u^{s0}, v^{s0})$ and subjected to the loading L^s can be formulated as:

$$(Q^s)\begin{cases} \text{Find } X^s := (u^s, v^s) \text{ sufficiently smooth in } \Omega^\varepsilon\times[0,T] \text{ such that} \\ \displaystyle\int_{\Omega^\varepsilon} j^\varepsilon(x^\varepsilon) h^s(\dot{v}^s)\cdot h^s(v) + m^\varepsilon(x^\varepsilon) k^s(u^s)\cdot k^s(v)\,\mathrm{d}x^\varepsilon = L^s(t)(v) \\ \text{for all } v \text{ sufficiently smooth in } \Omega^\varepsilon \text{ and almost every } t \text{ in } [0,T] \\ X^s(0) = X^{s0}. \end{cases} \tag{4.2}$$

As usual, we seek u^s on the form:

$$u^s(t) = u^{se}(t) + u^{sr}(t) \tag{4.3}$$

with u^{se} directly coming from the static study through the “quasi-static” formulation

$$u^{se}(t)\in H^2_{\Gamma^\varepsilon_D}(\Omega^\varepsilon,\mathbb{R}^3) =: \mathscr{U}^s; \quad \varphi^s(u^{se}(t),u) = L^s(t)(u), \quad \forall u\in\mathscr{U}^s,\ \forall t\in[0,T], \tag{4.4}$$

where

$$\varphi^s(u,u') = \frac{1}{\varepsilon\ell^2}\int_{\Omega^\varepsilon} m^\varepsilon(x^\varepsilon) k^s(u)\cdot k^s(u')\,\mathrm{d}x^\varepsilon, \quad \forall u,u'\in\mathscr{U}^s. \tag{4.5}$$

Clearly u^{se} is well-defined and satisfies

$$u^{se}\in C^{1,1}([0,T];\mathscr{U}^s) \tag{4.6}$$

and we set

$$X^{se} := (u^{se}, 0) \tag{4.7}$$

so that the remaining part u^{sr} is involved in an evolution equation on a space $\mathscr{H}^s$ of possible states with finite total (strain + kinetic) mechanical energy. If $\mathscr{T}^s$ is the following bilinear form associated with the kinetic energy

$$\mathscr{T}^s(v,v') := \frac{1}{\varepsilon\ell^2}\int_{\Omega^\varepsilon} j^\varepsilon(x^\varepsilon) h^s(v)\cdot h^s(v')\,\mathrm{d}x^\varepsilon, \quad \forall v,v'\in\mathscr{V}^s := H^1(\Omega^\varepsilon;\mathbb{R}^3) \tag{4.8}$$

then $\mathscr{H}^s$ reads as:

$$\mathscr{H}^s := \mathscr{U}^s\times\mathscr{V}^s \tag{4.9}$$

and is a Hilbert space if equipped with the following inner product and norm:

$$\begin{aligned}(X,X')^s &:= \varphi^s(u,u') + \mathcal{T}^s(v,v'), \quad \forall X=(u,v), X'=(u',v') \in \mathcal{H}^s \\ |X|^s &= [(X,X)^s]^{1/2}.\end{aligned} \tag{4.10}$$

As in [9] and due to (2.3), the scaling factor $1/\varepsilon\ell^2$ will appear as appropriate to determine quantitatively and qualitatively the asymptotic behavior of X^s when s goes to $\bar{s}$. So if A^s is the operator in $\mathcal{H}^s$ defined by

$$\begin{cases} D(A^s) := \left\{ X=(u,v) \in \mathcal{H}^s; \quad \begin{cases} \text{(i) } v \in \mathcal{U}^s \\ \text{(ii) } \exists!\, w \in \mathcal{V}^s;\ ((w,u),(v',v'))^s = 0, \quad \forall v' \in \mathcal{U}^s \end{cases} \right\} \\ A^s X := (v,w) \end{cases} \tag{4.11}$$

and which obviously satisfies

Proposition 4.1. *A^s is skew adjoint and for all $\Psi^s = (\Psi^s_u, \Psi^s_v)$ in $\mathcal{H}^s$*

$$\bar{X}^s - A^s\bar{X}^s = \Psi^s \Leftrightarrow \begin{cases} \bar{X}^s = (\bar{u}^s, \bar{v}^s) \\ \bar{u}^s = \bar{v}^s + \Psi^s_u \\ H^s(\bar{v}^s) \le H^s(v) := \frac{1}{2}[|(v,v)|^s]^2 + ((\Psi^s_u, -\Psi^s_v),(v,v))^s, \quad \forall v \in \mathcal{U}^s \end{cases} \tag{4.12}$$

then (Q^s) is formally equivalent to

$$(\mathcal{Q}^s)\begin{cases} \dfrac{\mathrm{d}X^s}{\mathrm{d}t} = A^s(X^s - X^{se}) \\ X^s(0) = X^{s0} \end{cases} \tag{4.13}$$

so that the following assumption

$$X^{s0} - X^{se}(0) \in D(A^s) \tag{H_{dyn_3}}$$

allows us to state the classical result:

Theorem 4.1. *Under assumptions (H_{stat_2}) and $(H_{\mathrm{dyn}_i})_{1\le i\le 3}$, the problem $(\mathcal{Q}^s)$ has a unique solution in $C^1([0,T],\mathcal{H}^s) \cap C^0([0,T],D(A^s))$.*

As in the static case, the essential task is—for various reasons, especially numerical ones—to propose a simplified but accurate enough model. This will also be done by considering $s=(\varepsilon,\ell)$ as a parameter satisfying the same conditions as in the static case and involving the four cases indexed by p defined in (H_{stat_1}) and (2.1).

