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The scientific legacy of Roland Glowinski / *L'héritage scientifique de Roland Glowinski*

Nonlinear iterative approximation of steady incompressible chemically reacting flows

Pablo Alexei Gazca-Orozco [a], Pascal Heid [b] and Endre Süli [*, b]

[a] Charles University, Faculty of Mathematics and Physics, Mathematical Institute, Sokolovská 83, Prague 186 75, Czech Republic

[b] Mathematical Institute, University of Oxford, Woodstock Road, Oxford OX2 6GG, UK

E-mails: gazcaorozco@karlin.mff.cuni.cz (P. A. Gazca-Orozco), pascal.heid@maths.ox.ac.uk (P. Heid), endre.suli@maths.ox.ac.uk (E. Süli)

Dedicated to the memory of Roland Glowinski

Abstract. We consider a system of nonlinear partial differential equations modelling steady flow of an incompressible chemically reacting non-Newtonian fluid, whose viscosity depends on both the shear-rate and the concentration; in particular, the viscosity is of power-law type, with a power-law index that depends on the concentration. We prove that the weak solution, whose existence was already established in the literature, is unique, given some strengthened assumptions on the diffusive flux and the stress tensor, for small enough data. We then show that the uniqueness result can be applied to a model describing the synovial fluid. Furthermore, in the latter context, we prove the convergence of a nonlinear iteration scheme; the proposed scheme is remarkably simple and it amounts to solving a linear Stokes–Laplace system at each step. Numerical experiments are performed, which confirm the theoretical findings.

Keywords. Fixed point iteration, Incompressible flow, Non-Newtonian fluids, Chemically reacting flow, Synovial fluid.

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1. Introduction

Non-Newtonian fluids play an important role in science and engineering, and the mathematical analysis and approximation of models of non-Newtonian fluids has been an active field of research. Some of the groundbreaking early contributions include Glowinski's work with Jean Céa [1] on the numerical approximation of viscoplastic (Bingham) fluids, motivated by the work of Duvaut and Glowinski's doctoral advisor Lions [2] on the minimization of nondifferentiable energy functionals. Glowinski's papers [3, 4] with Americo Marrocco were some of the earliest contributions to the finite element approximation of p-Laplace type nonlinear elliptic equations and associated convex energy-minimization problems for functionals with p-growth of the kind that appear in models of steady incompressible quasi-Newtonian fluids. Glowinski's subsequent work over the past five decades on the Bingham model [5–11] involved a range of new ideas, including domain decomposition and operator splitting methods, the analysis of qualitative

* Corresponding author.

properties of Bingham flows, particularly large-time stabilization and, most recently in [12], the numerical solution of the Bingham–Bratu–Gelfand problem, a non-smooth nonlinear eigenvalue problem associated with the total variation integral that includes an additional exponential nonlinearity.

In this work, we study the synovial fluid, which is a viscous, non-Newtonian fluid found in the cavities of synovial joints and whose function is to reduce friction during movement. The synovial fluid consists of an ultra filtrate of blood plasma that contains hyaluronic acid, whose concentration influences the shear-thinning property and helps to maintain a high viscosity; we refer the reader to [13–17] for more information about the biological properties of this fluid. Concerning the mathematical modelling of the rheological behaviour of the synovial fluid, we first point to the works [18, 19], in which a shear-thinning model with constant concentration was proposed; however, since the concentration may significantly vary throughout its domain, see, e.g. [20], such a model is not entirely appropriate. A model with a varying concentration was studied in [21], although the influence of the concentration on the shear rate was not considered; however, it was observed in laboratory experiments, see, e.g. [18], that such a description cannot reflect the true rheological response of the synovial fluid. On that account, Hron et al. [22] proposed a power-law type model whose index depends on the concentration of the hyaluronic acid. Finally, we point to the PhD thesis of Pustějovská [23] for an extensive characterization of the behaviour and the mathematical modelling of the viscous response of the synovial fluid and an overview of the existing literature. Furthermore, this reference also includes some experiments which support the power-law type model with concentration-dependent index as studied in [22].

Concerning the mathematical analysis of a coupled generalised Navier–Stokes system with a convection–diffusion equation we first refer to [24] where, however, the effects of the shear-rate and the concentration on the fluid viscosity were separated; in particular, the viscosity was modelled by a power-law type rheology with a fixed power-law index and a concentration-dependent multiplicative factor. The mathematical theory of a concentration-dependent power-law type model, as introduced in [22], was first established in [25] and further improved in [26]. The latter work employed a Lipschitz truncation and took advantage of the Hölder continuity of the concentration, which was shown by using De Giorgi's method.

The first study of finite element approximations of a concentration-dependent power-law type model of a chemically reacting fluid flow was conducted in [27], based on the theory from [26]. However, since—at that time—a finite element counterpart of De Giorgi's estimate was not available for three-dimensional space, the authors had to restrict themselves to the two-dimensional case. The finite element analysis in three space-dimensions was, subsequently, carried out in [28]. In order to circumvent the absence of a discrete De Giorgi regularity estimate, the authors exploited a more complicated limiting process invoking a regularised system. Finally, a discrete counterpart of the De Giorgi–Nash–Moser theory in three space-dimensions was established in [29], thus enabling the extension of the approach from [27] to the three-dimensional case (see [30]). We remark that the existence proof of a finite element solution in [27], as well in [30], has a minor flaw. For that reason, we will sketch an existence proof, which is based on the one from [27], in Appendix A for our simplified model in the discrete setting.

We note that the works mentioned above neither address the uniqueness of the weak solution nor the convergence of an iterative linearization scheme to a solution of the discrete or continuous problem. Those open questions motivated the work reported herein.

1.1. *Outline of the paper*

In Section 2 we introduce the necessary notations, state some auxiliary results, and define the weak formulation of the problem. Then, in Section 3 we prove the uniqueness of the steady state to an incompressible chemically reacting fluid flow problem under more restrictive assumptions than those under which existence of weak solutions was shown in [26–28,30]. Furthermore, it will be shown that the uniqueness result can be applied to a model of the synovial fluid. In Section 4 we show the convergence of an iteration scheme in the context of the model of the synovial fluid introduced in the previous section. The results of numerical experiments are reported in Section 5, and we conclude the paper with some closing remarks in Section 6.

2. Preliminaries

We assume throughout this work that $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, is a bounded open set with Lipschitz boundary.

2.1. *Basic notations*

For any $s \in [1,\infty)$ we denote by $L^s(\Omega) := L^s(\Omega;\mathbb{R})$ the Lebesgue space of s-integrable functions with corresponding norm $\|f\|_s := (\int_\Omega |f(\boldsymbol{x})|^s \,\mathrm{d}\boldsymbol{x})^{1/s}$. Moreover, $L^\infty(\Omega) := L^\infty(\Omega;\mathbb{R})$ denotes the Lebesgue space of essentially bounded functions with the norm $\|f\|_\infty := \operatorname{ess\,sup}_{\boldsymbol{x}\in\Omega}|f(\boldsymbol{x})|$, and $L^s_0(\Omega) := \{f \in L^s(\Omega) : \int_\Omega f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0\}$ denotes the set of functions (in the corresponding Lebesgue space) with zero integral mean value. We note that, for $s \in (1,\infty)$, $L^{s'}(\Omega)$ and $L^{s'}_0(\Omega)$ are the dual spaces of $L^s(\Omega)$ and $L^s_0(\Omega)$, respectively, where $s' \in (1,\infty)$ is the Hölder conjugate of s, i.e., the number $s' > 1$ that satisfies $1/s + 1/s' = 1$.

Likewise, for $s \in [1,\infty]$, we denote by $W^{1,s}(\Omega) := W^{1,s}(\Omega;\mathbb{R})$ the space of Sobolev functions, endowed with the norm

$$\|u\|_{1,s} := \|u\|_s + \|\nabla u\|_s. \tag{1}$$

Moreover, for $s \in [1,\infty]$, the space of Sobolev functions with zero trace along the boundary $\partial\Omega$ of Ω is denoted by $W^{1,s}_0(\Omega)$. Equivalently, for $s \in [1,\infty)$, $W^{1,s}_0(\Omega)$ is the closure in $W^{1,s}(\Omega)$ of $\mathscr{D}(\Omega) := C^\infty_0(\Omega)$, i.e., the space of smooth functions with compact support in Ω, and its dual space, for any $s \in (1,\infty)$, is denoted by $W^{-1,s'}(\Omega)$. Vector-valued Sobolev spaces will be denoted by $W^{1,s}(\Omega)^d := W^{1,s}(\Omega;\mathbb{R}^d)$. An important subspace will be the space of divergence-free Sobolev functions $W^{1,s}_{0,\mathrm{div}}(\Omega)^d := \{\mathbf{u} \in W^{1,s}_0(\Omega)^d : \mathrm{div}(\mathbf{u}) = 0\}$.

In the context of symmetric matrices, for $\boldsymbol{\delta},\boldsymbol{\kappa} \in \mathbb{R}^{d\times d}_{\mathrm{sym}} := \{\boldsymbol{\kappa} \in \mathbb{R}^{d\times d} : \boldsymbol{\kappa} = \boldsymbol{\kappa}^\top\}$ we denote by $\boldsymbol{\delta} : \boldsymbol{\kappa} := \mathrm{tr}(\boldsymbol{\delta}^\top\boldsymbol{\kappa})$ the Frobenius inner-product, and by $|\boldsymbol{\kappa}|$ the Frobenius norm.

2.2. *Auxiliary results*

Next, we list some well-known results that will play a crucial role in our analysis.

2.2.1. *Poincaré's inequality*

There exists a constant $C_P > 0$ (depending on Ω) such that

$$\|u\|_2 \le C_P\|\nabla u\|_2 \quad \text{for all } u \in W^{1,2}_0(\Omega); \tag{2}$$

see, e.g., [31, Corollary 9.19].

2.2.2. *Korn's inequality*

For all $\mathbf{u} \in W^{1,2}_0(\Omega)^d$ we have that

$$\|\mathrm{D}\mathbf{u}\|_2 \le \|\nabla\mathbf{u}\|_2 \le \sqrt{2}\|\mathrm{D}\mathbf{u}\|_2; \tag{3}$$

see, e.g., [32, Lemma 3.37].

2.2.3. *Sobolev embedding*

The injection $W^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$ is continuous for all $p \in [1,\infty)$ if $d = 2$, and $p \in [1,6]$ if $d = 3$, respectively; see, e.g., [31, Corollary 9.14].

2.2.4. *The Rellich–Kondrachov theorem*

If $q > d$, then the embedding $W^{1,q}(\Omega) \hookrightarrow C(\overline{\Omega})$ is compact, see, e.g., [31, Theorem 9.16] or [33, Theorem 2.35].

2.2.5. *Inf-sup condition*

For all $s, s' \in (1,\infty)$ with $1/s + 1/s' = 1$, there exists a constant $\nu_s > 0$ such that

$$\sup_{\mathbf{0}\neq\mathbf{v}\in W_0^{1,s}(\Omega)^d} \frac{\int_\Omega q \operatorname{div}\mathbf{v}\,\mathrm{d}\boldsymbol{x}}{\|\mathbf{v}\|_{1,s}} \geq \nu_s \|q\|_{s'} \quad \text{for all } q \in L_0^{s'}(\Omega); \tag{4}$$

see, e.g., [34, Section 5.1].