5. Asymptotic behavior for $(\mathcal{Q}^s)$

We will determine the asymptotic behavior by using the theory of Trotter of convergence of semi-groups of linear operators acting on variable Hilbert spaces (cf. [10,11]) which is particularly well-suited to sequences of time-dependent boundary value problems set on sequences of domains. By studying the asymptotic behavior of sequences with uniformly bounded mechanical energies, we will propose the functional framework, i.e. the spaces ${}^p\mathcal{H}$ of possible "limit" states with finite "limit" mechanical energy. Next, by considering the asymptotic behavior of sequences $(\mathcal{X}^s)$ in $\mathcal{H}^s$ such that $\mathcal{X}^s - \mathcal{A}^s\mathcal{X}^s$ is uniformly bounded, we will be able to define a "limit" operator ${}^pA$, skew-adjoint in a *special subspace* $\mathcal{S}\,{}^p\mathcal{H}$ of ${}^p\mathcal{H}$, which may govern an evolution equation in $\mathcal{S}\,{}^p\mathcal{H}$ similarly as A^s does in $\mathcal{H}^s$. It will be the problem $({}^p\mathcal{Q})$ which precisely describes the asymptotic behavior of the solution to $(\mathcal{Q}^s)$ by due account of a result of convergence in a special sense particularly suited to this situation, very common in the physics of continuous media, involving sequences of domains. As in the static case, the reader disinclined to rigorous mathematical developments may straight away direct himself to the crucial comment of Proposition 5.5, which clarifies the statements of Theorems 5.2 and 5.3, and then move to their mechanical interpretations detailed in Section 5.4.

5.1. *The limit space ${}^p\mathcal{H}$*

Similarly to the static case, it is convenient to use the "scaling operator" $\mathcal{S}_\varepsilon$ defined in (2.4) because

$$\begin{aligned}\varphi^s(u,u') &= \int_\Omega m(x)k_s(\mathcal{S}_\varepsilon u)\cdot k_s(\mathcal{S}_\varepsilon u')\,\mathrm{d}x, \quad \forall u,u' \in \mathcal{U}^s\\ \mathcal{T}^s(v,v') &= \int_\Omega j(x)h_s(\mathcal{S}_\varepsilon v)\cdot h_s(\mathcal{S}_\varepsilon v')\,\mathrm{d}x, \quad \forall v,v' \in \mathcal{V}^s\end{aligned} \tag{5.1}$$

with k_s also defined in (2.4) while

$$\begin{aligned} h_{s,\alpha}(v) &:= \frac{\varepsilon}{\ell}v_\alpha \qquad h_{s,3} := \frac{1}{\ell}v_3\\ h_{s,\alpha\beta}(v) &:= \varepsilon\partial_\beta v_\alpha \quad h_{s,\alpha 3} := \partial_3 v_\alpha \qquad \forall v \in H^1(\Omega,\mathbb{R}^3)\\ h_{s,3\beta}(v) &:= \partial_\beta v_3 \qquad h_{s,33} := \frac{1}{\varepsilon}\partial_3 v_3.\end{aligned} \tag{5.2}$$

To describe the asymptotic behavior for p in $\{1,2,3\}$, we need to introduce, in addition to (2.6)–(2.9), some notions and notations related to the velocities:

$$\begin{cases} {}^p\mathcal{V} := \left\{ V = (u^M,\zeta,v^F,\xi) \in \begin{cases} H^1(\omega,\mathbb{R}^2)\times L^2(\omega)\times H^1(\omega)\times L^2(\omega,\mathbb{R}^2), & p\in\{1,2\}\\ L^2(\Omega,\mathbb{R}^2)\times L^2(\Omega)\times L^2(\Omega)\times L^2(\Omega,\mathbb{R}^2);\ (\zeta,\xi) = \bar{\ell}\partial_3(v^M,v^F), & p=3\end{cases}\right\}\\ {}^ph(V) := \left((v^M,v^F), \begin{bmatrix} \bar{\ell}\widehat{\nabla} v^M & \xi\\ \bar{\ell}\widehat{\nabla} v^F & \zeta\end{bmatrix}\right)\\ {}^p\mathcal{T}(V,V') := \int_\Omega j\,{}^ph(V)\cdot{}^ph(V')\,\mathrm{d}x, \quad \forall V,V' \in {}^p\mathcal{V}\end{cases} \tag{5.3}$$

and

$$\begin{cases} {}^p\varphi(U,U') := \int_\Omega {}^p\widetilde{m}\,{}^pk(U)\cdot{}^pk(U')\,\mathrm{d}x, \quad \forall U,U' \in {}^p\mathcal{U}\\ {}^p\mathcal{H} := {}^p\mathcal{U}\times{}^p\mathcal{V}\\ {}^p(X,X') := {}^p\varphi(U,U') = {}^p\mathcal{T}(V,V'), \quad \forall X=(U,V), X'=(U',V') \in {}^p\mathcal{H}\\ {}^p|X| := [{}^p(X,X)]^{1/2}.\end{cases} \tag{5.4}$$

The next proposition shows that the spaces ${}^p\mathcal{H}$ are suitable to describe the asymptotic behavior:

Proposition 5.1. *For all sequences $\mathcal{X}^s = (\mathcal{X}^s_u,\mathcal{X}^s_v)$ in $\mathcal{H}^s$ such that $|\mathcal{X}^s|^s$ is uniformly bounded, there exists a not relabeled subsequence, ${}^p\Xi = ({}^p\Xi_u,{}^p\Xi_v)$ in ${}^p\mathcal{H}$ and ${}^p\bar{k}$ in $L^2(\Omega,\mathbb{K})$ such that, if $\mathcal{X}_s = (\mathcal{X}_{su},\mathcal{X}_{sv}) := (\mathcal{S}_\varepsilon\mathcal{X}^s_u,\mathcal{S}_\varepsilon\mathcal{X}^s_v)$, one has:*

(i) *$((\varepsilon/\ell)\hat{\mathcal{X}}_{su},(1/\ell\varepsilon)\partial_3\mathcal{X}_{su_3},\mathcal{X}_{su_3},(1/\ell)e_{\alpha 3}(\mathcal{X}_{su}))$, $k_s(\mathcal{X}_{su})$, $h_s(\mathcal{X}_{sv})$ weakly converge in ${}^p\mathcal{H}\times L^2(\Omega,\mathbb{K})\times L^2(\Omega,\mathbb{R}^3\times\mathbb{M}^3)$ toward $({}^p\Xi_u,{}^p\bar{k},{}^ph({}^p\Xi_v))$ with ${}^p({}^p\bar{k})^+ = {}^pk({}^p\Xi_u)$,*

(ii) *${}^p|{}^p\Xi| \le \varliminf_{s\to\bar{s}}|\Xi^s|^s$.*

Proof. Point (i) for the displacements $\mathcal{X}^s_u$ is nothing but rephrasing Proposition 2.1 while for velocities it stems from the boundedness of $(\partial_3\hat{\mathcal{X}}_{sv},(1/\varepsilon)\partial_3\mathcal{X}_{sv_3})$ in $L^2(\Omega,\mathbb{R}^3)$. A standard argument of lower semi-continuity and the very definition of ${}^p\widetilde{m}$ yield point (ii). □