2.3. *Problem formulation*

In this work, we consider the following model of an incompressible chemically reacting non-Newtonian fluid flow in steady state:

$$-\operatorname{div}(\mathbf{S}(c, \mathrm{D}\mathbf{u}) - \mathbf{u}\otimes\mathbf{u} - p\mathbf{I}) = \mathbf{f} \quad \text{in } \Omega, \tag{5}$$

$$\operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega, \tag{6}$$

$$-\operatorname{div}(\mathbf{q}_c(c, \nabla c, \mathrm{D}\mathbf{u}) - c\mathbf{u}) = 0 \quad \text{in } \Omega, \tag{7}$$

where $\mathbf{u}\colon \overline{\Omega} \to \mathbb{R}^d$, $p\colon \Omega \to \mathbb{R}$, $c\colon \overline{\Omega} \to \mathbb{R}_{\geq 0}$ are the unknown velocity, pressure, and concentration fields, respectively. Moreover, $\mathbf{f}\colon \Omega \to \mathbb{R}^d$ denotes a given external force, $\mathrm{D}\mathbf{u}$ signifies the symmetric part of the velocity gradient $\nabla\mathbf{u}$, i.e., $\mathrm{D}\mathbf{u} := 1/2(\nabla\mathbf{u} + \nabla\mathbf{u}^\top)$, and $\mathbf{S}(c, \mathrm{D}\mathbf{u})$ and $\mathbf{q}_c(c, \nabla c, \mathrm{D}\mathbf{u})$ are the shear stress tensor and the diffusive flux, respectively. To complete the problem (5)–(7), we consider the Dirichlet boundary conditions:

$$\mathbf{u} = \mathbf{0} \quad \text{and} \quad c = c_d \quad \text{on } \partial\Omega,$$

where $c_d \in W^{1,q}(\Omega)$ for some $q > d$ and $c_d \geq 0$ a.e. in Ω. We further let

$$c^- := \min_{\boldsymbol{x}\in\partial\Omega} c_d(\boldsymbol{x}) \quad \text{and} \quad c^+ := \max_{\boldsymbol{x}\in\partial\Omega} c_d(\boldsymbol{x}), \tag{8}$$

which are well-defined thanks to the continuous embedding $W^{1,q}(\Omega) \hookrightarrow C(\overline{\Omega})$, cf. Section 2.2.4. Furthermore, we impose the following structural assumptions on the shear stress tensor and the diffusive flux.

(AS) The shear stress tensor $\mathbf{S}\colon \mathbb{R}_{\geq 0} \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ is continuous, and satisfies the following growth, strong monotonicity, and coercivity conditions, respectively: there exist positive constants C_{SG}, C_{SM}, and C_{SC} such that

$$|\mathbf{S}(c, \boldsymbol{\kappa})| \leq C_{SG}(|\boldsymbol{\kappa}| + 1), \tag{9}$$

$$(\mathbf{S}(c, \boldsymbol{\kappa}) - \mathbf{S}(c, \boldsymbol{\delta})) : (\boldsymbol{\kappa} - \boldsymbol{\delta}) \geq C_{SM}|\boldsymbol{\kappa} - \boldsymbol{\delta}|^2, \tag{10}$$

$$\mathbf{S}(c, \boldsymbol{\kappa}) : \boldsymbol{\kappa} \geq C_{SC}|\boldsymbol{\kappa}|^2 \tag{11}$$

for all $c \in \mathbb{R}_{\geq 0}$ and $\boldsymbol{\kappa}, \boldsymbol{\delta} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$.

(AQ) The diffusive flux $\mathbf{q}_c\colon \mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^{d\times d}_{\mathrm{sym}}\to\mathbb{R}^d$ is continuous, and, in addition, linear with respect to the second argument. Moreover, there exist positive constants C_{qG} and C_{qC} such that

$$|\mathbf{q}_c(c,\mathbf{g},\boldsymbol{\kappa})| \le C_{qG}|\mathbf{g}|, \tag{12}$$

$$\mathbf{q}_c(c,\mathbf{g},\boldsymbol{\kappa})\cdot\mathbf{g} \ge C_{qC}|\mathbf{g}|^2 \tag{13}$$

for all $c\in\mathbb{R}_{\ge0}$, $\mathbf{g}\in\mathbb{R}^d$, and $\boldsymbol{\kappa}\in\mathbb{R}^{d\times d}_{\mathrm{sym}}$.

Remark 1. Compared to the works [25–28, 30], we imposed slightly stronger conditions on the stress tensor $\mathbf{S}$. Most notably, by assuming (implicitly) an infinite shear plateau for the viscosity (cf. the paragraph following (44)), we circumvent the difficulty of dealing with Lebesgue and Sobolev spaces with variable exponents.

In regard to the weak formulation of our problem (5)–(7) in the discrete case, we will first define two trilinear forms for dealing with the convection terms in the momentum and concentration equations, respectively. For any $\mathbf{u},\mathbf{v},\mathbf{w}\in W^{1,2}_0(\Omega)^d$, integration by parts yields that

$$\int_\Omega(\mathbf{u}\otimes\mathbf{v}):\nabla\mathbf{w}\,\mathrm{d}\boldsymbol{x} = -\int_\Omega(\mathbf{u}\otimes\mathbf{w}):\nabla\mathbf{v}\,\mathrm{d}\boldsymbol{x} - \int_\Omega(\mathbf{v}\cdot\mathbf{w})\operatorname{div}\mathbf{u}\,\mathrm{d}\boldsymbol{x}, \tag{14}$$