Indeed ${}^p\mathcal{H}$ is *exactly* the appropriate space because any of its element Ξ admits a representative ${}^pP^s\,\Xi$ in $\mathcal{H}^s$ which is energetically very close to Ξ:

Proposition 5.2. *For all $\Xi = (U,V)$ in ${}^p\mathcal{H}$, let ${}^pP^s\,\Xi = ({}^pP^s_u\,U,{}^pP^s_v\,V)$ in $\mathcal{H}^s$ be defined by:*

$$({}^pP^s\,\Xi,\mathcal{X}')^s = \int_\Omega {}^p\widetilde{m}\,{}^pk(U)\cdot{}^p(k_s(S_\varepsilon u'))^+ + j\,{}^ph(V)\cdot h_s(S_\varepsilon v')\,\mathrm{d}x =: F^s_\Xi(\mathcal{X}'), \quad \forall\mathcal{X}' = (u',v') \in \mathcal{H}^s \tag{5.5}$$

then

(i) $\exists C>0$ *s.t.* $|{}^pP^s\,\Xi|^s \le C\,{}^p|\Xi|, \quad \forall \Xi \in {}^p\mathcal{H}, \quad \forall s\in(0,+\infty)^2$,

(ii) $\lim_{s\to\bar{s}} |{}^pP^s\,\Xi|^s = {}^p|\Xi|, \quad \forall \Xi \in {}^p\mathcal{H}$,

(iii) $$\lim_{s\to\bar{s}} \frac{1}{\varepsilon\ell^2}\int_{\Omega^\varepsilon} m^\varepsilon(k^s({}^pP^s_u U) - {}^pk^s(U))\cdot(k^s({}^pP^s_u U) - {}^pk^s(U)) + j^\varepsilon(h^s({}^pP^s_v V) - {}^ph^s(V))\cdot(h^s({}^pP^s_v V) - {}^ph^s(V))\,\mathrm{d}x = 0,$$

where ${}^pk^s(U)$ *stands for* ${}^pK^s({}^pk(U))$ *(see Theorem 2.1) and* ${}^ph^s(V)(x^\varepsilon) := \ell\,{}^ph(V)(x)$, $V\in{}^p\mathcal{V}$, *a.e.* $x\in\Omega^\varepsilon$.

Proof. As regard displacement fields, the result is a variant of Theorem 2.1 with the linear form L_s replaced by

$$H^2_{\Gamma_D}(\Omega,\mathbb{R}^2) \ni w \mapsto \int_\Omega {}^p\widetilde{m}\,{}^pk(U)\cdot {}^p(k_s(w))^+\,\mathrm{d}x \tag{5.6}$$

while for the velocities it participates from the same strategy as the one used in Theorem 2.1 with the following test functions to be used in its second step:

$$\begin{aligned} v_\varepsilon &= \left(\frac{\ell}{\varepsilon}v^M + \int_{-1}^{x_3}\xi(\hat{x},\tau)\,\mathrm{d}\tau, \ell v^F + \int_{-1}^{x_3}\zeta(\hat{x},\tau)\,\mathrm{d}\tau\right) \quad \text{if } p\in\{1,2\},\\ v_\varepsilon &= \left(\frac{\ell}{\varepsilon}v^M, v^F\right) \quad \text{if } p=3 \end{aligned} \tag{5.7}$$

with (v^M,ζ,v^F,ξ) arbitrary in ${}^p\mathcal{V}$. □

5.2. *The limit problem* $({}^p\mathcal{Q})$

5.2.1. *The limit operator* ${}^pA$

To guess and define the limit operator ${}^pA$ it suffices by (4.12) to consider sequences such that $\mathcal{T}^s(w^s,w^s)+\varphi^s(w^s,w^s)$ is uniforly bounded *because* ${}^pA$ is such that its resolvent should be the limit of the resolvent of A^s according to Trotter's theory of convergence of semi-groups (see [10, 11]). We will see that the sequence (w_s) with $w_s := \mathcal{S}_\varepsilon w^s$ will have a "limit" ${}^pW$ in a *special subspace* $\mathcal{S}\,{}^p\mathcal{U}$ of ${}^p\mathcal{U}$ that will define the space of "*virtual* generalized velocities" while the limit of the kinetic scaled enery $\mathcal{T}_s(w_s,w_s) := \int_\Omega j\,h_s(w_s)\cdot h_s(w_s)\,\mathrm{d}x$ involves ${}^p\check{W}$ defined from ${}^pW$ by:

$$\begin{cases} {}^pW = ({}^pw^M, {}^pz, {}^pw^F, {}^py) \in \mathcal{S}\,{}^p\mathcal{U} \mapsto {}^p\check{W} := \begin{cases} ({}^pw^M, \bar{\ell}\,{}^pz, (1/\bar{\ell})\,{}^pw^F, 2\bar{\ell}\,{}^py - \hat{\nabla}{}^pw^F) \in {}^p\mathcal{V}, & p=1\\ ({}^pw^M,0,0,0)\in{}^p\mathcal{V}, & p\in\{2,3\}\end{cases}\\ \mathcal{S}\,{}^p\mathcal{U} := \left\{ U=(u^M,z,u^F,y) \in \begin{cases} {}^p\mathcal{U}, & p=1\\ {}^p\mathcal{U};\ u^F=0, & p=2,3\end{cases}\right\}. \end{cases} \tag{5.8}$$

There holds:

Proposition 5.3. *For all* p *in* $\{1,2,3\}$ *and all sequence* (w^s) *in* $\mathcal{H}^s$ *such that* $\mathcal{T}^s(w^s,w^s)+\varphi^s(w^s,w^s)\le C$, *there exists a not relabeled subsequence and* $({}^pW,{}^pW^\star)$ *in* ${}^p\mathcal{H}$ *such that, if* $w_s := \mathcal{S}_\varepsilon w^s$, *one has:*

$$\left(\left(\frac{\varepsilon}{\ell}\widehat{w}_s, \frac{1}{\varepsilon\ell}\partial_3 w_{s_3}, w_{s_3}, \frac{1}{\ell}e_{\alpha3}(w_s)\right), k_s(w_s), h_s(w_s)\right) \rightharpoonup ({}^pW, {}^p\bar{k}, {}^ph(W^\star)) \quad in\ {}^pH\times L^2(\Omega,\mathbb{K})\times L^2(\Omega,\mathbb{R}^3\times\mathbb{M}^3) \tag{5.9}$$

with

$$\begin{cases} {}^p({}^p\bar{k})^+ = {}^pk({}^pW)\\ {}^pW^\star \in {}^p\check{W} + \begin{cases}\{0\} & \textit{if } p=1\\ (0,0,L^2(\omega),0) & \textit{if } p\in\{2,3\}.\end{cases}\end{cases} \tag{5.10}$$

Proof. By Proposition 4.1, there exists some ${}^pW^\star$ such that for a not relabeled subsequence $h_s(w_s)$ weakly converges in $L^2(\Omega, \mathbb{R}^3 \times \mathbb{M}^3)$ toward ${}^ph({}^pW^\star)$, but Proposition 2.1 and $\partial_3 w_{s_\alpha} = 2e_{\alpha 3}(w_s) - \partial_\alpha w_{s_3}$ imply both that ${}^pW$ belongs to $\mathscr{S}\,{}^p\mathscr{U}$ and (5.10). □

However, to avoid using multi-valued operators, we consider a special subspace $\mathscr{S}\,{}^p\mathscr{V}$ of ${}^p\mathscr{V}$ and, consequently, a special subspace $\mathscr{S}\,{}^p\mathscr{H}$ of ${}^p\mathscr{H}$ of limit possible states with finite energy:

$$\begin{aligned} \mathscr{S}\,{}^p\mathscr{V} &:= \{V = (v^M, \zeta, v^F, \xi) \in {}^p\mathscr{V}\ ;\ v^F = 0\} \\ \mathscr{S}\,{}^p\mathscr{H} &:= {}^p\mathscr{U} \times \mathscr{S}\,{}^p\mathscr{V} \end{aligned} \tag{5.11}$$

and we define the unbounded linear operator ${}^pA$ in $\mathscr{S}\,{}^p\mathscr{H}$ by:

$$\begin{cases} D({}^pA) := \left\{ X = (U,V) \in \mathscr{S}\,{}^p\mathscr{H};\ \begin{cases} \text{(i) } \exists! \widetilde{V} \in \mathscr{S}\,{}^p\mathscr{U} \text{ such that } {}^p\check{\widetilde{V}} = V \\ \text{(ii) } \exists! W \in \mathscr{S}\,{}^p\mathscr{V} \text{ such that for all } V' \in \mathscr{S}\,{}^p\mathscr{U} \\ \quad \displaystyle\int_\Omega j\,{}^ph(W)\cdot {}^ph({}^p\check{V}') + {}^p\widetilde{m}\,{}^pk(U)\cdot {}^pk(V') = 0, \end{cases} \right\} \\ {}^pA\,X := (\widetilde{V}, W) \end{cases} \tag{5.12}$$

As for A^s, it is routine to check

Proposition 5.4. *${}^pA$ is skew-adjoint and for all $\Psi = (\Psi_u, \Psi_v)$ in $\mathscr{S}\,{}^p\mathscr{H}$*

$${}^p\bar{X} - {}^pA\,{}^p\bar{X} = \Psi \Leftrightarrow \begin{cases} {}^p\bar{X} = ({}^p\bar{U}, {}^p\bar{V}) \text{ with } {}^p\bar{U} = \widetilde{V} + \Psi_u,\ {}^p\bar{V} = {}^p\check{\widetilde{V}} \\ {}^pH(\widetilde{W}) \le {}^pH(V) := \frac{1}{2}[{}^p|(V, {}^p\check{V})|]^2 + {}^p((\Psi_u, -\Psi_v), (V,V)), \quad \forall V \in \mathscr{S}\,{}^p\mathscr{U}. \end{cases} \tag{5.13}$$

5.2.2. *The limit problem* $({}^p\mathscr{Q})$

As for X^{se}, we consider ${}^pX^e := ({}^pU^e, 0)$ where ${}^pU^e$ is the solution to

$${}^pU^e(t) \in {}^p\mathscr{U}\ ;\ {}^p\varphi({}^pU^e, U) = {}^pL(t)(U), \quad \forall U \in {}^p\mathscr{U}, \quad \forall t \in [0,T] \tag{5.14}$$

that is to say ${}^pU^e$ is the solution to the static problem $({}^p\mathscr{P})$ given by Theorem 2.1. If we make the following time-dependent variant of (H_{stat_3}):

There exists $\hat{L}(t), L_3(t)$ in $C^{1,1}([0,T];\ H^2_{\Gamma_D}(\Omega, \mathbb{R}^3)')$ such that

$$(\hat{L}_s, L_{s_3}) \text{ converges strongly in } C^{1,1}([0,T];\ H^2_{\Gamma_D}(\Omega,\mathbb{R}^3)') \text{ toward } (\hat{L}, L_3) \tag{H_{dyn_4}}$$

we get

$${}^pX^e \in C^{1,1}([0,T];\ {}^p\mathscr{H}) \tag{5.15}$$

and consequently

Theorem 5.1. *Under assumptions (H_{stat_2}), $(H_{\text{dyn}_i})_{1\le i\le 4}$ and*

$${}^pX^0 \in {}^pX^e(0) + D({}^pA) \tag{H_{dyn_5}}$$

the differential equation in $\mathscr{S}\,{}^p\mathscr{H}$

$$({}^p\mathscr{Q}) \begin{cases} \dfrac{\mathrm{d}\,{}^pX}{\mathrm{d}t} = {}^pA({}^pX - {}^pX^e) \\ {}^pX(0) = {}^pX^0 \end{cases} \tag{5.16}$$

has a unique solution in $C^1([0,T], \mathscr{S}\,{}^p\mathscr{H}) \cap C^0([0,T], D({}^pA))$

5.3. *A convergence result*

To express it, we introduce the following fundamental notion (see [10, 11]):