where we have used that the functions involved have zero trace along the boundary $\partial\Omega$; here we employ the convention that $(\nabla\mathbf{v})_{ij} := \partial_i\mathbf{v}_j$, for $i,j\in\{1,\dots,d\}$. Hence, if $\operatorname{div}\mathbf{u}\equiv0$, i.e., $\mathbf{u}\in W^{1,2}_{0,\mathrm{div}}(\Omega)^d$, the last term vanishes. In turn, the convection term in the momentum equation is skew-symmetric with respect to the second and third argument. In order to preserve this property in the discrete setting, when the function $\mathbf{u}$ is replaced by a function from a finite-element subspace of $W^{1,2}_0(\Omega)^d$ that isn't necessarily pointwise divergence-free, we define the trilinear form

$$B_u[\mathbf{u},\mathbf{v},\mathbf{w}] := \frac12\int_\Omega(\mathbf{u}\otimes\mathbf{w}):\nabla\mathbf{v}-(\mathbf{u}\otimes\mathbf{v}):\nabla\mathbf{w}\,\mathrm{d}\boldsymbol{x}, \tag{15}$$

which coincides with (14) for $\mathbf{u}\in W^{1,2}_{0,\mathrm{div}}(\Omega)^d$. For the same reason we consider the following trilinear form for the convection term in the convection–diffusion equation:

$$B_c[c,\mathbf{u},z] := \frac12\int_\Omega(z\mathbf{u}\cdot\nabla c - c\mathbf{u}\cdot\nabla z)\,\mathrm{d}\boldsymbol{x}, \tag{16}$$

for all $\mathbf{u}\in W^{1,2}_0(\Omega)^d$ and $c,z\in W^{1,2}(\Omega)$. Then, the weak formulation of our problem reads as follows.

Problem (W1). For $\mathbf{f}\in W^{-1,2}(\Omega)^d$ and $c_d\in W^{1,q}(\Omega)$, $q>d$, find $(c-c_d)\in W^{1,2}_0(\Omega)$, $\mathbf{u}\in W^{1,2}_0(\Omega)^d$, and $p\in L^2_0(\Omega)$ such that

$$\int_\Omega\mathbf{S}(c,\mathrm{D}\mathbf{u}):\mathrm{D}\mathbf{v}\,\mathrm{d}\boldsymbol{x}+B_u[\mathbf{u},\mathbf{u},\mathbf{v}]-\int_\Omega p\operatorname{div}\mathbf{v}\,\mathrm{d}\boldsymbol{x} = \langle\mathbf{f},\mathbf{v}\rangle\quad\forall\,\mathbf{v}\in W^{1,2}_0(\Omega)^d, \tag{17}$$

$$\int_\Omega q\operatorname{div}\mathbf{u}\,\mathrm{d}\boldsymbol{x} = 0\quad\forall\,q\in L^2_0(\Omega), \tag{18}$$

$$\int_\Omega\mathbf{q}_c(c,\nabla c,\mathrm{D}\mathbf{u})\cdot\nabla z\,\mathrm{d}\boldsymbol{x}+B_c[c,\mathbf{u},z] = 0\quad\forall\,z\in W^{1,2}_0(\Omega). \tag{19}$$

Furthermore, thanks to the inf-sup condition (4), we can restate Problem (W1) in the following divergence-free form.

Problem (W2). For $\mathbf{f}\in W^{-1,2}(\Omega)^d$ and $c_d\in W^{1,q}(\Omega)$, $q>d$, find $(c-c_d)\in W^{1,2}_0(\Omega)$ and $\mathbf{u}\in W^{1,2}_{0,\mathrm{div}}(\Omega)^d$ such that

$$\int_\Omega\mathbf{S}(c,\mathrm{D}\mathbf{u}):\mathrm{D}\mathbf{v}\,\mathrm{d}\boldsymbol{x}+B_u[\mathbf{u},\mathbf{u},\mathbf{v}] = \langle\mathbf{f},\mathbf{v}\rangle\quad\forall\,\mathbf{v}\in W^{1,2}_{0,\mathrm{div}}(\Omega)^d, \tag{20}$$

$$\int_\Omega\mathbf{q}_c(c,\nabla c,\mathrm{D}\mathbf{u})\cdot\nabla z\,\mathrm{d}\boldsymbol{x}+B_c[c,\mathbf{u},z] = 0\quad\forall\,z\in W^{1,2}_0(\Omega). \tag{21}$$

3. Uniqueness of the solution to the weak problem

Let us recall that the existence of a weak solution to Problem (W2), or, equivalently (W1), was already established in the works [25–28, 30]; see also Appendix A for the discrete setting. Concerning the existence of classical solutions, we refer to [35]. In this section we will show that, under certain more restrictive assumptions, cf. Remark 1, the solution is unique for small enough data. In a first step this will be done for the general setting introduced before, and subsequently we will apply our findings to a model of the synovial fluid.

3.1. *Uniqueness of the solution—general framework*

Before addressing the uniqueness of the solution, we will first state some bounds on the convection terms (15) and (16), respectively, which will be crucial for our analysis below.

Lemma 2. *We have that*

$$|B_c[c,\mathbf{u},z]| \leq C_c^2\|\nabla c\|_2\|\nabla\mathbf{u}\|_2\|\nabla z\|_2, \tag{22}$$

$$|B_c[c_d,\mathbf{u},z]| \leq C_B\|\nabla\mathbf{u}\|_2\|\nabla z\|_2 \tag{23}$$

for all $c, z \in W_0^{1,2}(\Omega)$ and $\mathbf{u} \in W_0^{1,2}(\Omega)^d$, where $C_c := C_{SE}(1+C_P) > 0$ and

$$C_B := \min\left\{\max\{C_c^2\|\nabla c_d\|_2, C_P^2\|c_d\|_\infty\}, 3/2C_P\|c_d\|_\infty\right\}. \tag{24}$$

Moreover, if $\mathbf{u} \in W_{0,\mathrm{div}}^{1,2}(\Omega)^d$, then the bound (22) remains valid for $c \in W^{1,2}(\Omega)$, i.e., even if $c|_{\partial\Omega} \neq 0$. Finally, we have that

$$|B_c[c,\mathbf{u},z]| \leq C_cC_{SE}\|\nabla c\|_2\|\nabla\mathbf{u}\|_2\|z\|_{1,2} \tag{25}$$

for all $\mathbf{u} \in W_{0,\mathrm{div}}^{1,2}(\Omega)^d$ and $c, z \in W^{1,2}(\Omega)$.

Proof. We will first establish the bound (22) for $\mathbf{u} \in W_0^{1,2}(\Omega)^d$. For $d \in \{2,3\}$, the Sobolev embedding theorem, cf. Section 2.2.3, yields the existence of a constant $C_{SE} > 0$ such that

$$\|u\|_4 \leq C_{SE}\|u\|_{1,2} \quad \text{for all } u \in W^{1,2}(\Omega). \tag{26}$$

In turn, by Poincaré's inequality (2), we find that

$$\|u\|_4 \leq C_{SE}(1+C_P)\|\nabla u\|_2 = C_c\|\nabla u\|_2 \tag{27}$$

for all $u \in W_0^{1,2}(\Omega)$. Hence, by first applying the Cauchy–Schwarz inequality twice and subsequently invoking (27), we find that

$$|B_c[c,\mathbf{u},z]| \leq 1/2(\|\nabla c\|_2\|z\|_4\|\mathbf{u}\|_4 + \|\nabla z\|_2\|c\|_4\|\mathbf{u}\|_4) \leq C_c^2\|\nabla c\|_2\|\nabla z\|_2\|\nabla\mathbf{u}\|_2, \tag{28}$$

which proves the bound (22).

In order to establish the first bound in (23), (24), we will apply Hölder's inequality to each of the two terms on the right-hand side of (16), which yields

$$|B_c[c_d,\mathbf{u},z]| \leq 1/2(\|\nabla c_d\|_2\|\mathbf{u}\|_4\|z\|_4 + \|c_d\|_\infty\|\mathbf{u}\|_2\|z\|_2).$$

Next, by invoking the upper bounds (26) and (2), respectively, we further find that

$$|B_c[c_d,\mathbf{u},z]| \leq 1/2(C_c^2\|\nabla c_d\|_2\|\nabla\mathbf{u}\|_2\|\nabla z\|_2 + C_P^2\|c_d\|_\infty\|\nabla\mathbf{u}\|_2\|\nabla z\|_2),$$

which, in turn, leads to

$$|B_c[c_d,\mathbf{u},z]| \leq \max\{C_c^2\|\nabla c_d\|_2, C_P^2\|c_d\|_\infty\}\|\nabla\mathbf{u}\|_2\|\nabla z\|_2.$$

To prove the second bound in (23), (24), we will proceed along the lines of [36, p. 530]. We note that the definition of B_c, cf. (16), together with the divergence theorem, imply that

$$B_c[c_d,\mathbf{u},z] = -\int_\Omega c_d\mathbf{u}\cdot\nabla z\,\mathrm{d}\boldsymbol{x} - \frac{1}{2}\int_\Omega c_d z\,\mathrm{div}\,\mathbf{u}\,\mathrm{d}\boldsymbol{x}. \tag{29}$$

Thus, it follows from Hölder's inequality that

$$\begin{aligned}|B_c[c_d,\mathbf{u},z]| &\le \|c_d\|_\infty\|\mathbf{u}\|_2\|\nabla z\|_2+\tfrac{1}{2}\|c_d\|_\infty\|z\|_2\|\nabla\mathbf{u}\|_2\\ &\le C_P\|c_d\|_\infty\|\nabla\mathbf{u}\|_2\|\nabla z\|_2+C_P\tfrac{1}{2}\|c_d\|_\infty\|\nabla z\|_2\|\nabla\mathbf{u}\|_2,\end{aligned}$$

where we employed Poincaré's inequality in the second step; this gives rise to the other bound in (23), (24).

If $\mathbf{u}\in W^{1,2}_{0,\mathrm{div}}(\Omega)^d$, then (29) implies that

$$|B_c[c,\mathbf{u},z]| = |B_c[z,\mathbf{u},c]| = \left|\int_\Omega z\mathbf{u}\cdot\nabla c\,\mathrm{d}\boldsymbol{x}\right| \le C_c^2\|\nabla c\|_2\|\nabla\mathbf{u}\|_2\|\nabla z\|_2$$

for all $z\in W^{1,2}_0(\Omega)$ and merely $c\in W^{1,2}(\Omega)$; here, we applied as above the Cauchy–Schwarz inequality twice and subsequently the bound (27). The inequality (25) follows in the same manner. □

Remark 3. We note that the bounds (22), (23) (for $\mathbf{u}\in W^{1,2}(\Omega)^d$) are equally valid on finite-dimensional subspaces. However, the other two bounds in Lemma 2 only remain valid in the discrete setting for pointwise divergence-free functions, but not for functions from a finite element subspace that are only discretely/approximately divergence-free.

Lemma 4. *We have that*

$$|B_u[\mathbf{u},\mathbf{v},\mathbf{w}]| \le C_u\|\mathrm{D}\mathbf{u}\|_2\|\mathrm{D}\mathbf{v}\|_2\|\mathrm{D}\mathbf{w}\|_2 \tag{30}$$

for all $\mathbf{u},\mathbf{v},\mathbf{w}\in W^{1,2}_0(\Omega)^d$, *where* $C_u := 2\sqrt{2}C_c^2$.