Definition 5.1. A sequence $(\mathscr{X}^s)$ in $\mathscr{H}^s$ is said to converge in the sense of Trotter toward Ξ in ${}^p\mathscr{H}$ if and only if

$$\lim_{s\to\bar{s}} |{}^pP^s\Xi - \mathscr{X}^s|^s = 0. \tag{5.17}$$

So, Proposition 5.2 implies

Proposition 5.5. *The sequence $\mathscr{X}^s = (\mathscr{X}^s_u, \mathscr{X}^s_u)$ in $\mathscr{H}^s$ converges in the sense of Trotter toward $\Xi = (\Xi_u, \Xi_v)$ in ${}^p\mathscr{H}$ if and only if:*

$$\begin{aligned}\lim_{s\to\bar{s}} \frac{1}{\varepsilon\ell^2}\int_{\Omega^\varepsilon} & m^\varepsilon(k^s(\mathscr{X}^s_u) - {}^pk^s(\Xi_u))\cdot(k^s(\mathscr{X}^s_u) - {}^pk^s(\Xi_u)) \\ &+ j^\varepsilon(h^s(\mathscr{X}^s_v) - {}^ph^s(\Xi_v))\cdot(h^s(\mathscr{X}^s_v) - {}^ph^s(\Xi_v))\,\mathrm{d}x = 0.\end{aligned} \tag{5.18}$$

This is a convergence result of relative energetic gaps, measured on the real physical plate (the only one which has a meaning because the total mechanical energies are going to zero!), between the state $\mathscr{X}^s$ and the image on the intial physical configuration Ω^ε of the limit state Ξ.

According to Trotter's theory (see [10, 11]), to affirm the convergence in the sense of Trotter of $X^s(t)$ toward ${}^pX(t)$ uniformly on $[0, T]$, it suffices to make the additional assumption:

$$\exists\, {}^pX^0 \in {}^pX^e(0) + D({}^pA)\,;\ \lim_{s\to\bar{s}} |{}^pP^s\,{}^pX^0 - X^{s0}|^s = 0 \tag{H_{dyn_6}}$$

and to establish

Proposition 5.6. *There holds*

(i) $\lim_{s\to\bar{s}} |{}^pP^s(I - {}^pA)^{-1}\Psi - (I - A^s)^{-1}\,{}^pP^s\,\Psi|^s = 0, \quad \forall\Psi \in \mathscr{S}\,{}^p\mathscr{H}$

(ii) $\lim_{s\to\bar{s}} |{}^pP^s X^e(t) - X^{se}(t)|^s = 0, \quad \forall t \in [0, T].$

Proof. First let $\Psi = (\Psi_u, \Psi_v)$ in $\mathscr{S}\,{}^p\mathscr{H}$, according to (5.5) and Proposition 5.1, $\bar{X}^s = (\bar{U}^s, \bar{V}^s) := (I - A^s)^{-1}\,{}^pP^s\Psi$ is such that $\bar{U}^s = \bar{V}^s + {}^pP^s_u\Psi_u$ and $\bar{V}^s$ is the unique minimizer on $\mathscr{U}^s$ of $\widetilde{H}^s$ defined by $\widetilde{H}^s(v) := 1/2[(v, v)]^2 + F^s_{(\Psi_u, -\Psi_v)}(v)$ for all v in $\mathscr{U}^s$. So $\bar{V}^s$ is bounded in both $\mathscr{U}^s$ and $\mathscr{V}^s$ and Proposition 5.3 implies that there exists $({}^pW, {}^pW^\star)$ in $(\mathscr{S}\,{}^p\mathscr{U}, {}^p\mathscr{V})$ such that

$$\tfrac{1}{2}[{}^p|(W, {}^pW^\star)|]^2 + {}^p((\Psi_u, -\Psi_v), ({}^pW, {}^p\check{W}^\star)) \le \varliminf_{s\to\bar{s}} \widetilde{H}^s(\bar{v}_s) \tag{5.19}$$

and the crucial point is to note, as $\psi^F_v = 0$ if $p = 2, 3$, that one has:

$${}^pH({}^pW) \le \tfrac{1}{2}[{}^p|(W, {}^pW^\star)|]^2 + {}^p((\Psi_u, -\Psi_v), ({}^pW, {}^p\check{W}^\star)), \quad \forall p \in \{1, 2, 3\}. \tag{5.20}$$

To conclude that the whole sequence converges toward $\bar{V}$ the unique minimizer on $\mathscr{S}\,{}^p\mathscr{U}$ of the strictly convex function ${}^pH$, it remains to show that for all V in $\mathscr{S}\,{}^p\mathscr{U}$ there exists a sequence (v^s) in $\mathscr{U}^s$ such that if $v_s := \mathscr{S}_\varepsilon v^s$ one has $\varlimsup_{s\to\bar{s}} \widetilde{H}^s(v^s) \le {}^pH(V)$ and it is straightforward to check that v_s given by (2.18) suits.

Eventually the convergence (ii) is nothing but the one stated in Theorem 2.1. □

Thus one has the final convergence result:

Theorem 5.2. *Under assumptions $(H_{\mathrm{stat}_i})_{1\le i\le 2}$ and $(H_{\mathrm{dyn}_i})_{1\le i\le 6}$, the solution X^s to $(\mathscr{Q}^s)$ converges in the sense of Trotter toward the solution ${}^pX$ to $({}^p\mathscr{Q})$, p in $\{1, 2, 3\}$.*

5.4. *The asymptotic behaviors and models*

First when $p = 4$, the Duhamel formula involving the unitary group generated by A^s, assumption (H_{dyn_2}) and

$$\lim_{s\to\bar{s}} |X^{s0}|^s = 0 \qquad (H_{\text{dyn}_7})$$

imply

Theorem 5.3. *When $p = 4$, under assumptions $(H_{\text{stat}_i})_{1\le i\le 2}$ and $(H_{\text{dyn}_i})_{i\in\{1,2,3,4,5\}\cup\{7\}}$, one has:*

$$\lim_{s\to\bar{s}} |X^s(t)|^s = 0 \quad \textit{uniformly on } [0,T]. \tag{5.21}$$

When $p \in \{1,2,3\}$, Theorem 5.2 and Proposition 5.5 supply a “simplified and accurate” equivalent $({}^p k^s, {}^p h^s)$ of the generalized physical strain $k^s(u^s)$ and generalized velocities $h^s(u^s)$ by a convergence of *relative* energetic gaps measured on the real physical plate. It can be viewed as simplified because the fields $({}^p k^s, {}^p h^s)$ are obtained through the formula of Proposition 5.2 by solving a linear time-dependent problem set on the abstract fixed domain Ω which has valuable features.

For the sake of brevity, we skip the left superscripts 1, 2 and 3 for the solutions of the following limit formulations. Problem $({}^1\mathcal{Q})$ is posed over the bidimensional set ω and reads as:

$$\left\{\begin{aligned}
&\text{Find } (u^M, z, u^F, y) \text{ in } {}^1\mathcal{U} \text{ such that for all } V' = (v'^M, z', v'^F, y') \text{ in } \mathcal{S}\,{}^1\mathcal{U}\\
&\int_\omega \left[\int_{-1}^1 j(\hat{x}, x_3)\,\mathrm{d}x_3\right] \left(\left(\ddot{u}^M, \frac{1}{\bar\ell}\ddot{u}^F\right) \begin{bmatrix} \bar\ell\hat\nabla\ddot{u}^M & 2\bar\ell\ddot{y} - \hat\nabla\ddot{u}^F \\ \bar\ell\hat\nabla\ddot{u}^F & \bar\ell\ddot{z}\end{bmatrix}\right) \cdot \left(\left(v'^M, \frac{1}{\bar\ell}v'^F\right) \begin{bmatrix} \bar\ell\hat\nabla v'^M & 2\bar\ell y' - \hat\nabla v'^F \\ \bar\ell\hat\nabla v'^F & \bar\ell z'\end{bmatrix}\right)\\
&\quad + \left[\int_{-1}^1 {}^1\widetilde{m}(\hat{x}, x_3)\,\mathrm{d}x_3\right] \left(\begin{bmatrix} e_{\alpha\beta}(u^M) & y \\ y & z\end{bmatrix}, \begin{bmatrix} \bar\ell\partial_\gamma e_{\alpha\beta}(u^M) & 2\bar\ell e_{\alpha\beta}(y) - \partial^2_{\alpha\beta}u^F \\ \bar\ell\partial_\gamma y_\alpha & \bar\ell\partial_\gamma z\end{bmatrix}\right)\cdot\\
&\quad \cdot\left(\begin{bmatrix} e_{\alpha\beta}(v'^M) & y' \\ y' & z'\end{bmatrix}, \begin{bmatrix} \bar\ell\partial_\gamma e_{\alpha\beta}(v'^M) & 2\bar\ell e_{\alpha\beta}(y') - \partial^2_{\alpha\beta}u'^F \\ \bar\ell\partial_\gamma y'_\alpha & \bar\ell\partial_\gamma z'\end{bmatrix}\right) \mathrm{d}\hat{x} = \bar\ell L(v'^M) + L_3(v'^F),\\
&\quad + \text{initial conditions}
\end{aligned}\right. \tag{5.22}$$

Problem $({}^2\mathcal{Q})$ is also a problem posed over the bidimensional set ω, it reads as:

$$\left\{\begin{aligned}
&\text{Find } (u^M, z, u^F, y) \text{ in } {}^2\mathcal{U} \text{ such that for all } V' = (v'^M, z', v'^F, y') \text{ in } \mathcal{S}\,{}^2\mathcal{U}\\
&\int_\omega \left[\int_{-1}^1 j(\hat{x}, x_3)\,\mathrm{d}x_3\right] \left((\ddot{u}^M, 0) \begin{bmatrix} 0 & 0 \\ 0 & 0\end{bmatrix}\right) \cdot \left((v'^M, 0) \begin{bmatrix} 0 & 0 \\ 0 & 0\end{bmatrix}\right)\\
&\quad + \left[\int_{-1}^1 {}^2\widetilde{m}(\hat{x}, x_3)\,\mathrm{d}x_3\right] \left(\begin{bmatrix} e_{\alpha\beta}(u^M) & y \\ y & z\end{bmatrix}, \partial^2_{\alpha\beta}u^{F0}\right) \cdot \left(\begin{bmatrix} e_{\alpha\beta}(v'^M) & y' \\ y' & z'\end{bmatrix}, 0\right) \mathrm{d}\hat{x} = 0\\
&\quad + \text{initial conditions}
\end{aligned}\right. \tag{5.23}$$

The acceleration term involves u^M only while u^F is frozen, as is also y in the cases implying ${}^2\widetilde{m}$-polarity (see Remark 3.3).

Problem $({}^3\mathcal{Q})$ reads as:

$$\left\{\begin{aligned}
&\text{Find } (u^M, z, u^F, y) \text{ in } {}^3\mathcal{U} \text{ such that for all } V' = (v'^M, z', v'^F, y') \text{ in } \mathcal{S}\,{}^3\mathcal{U}\\
&\int_\Omega j(\hat{x}, x_3) \left((\ddot{u}^M, 0) \begin{bmatrix} 0 & 0 \\ 0 & 0\end{bmatrix}\right) \cdot \left((v'^M, 0) \begin{bmatrix} 0 & 0 \\ 0 & 0\end{bmatrix}\right)\\
&\quad + {}^3\widetilde{m} \left(\begin{bmatrix} e_{\alpha\beta}(u^M) & y \\ y & z\end{bmatrix} \begin{bmatrix} -\partial^2_{\alpha\beta}u^{F0} \\ \bar{\bar\ell}\partial_3 y_\alpha \\ \bar{\bar\ell}\partial_3 z\end{bmatrix}\right) \cdot \left(\begin{bmatrix} e_{\alpha\beta}(v'^M) & y' \\ y' & z'\end{bmatrix} \begin{bmatrix} 0 \\ \bar{\bar\ell}\partial_3 y'_\alpha \\ \bar{\bar\ell}\partial_3 z'\end{bmatrix}\right) \mathrm{d}x = 0\\
&\quad + \text{initial conditions}
\end{aligned}\right. \tag{5.24}$$

Here again u^F is frozen and the acceleration term involves only u^M thus $u^{M0} = u^M + (x_3/2)\widehat{\nabla} u^{F0}$. The problem is tridimensional but (y,z) may be eliminated as in the static case and $({}^3\mathscr{Q})$ reduces to a purely membrane dynamical problem as in the Kirchhoff–Love theory of plates (see [9, 12]).

Remark 5.1. Even if from the physical point of view it seems to us not clear at all to introduce a new parameter—say ℓ_{dyn}—in place of ℓ (which from now on we rename ℓ_{stat} for purposes of clarity) in the expression of $h^s(v^s)$, it could be interesting to deal with it from a mathematical point of view or to examine the relevance of the effect of the gradient of velocity on the kinetic energy or of the relative magnitudes of the elements ρ, d, J of $j\dots$ It is straightforward to get that k_s, ${}^pk$ keep the same expressions with ℓ replaced by ℓ_{stat} and for k_s, ${}^pk$ with ℓ replaced by ℓ_{dyn}. So we obtain new cases indexed by $p = (p_1, p_2) \in \{1,2,3,4\}^2$ corresponding to the previous definition of p but with ℓ replaced by ℓ_{stat}, ℓ_{dyn} for p_1, p_2 respectively. We left the easy details to the interested reader and simply mention that one may obtain "crossbred" behavior when $p_1 \neq p_2$. For example when $p = (2,1)$ full gradient of acceleration terms coexist with reduced terms of strain gradient whereas when $p = (1,2)$ no gradient of acceleration terms mixes with a full strain gradient term.