Proof. By invoking the definition of $B_u[\cdot,\cdot,\cdot]$, cf. (15), the Cauchy–Schwarz inequality, the inequality (27), and Korn's inequality (3), we immediately find that

$$|B_u[\mathbf{u},\mathbf{v},\mathbf{w}]| \le \tfrac{1}{2}(\|\mathbf{u}\|_4\|\mathbf{v}\|_4\|\nabla\mathbf{w}\|_2+\|\mathbf{u}\|_4\|\mathbf{w}\|_4\|\nabla\mathbf{v}\|_2) \le 2\sqrt{2}C_c^2\|\mathrm{D}\mathbf{u}\|_2\|\mathrm{D}\mathbf{v}\|_2\|\mathrm{D}\mathbf{w}\|_2,$$

which proves the claim. □

Next, we will show that any solution of (W2) can be bounded from above in terms of the source function $\mathbf{f}$ and the boundary datum c_d.

Proposition 5. *Let* $(\mathbf{u}^\star,c^\star)\in W^{1,2}_{0,\mathrm{div}}(\Omega)^d\times W^{1,2}(\Omega)$ *be a solution of (W2). Then, we have that*

$$\|\mathrm{D}\mathbf{u}^\star\|_2 \le \frac{\|\mathbf{f}\|_\star}{C_{SC}} =: C_{ub}, \tag{31}$$

where

$$\|\mathbf{f}\|_\star := \sup_{0\ne\mathbf{v}\in W^{1,2}_0(\Omega)^d}\frac{\langle\mathbf{f},\mathbf{v}\rangle}{\|\mathrm{D}\mathbf{v}\|_2}. \tag{32}$$

Moreover, the concentration can be bounded by

$$\|\nabla c^\star\|_2 \le C_{qC}^{-1}(C_{qG}\|\nabla c_d\|_2+\sqrt{2}C_cC_{SE}C_{ub}\|c_d\|_{1,2}) =: C_{cb}. \tag{33}$$

Proof. Using the test function $\mathbf{v}=\mathbf{u}^\star$ in (20) and applying the coercivity property (11) implies that

$$C_{SC}\|\mathrm{D}\mathbf{u}^\star\|_2^2 \le \langle\mathbf{f},\mathbf{u}^\star\rangle,$$

which immediately yields the bound (31).

Next, we use the admissible test function $c^\star-c_d\in W^{1,2}_0(\Omega)$ in (21), which, after a simple manipulation of the terms, leads to

$$\begin{aligned}\int_\Omega\mathbf{q}_c(c^\star,\nabla c^\star,\mathrm{D}\mathbf{u}^\star)\cdot\nabla c^\star\,\mathrm{d}\boldsymbol{x} &= \int_\Omega\mathbf{q}_c(c^\star,\nabla c^\star,\mathrm{D}\mathbf{u}^\star)\cdot\nabla c_d\,\mathrm{d}\boldsymbol{x}-B_c[c^\star,\mathbf{u}^\star,c^\star-c_d]\\ &= \int_\Omega\mathbf{q}_c(c^\star,\nabla c^\star,\mathrm{D}\mathbf{u}^\star)\cdot\nabla c_d\,\mathrm{d}\boldsymbol{x}+B_c[c^\star,\mathbf{u}^\star,c_d];\end{aligned}$$

here, we employed the linearity and the anti-symmetry of the convection term in the second step. Consequently, by invoking (13), (12), (25), and the Cauchy–Schwarz inequality, we find that

$$C_{qC}\|\nabla c^\star\|_2^2 \le C_{qG}\|\nabla c^\star\|_2\|\nabla c_d\|_2 + C_c C_{SE}\|\nabla \mathbf{u}^\star\|_2\|\nabla c^\star\|_2\|c_d\|_{1,2}.$$

Hence, from the first part and Korn's inequality (3) it follows that

$$\|\nabla c^\star\|_2 \le C_{qC}^{-1}(C_{qG}\|\nabla c_d\|_2 + \sqrt{2}C_c C_{SE} C_{ub}\|c_d\|_{1,2}); \tag{34}$$

this finishes the proof. □

Now, we shall finally establish the uniqueness of the solution of (W2) under suitable assumptions. For that purpose, let us assume that $(\mathbf{u}^\star, c^\star), (\mathbf{u}^\diamond, c^\diamond) \in W^{1,2}_{0,\mathrm{div}}(\Omega)^d \times W^{1,2}(\Omega)$ are two solutions of the weak problem (W2). Then, we have that

$$\begin{aligned} 0 \ &= \int_\Omega (\mathbf{S}(c^\star, \mathrm{D}\mathbf{u}^\star) - \mathbf{S}(c^\diamond, \mathrm{D}\mathbf{u}^\diamond)) : \mathrm{D}(\mathbf{u}^\star - \mathbf{u}^\diamond)\,\mathrm{d}\boldsymbol{x} \\ &\quad + B_u[\mathbf{u}^\star, \mathbf{u}^\star, \mathbf{u}^\star - \mathbf{u}^\diamond] - B_u[\mathbf{u}^\diamond, \mathbf{u}^\diamond, \mathbf{u}^\star - \mathbf{u}^\diamond] \\ &\quad + B_c[c^\star, \mathbf{u}^\star, c^\star - c^\diamond] - B_c[c^\diamond, \mathbf{u}^\diamond, c^\star - c^\diamond] \\ &\quad + \int_\Omega (\mathbf{q}_c(c^\star, \nabla c^\star, \mathrm{D}\mathbf{u}^\star) - \mathbf{q}_c(c^\diamond, \nabla c^\diamond, \mathrm{D}\mathbf{u}^\diamond)) \cdot \nabla(c^\star - c^\diamond)\,\mathrm{d}\boldsymbol{x} \\ &=: \ \mathrm{(I)} + \mathrm{(II)} + \mathrm{(III)} + \mathrm{(IV)}. \end{aligned} \tag{35}$$