Conflicts of interest

Authors have no conflict of interest to declare.

Acknowledgments

The authors would like to warmly thank Jean-Baptiste Leblond for his interest and his many insightful comments which helped to considerably improve our manuscript.

Appendix A. Influence of symmetries

It is interesting to show how the elastic coupling and the second-order elastic tensors interact with the various components of the gradient of the strain tensor. We define

$$\mathbb{S}^3_M := \left\{(\sigma_{ij}) \in \mathbb{S}^3; \sigma_{31} = \sigma_{23} = 0\right\}, \quad \mathbb{S}^3_F := \left\{(\sigma_{ij}) \in \mathbb{S}^3; \sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{12} = 0\right\}$$

and in the following each element of $\mathbb{S}^3$ is understood as a sextuplet $(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{31}, \sigma_{12})$ of $\mathbb{R}^6$.

For p in $\{1,2\}$ and all $U = (u^M, z, u^F, y)$ in ${}^p\mathscr{U}$ we introduce the following notations and spaces:

$${}^pg_M(U) := {}^pg(u^M, z, 0, 0) \in {}^p\mathscr{U}_M, \quad {}^pg_F(U) := {}^pg(0, 0, u^F, y) \in {}^p\mathscr{U}_F.$$

and recall that:

$${}^p\widetilde{m}\,{}^pk(U)\cdot{}^pk(V) = {}^p\widetilde{a}\,{}^pe(U)\cdot{}^pe(V) + {}^p\widetilde{b}\,{}^pg(U)\cdot{}^pe(V) + {}^p\widetilde{b}^T\,{}^pe(U)\cdot{}^pg(V) + {}^p\widetilde{c}\,{}^pg(U)\cdot{}^pg(V),$$
$$\forall U, V \in {}^p\mathscr{U}.$$

In the case $p = 1$ we associate with ${}^1\widetilde{b}$ the sub-operators ${}^1\widetilde{b}_M$, ${}^1\widetilde{b}_F$ and focus our attention on the expression:

$${}^1\widetilde{b}\,{}^1g(U) = {}^1\widetilde{b}_M\,{}^1g_M(U) + {}^1\widetilde{b}_F\,{}^1g_F(U), \quad \forall U \in {}^1\mathscr{U}.$$

Considering an hexagonal material whose symmetry class is D_6, we get:

$$
{}^1\widetilde{b}\,{}^1g(U) = \overbrace{\begin{pmatrix} x & x & x & x & 0 & 0 & 0 & 0 \\ x & x & x & x & 0 & 0 & 0 & 0 \\ x & x & x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x & x & x & x \end{pmatrix}}^{{}^1\widetilde{b}_M} \underbrace{\begin{pmatrix} g_{111}(U) \\ g_{221}(U) \\ g_{122}(U) \\ g_{331}(U) \\ g_{222}(U) \\ g_{112}(U) \\ g_{121}(U) \\ g_{332}(U) \end{pmatrix}}_{{}^1g_M(U)} + \overbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x & x & x \\ x & x & x & x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}^{{}^1\widetilde{b}_F} \underbrace{\begin{pmatrix} g_{113}(U) \\ g_{131}(U) \\ g_{223}(U) \\ g_{232}(U) \\ g_{231}(U) \\ g_{132}(U) \\ g_{123}(U) \end{pmatrix}}_{{}^1g_F(U)} \tag{A.1}
$$

where we use the letter x to denote non-vanishing components non-necessarily identical. It is therefore clear that ${}^1\widetilde{b}_M{}^1g_M(U)$ belongs to $\mathbb{S}^3_M$ while ${}^1\widetilde{b}_F{}^1g_F(U)$ belongs to $\mathbb{S}^3_F$. In other words, ${}^1\mathcal{U}_M$ and ${}^1\mathcal{U}_F$ are ${}^1\widetilde{b}$-polar in the sense that ${}^1\widetilde{b}\,{}^1g(U)\cdot\,{}^1e(V) = {}^1\widetilde{b}\,{}^1g(V)\cdot\,{}^1e(U) = 0$ for all (U,V) in ${}^1\mathcal{U}_M \times {}^1\mathcal{U}_F$. This is a rather infrequent feature as it occurs only for D_6, Z_2^-, Z_6^-, D_6^h and D_{10}^h symmetry classes materials.

Let us now consider a monoclinic material whose symmetry class is Z_2, we have:

$$
{}^1\widetilde{b}\,{}^1g(U) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x & x & x & x & x & x & x & x \\ x & x & x & x & x & x & x & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} g_{111}(U) \\ g_{221}(U) \\ g_{122}(U) \\ g_{331}(U) \\ g_{222}(U) \\ g_{112}(U) \\ g_{121}(U) \\ g_{332}(U) \end{pmatrix} + \begin{pmatrix} x & x & x & x & x & x \\ x & x & x & x & x & x \\ x & x & x & x & x & x \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ x & x & x & x & x & x \end{pmatrix} \begin{pmatrix} g_{113}(U) \\ g_{131}(U) \\ g_{223}(U) \\ g_{232}(U) \\ g_{231}(U) \\ g_{132}(U) \\ g_{123}(U) \end{pmatrix} \tag{A.2}
$$

which means that ${}^1\mathcal{U}_M$ together with ${}^1\mathcal{U}_F$ are ${}^1\widetilde{b}$-polar to themselves! This is unexpected but surprisingly quite common: it occurs for all other possible symmetry classes except Z_3, Z_5, D_3, D_5, D_3^v and D_5^v (i.e. trigonal and pentagonal ones). It can also be shown that ${}^1\mathcal{U}_M$ and ${}^1\mathcal{U}_F$ are ${}^1\widetilde{c}$-polar (this time to one another) for all symmetry classes except the latter.

It can be shown that ${}^2\mathcal{U}_M$ and ${}^2\mathcal{U}_F$ are ${}^2\widetilde{b}$-polar for D_6, Z_2^-, Z_6^- and D_6^h symmetry classes while they are ${}^2\widetilde{b}$-polar to themselves in all other symmetry classes except trigonal ones.

In the case $p = 3$ the situation is a bit more peculiar. Considering a material whose symmetry class is Z_2^-, we get:

$$
{}^3\widetilde{b}\,{}^3g(U) = \begin{pmatrix} 0 & 0 & 0 & x & x & 0 \\ 0 & 0 & 0 & x & x & 0 \\ 0 & 0 & 0 & x & x & 0 \\ x & x & x & 0 & 0 & x \\ x & x & x & 0 & 0 & x \\ 0 & 0 & 0 & x & x & 0 \end{pmatrix} \begin{pmatrix} g_{113}(U) \\ g_{223}(U) \\ g_{123}(U) \\ g_{133}(U) \\ g_{233}(U) \\ g_{333}(U) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & x & x & 0 \\ 0 & 0 & 0 & x & x & 0 \\ 0 & 0 & 0 & x & x & 0 \\ x & x & x & 0 & 0 & x \\ x & x & x & 0 & 0 & x \\ 0 & 0 & 0 & x & x & 0 \end{pmatrix} \begin{pmatrix} -\partial^2_{11}u^F \\ -\partial^2_{22}u^F \\ -\partial^2_{12}u^F \\ \bar{\bar{l}}\partial_3 y_1 \\ \bar{\bar{l}}\partial_3 y_2 \\ \bar{\bar{l}}\partial_3 z \end{pmatrix}, \quad \forall U = (u^M, z, u^F, y) \in {}^3\mathcal{U} \tag{A.3}
$$

so that ${}^3\widetilde{b}\,{}^3g(u^M, z, u^F, 0)$ belongs to $\mathbb{S}^3_F$ while ${}^3\widetilde{b}\,{}^3g(0,0,0,y)$ belongs to $\mathbb{S}^3_M$. We therefore obtain:

$$
\begin{aligned}
{}^3\widetilde{b}\,{}^3g(u^M, z, u^F, 0)\cdot{}^3e(u'^M, z', 0, 0) &= {}^3\widetilde{b}\,{}^3g(0,0,0,y)\cdot{}^3e(0,0,u'^F, y') = 0, \\
&\forall U = (u^M, z, u^F, y),\ \forall U' = (u'^M, z', u'^F, y') \in {}^3\mathcal{U}.
\end{aligned} \tag{A.4}
$$

This somewhat remarkable result occurs for materials whose symmetry class is either Z_2^-, Z_4^-, Z_6^-, D_6 or D_6^h.

If we consider a monoclinic material whose symmetry class is Z_2 we have:

$$
{}^3\widetilde{b}\,{}^3g(U) = \begin{pmatrix} x & x & x & 0 & 0 & x \\ x & x & x & 0 & 0 & x \\ x & x & x & 0 & 0 & x \\ 0 & 0 & 0 & x & x & 0 \\ 0 & 0 & 0 & x & x & 0 \\ x & x & x & 0 & 0 & x \end{pmatrix} \begin{pmatrix} -\partial^2_{11} u^F \\ -\partial^2_{22} u^F \\ -\partial^2_{12} u^F \\ \bar{\bar{l}}\partial_3 y_1 \\ \bar{\bar{l}}\partial_3 y_2 \\ \bar{\bar{l}}\partial_3 z \end{pmatrix}, \quad \forall U = (u^M, z, u^F, y) \in {}^3\mathscr{U} \tag{A.5}
$$

so that this time we obtain

$$
{}^3\widetilde{b}\,{}^3g(u^M, z, u^F, 0)\cdot{}^3e(0,0,u'^F, y') = {}^3\widetilde{b}\,{}^3g(0,0,0,y)\cdot{}^3e(u'^M, z', 0, 0) = 0, \quad \forall U, U' \in {}^3\mathscr{U}! \tag{A.6}
$$

This result occurs for all other symmetry classes that are not trigonal.

Lastly, it can be shown in a similar manner that for all microstructure which is not trigonal, we have ${}^3\widetilde{c}\,{}^3g(u^M, z, u^F, 0)\cdot{}^3g(0,0,0,y') = 0$ for all U, U' in ${}^3\mathscr{U}$!

References

[1] P. Germain, "La méthode des puissances virtuelles en mécanique des milieux continus premiere partie : théorie du second gradient", *J. de Méc.* **12** (1973), p. 235-274.

[2] P. G. Ciarlet, *Mathematical Elasticity. Volume II: Theory of Plates*, North-Holland, Amsterdam, The Netherlands, 1997.

[3] M. Serpilli, F. Krasucki, G. Geymonat, "An asymptotic strain gradient Reissner–Mindlin plate model", *Meccanica* **48** (2013), p. 2007-2018.

[4] N. Auffray, H. Le Quang, Q. C. He, "Matrix representations for 3D strain-gradient elasticity", *J. Mech. Phys. Solids* **61** (2013), p. 1202-1223.

[5] R. Paroni, P. Podio-Guidugli, G. Tomassetti, "A justification of the Reissner–Mindlin plate theory through variational convergence", *Anal. Appl.* **5** (2007), p. 165-182.

[6] C. Licht, T. Weller, "Asymptotic modeling of piezoelectric plates with electric field gradient", *C. R. Méc.* **340** (2012), p. 405-410.

[7] C. Licht, T. Weller, "Modeling of linearly electromagneto-elastic thin plates", *C. R. Méc.* **335** (2007), p. 201-206.

[8] G. Rossi, L. Placidi, N. Auffray, "On the validity range of strain-gradient elasticity: a mixed static-dynamic identification procedure", *Eur. J. Mech. A: Solids* **69** (2017), p. 179-191.

[9] O. Iosifescu, C. Licht, "Transient response of a thin linearly elastic plate with Norton or Tresca friction", *Asymptot. Anal.* **128** (2022), no. 4, p. 555-570.

[10] H. F. Trotter, "Approximation of semi-groups of operators", *Pac. J. Math.* **28** (1958), p. 897-919.

[11] C. Licht, T. Weller, "Approximation of semi-groups in the sense of Trotter and asymptotic mathematical modeling in physics of continuous media", *Discrete Contin. Dyn. Syst.* **12** (2019), p. 1709-1741.

[12] T. Weller, C. Licht, "Asymptotic modeling of thin piezoelectric plates", *Ann. Solid Struct. Mech.* **1** (2010), p. 173-188.