The goal is to show that $(\mathbf{u}^\star, c^\star) \ne (\mathbf{u}^\diamond, c^\diamond)$ implies that the sum above is positive, which is a contradiction. To that end, we will bound the summands (I)–(IV) individually. We will start with (II): using the anti-symmetry and the linearity of the convection term B_u, we find that

$$\mathrm{(II)} = B_u[\mathbf{u}^\star - \mathbf{u}^\diamond, \mathbf{u}^\star, \mathbf{u}^\star - \mathbf{u}^\diamond].$$

Hence, thanks to Lemma 4 and Proposition 5, we obtain the bound

$$|\mathrm{(II)}| \le C_{ub} C_u \|\mathrm{D}(\mathbf{u}^\star - \mathbf{u}^\diamond)\|_2^2.$$

Next, we will take care of (III). Similarly as before, by employing the anti-symmetry and the linearity of the convection term B_c, we obtain that

$$\mathrm{(III)} = B_c[c^\star, \mathbf{u}^\star - \mathbf{u}^\diamond, c^\star - c^\diamond].$$

Applying first the bound (22) (for a divergence-free velocity vector) in combination with Korn's inequality (3) and the bound (33) yields

$$|\mathrm{(III)}| \le \sqrt{2}C_c^2\|\nabla c^\star\|_2\|\mathrm{D}(\mathbf{u}^\star - \mathbf{u}^\diamond)\|_2\|\nabla(c^\star - c^\diamond)\|_2 \le \sqrt{2}C_c^2 C_{cb}\|\mathrm{D}(\mathbf{u}^\star - \mathbf{u}^\diamond)\|_2\|\nabla(c^\star - c^\diamond)\|_2.$$

Let us recall that, for fixed real numbers $a, b > 0$, we have $ab \le 2^{-1}\epsilon a^2 + (2\epsilon)^{-1}b^2$ for all $\epsilon > 0$. Hence, we may obtain the bound

$$|\mathrm{(III)}| \le \frac{\varepsilon}{\sqrt{2}} C_c^2 C_{cb}\|\mathrm{D}(\mathbf{u}^\star - \mathbf{u}^\diamond)\|_2^2 + \frac{1}{\sqrt{2}\varepsilon} C_c^2 C_{cb}\|\nabla(c^\star - c^\diamond)\|_2^2.$$

In order to control the terms (I) and (IV), we need to impose additional continuity assumptions on the stress tensor and the diffusive flux, respectively.

(AS$^+$) There exists a continuous, non-decreasing function $\varphi_L : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ such that $\varphi_L(0) = 0$ and

$$\left| \int_\Omega (\mathbf{S}(c, \mathrm{D}\mathbf{u}) - \mathbf{S}(z, \mathrm{D}\mathbf{u})) : \mathrm{D}\mathbf{v}\,\mathrm{d}\boldsymbol{x} \right| \le \varphi_L(\|\mathrm{D}\mathbf{u}\|_2)\|\nabla(c - z)\|_2\|\mathrm{D}\mathbf{v}\|_2 \tag{36}$$

for all $(c - c_d), (z - c_d) \in W_0^{1,2}(\Omega)$ and $\mathbf{u}, \mathbf{v} \in W_0^{1,2}(\Omega)^d$.

(AQ$^+$) There exist continuous, non-decreasing functions $\psi_L : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and $\widetilde{\psi}_L : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, with $\psi_L(0) = \widetilde{\psi}_L(0,0) = 0$, such that

$$\left| \int_\Omega (\mathbf{q}_c(c, \nabla c, \mathrm{D}\mathbf{u}) - \mathbf{q}_c(c, \nabla c, \mathrm{D}\mathbf{v})) \cdot \nabla z \, \mathrm{d}\boldsymbol{x} \right| \leq \psi_L(\|\nabla c\|_2) \|\mathrm{D}(\mathbf{u} - \mathbf{v})\|_2 \|\nabla z\|_2, \tag{37}$$

$$\left| \int_\Omega (\mathbf{q}_c(c, \nabla c, \mathrm{D}\mathbf{u}) - \mathbf{q}_c(\widetilde{c}, \nabla c, \mathrm{D}\mathbf{u})) \cdot \nabla z \, \mathrm{d}\boldsymbol{x} \right| \leq \widetilde{\psi}_L(\|\nabla c\|_2, \|\mathrm{D}\mathbf{u}\|_2) \|\nabla(c - \widetilde{c})\|_2 \|\nabla z\|_2 \tag{38}$$

for all $(c - c_d)$, $(\widetilde{c} - c_d)$, $z \in W^{1,2}_0(\Omega)$ and $\mathbf{u}, \mathbf{v} \in W^{1,2}_0(\Omega)^d$.

Assuming (AS$^+$), we will first bound (I). For that purpose we decompose the integral into two parts,

$$(\mathrm{I}) = \int_\Omega (\mathbf{S}(c^\star, \mathrm{D}\mathbf{u}^\star) - \mathbf{S}(c^\star, \mathrm{D}\mathbf{u}^\diamond)) : \mathrm{D}(\mathbf{u}^\star - \mathbf{u}^\diamond) \, \mathrm{d}\boldsymbol{x} + \int_\Omega (\mathbf{S}(c^\star, \mathrm{D}\mathbf{u}^\diamond) - \mathbf{S}(c^\diamond, \mathrm{D}\mathbf{u}^\diamond)) : \mathrm{D}(\mathbf{u}^\star - \mathbf{u}^\diamond) \, \mathrm{d}\boldsymbol{x}.$$

By invoking (10), (36), and (31), we find that

$$\begin{aligned}(\mathrm{I}) &\geq C_{SM} \|\mathrm{D}(\mathbf{u}^\star - \mathbf{u}^\diamond)\|_2^2 - \varphi_L(\|\mathrm{D}\mathbf{u}^\diamond\|_2) \|\nabla(c^\star - c^\diamond)\|_2 \|\mathrm{D}(\mathbf{u}^\star - \mathbf{u}^\diamond)\|_2 \\ &\geq (C_{SM} - 2^{-1}\delta \varphi_L(C_{ub})) \|\mathrm{D}(\mathbf{u}^\star - \mathbf{u}^\diamond)\|_2^2 - (2\delta)^{-1} \varphi_L(C_{ub}) \|\nabla(c^\star - c^\diamond)\|_2^2\end{aligned}$$

for any $\delta > 0$. In a similar manner, by first decomposing the integral

$$\begin{aligned}(\mathrm{IV}) = &\int_\Omega (\mathbf{q}_c(c^\star, \nabla c^\star, \mathrm{D}\mathbf{u}^\star) - \mathbf{q}_c(c^\star, \nabla c^\diamond, \mathrm{D}\mathbf{u}^\star)) \cdot \nabla(c^\star - c^\diamond) \, \mathrm{d}\boldsymbol{x} \\ &+ \int_\Omega (\mathbf{q}_c(c^\star, \nabla c^\diamond, \mathrm{D}\mathbf{u}^\star) - \mathbf{q}_c(c^\diamond, \nabla c^\diamond, \mathrm{D}\mathbf{u}^\star)) \cdot \nabla(c^\star - c^\diamond) \, \mathrm{d}\boldsymbol{x} \\ &+ \int_\Omega (\mathbf{q}_c(c^\diamond, \nabla c^\diamond, \mathrm{D}\mathbf{u}^\star) - \mathbf{q}_c(c^\diamond, \nabla c^\diamond, \mathrm{D}\mathbf{u}^\diamond)) \cdot \nabla(c^\star - c^\diamond) \, \mathrm{d}\boldsymbol{x}\end{aligned}$$

and then employing the linearity of the diffusive flux in the second argument in combination with (13), as well as (38), (37), and (33), we obtain the lower bound

$$\begin{aligned}(\mathrm{IV}) &\geq C_{qC} \|\nabla(c^\star - c^\diamond)\|_2^2 - \widetilde{\psi}_L(C_{cb}, C_{ub}) \|\nabla(c^\star - c^\diamond)\|_2^2 - \psi_L(C_{cb}) \|\mathrm{D}(\mathbf{u}^\star - \mathbf{u}^\diamond)\|_2 \|\nabla(c^\star - c^\diamond)\|_2 \\ &\geq (C_{qC} - \widetilde{\psi}_L(C_{cb}, C_{ub}) - (2\gamma)^{-1}\psi_L(C_{cb})) \|\nabla(c^\star - c^\diamond)\|_2^2 - 2^{-1}\gamma \psi_L(C_{cb}) \|\mathrm{D}(\mathbf{u}^\star - \mathbf{u}^\diamond)\|_2^2,\end{aligned}$$

for any $\gamma > 0$. Combining all of the above inequalities implies that

$$\begin{aligned}0 \geq &\left(C_{qC} - (2\delta)^{-1}\varphi_L(C_{ub}) - \frac{1}{\sqrt{2\varepsilon}} C_c^2 C_{cb} - \widetilde{\psi}_L(C_{cb}, C_{ub}) - (2\gamma)^{-1}\psi_L(C_{cb}) \right) \|\nabla(c^\star - c^\diamond)\|_2^2 \\ &+ \left(C_{SM} - 2^{-1}\delta\varphi_L(C_{ub}) - C_{ub} C_u - \frac{\varepsilon}{\sqrt{2}} C_c^2 C_{cb} - 2^{-1}\gamma\psi_L(C_{cb}) \right) \|\mathrm{D}(\mathbf{u}^\star - \mathbf{u}^\diamond)\|_2^2.\end{aligned}$$

For simplicity, we set $\gamma = \delta = \epsilon = 1$ so that

$$\begin{aligned}0 \geq &\underbrace{\left(C_{qC} - 2^{-1}\varphi_L(C_{ub}) - \frac{1}{\sqrt{2}} C_c^2 C_{cb} - \widetilde{\psi}_L(C_{cb}, C_{ub}) - 2^{-1}\psi_L(C_{cb}) \right)}_{=:\nu_c} \|\nabla(c^\star - c^\diamond)\|_2^2 \\ &+ \underbrace{\left(C_{SM} - 2^{-1}\varphi_L(C_{ub}) - C_{ub} C_u - \frac{1}{\sqrt{2}} C_c^2 C_{cb} - 2^{-1}\psi_L(C_{cb}) \right)}_{=:\nu_{\mathbf{u}}} \|\mathrm{D}(\mathbf{u}^\star - \mathbf{u}^\diamond)\|_2^2.\end{aligned} \tag{39}$$

Recall that $\widetilde{\psi}_L, \psi_L, \varphi_L$ are continuous (non-decreasing) functions with $\widetilde{\psi}_L(0,0) = \psi_L(0) = \varphi_L(0) = 0$. Moreover, thanks to Proposition 5, we have that $C_{ub} \to 0$ as $\|\mathbf{f}\|_\star \to 0$, and $C_{cb} \to 0$ as $\|\mathbf{f}\|_\star$, $\|c_d\|_{1,q} \to 0$ with $q > d$. Consequently, we find that

$$\widetilde{\psi}_L(C_{cb}, C_{ub}), \quad \psi_L(C_{cb}), \quad \varphi_L(C_{ub}) \to 0 \quad \text{as } \|\mathbf{f}\|_\star \to 0 \quad \text{and} \quad \|c_d\|_{1,q} \to 0. \tag{40}$$

Thus, since C_{qC} and C_{SM} are fixed positive constants, we have that the factors ν_c and $\nu_{\mathbf{u}}$ of $\|\nabla(c^\star - c^\diamond)\|_2^2$ and $\|\mathrm{D}(\mathbf{u}^\star - \mathbf{u}^\diamond)\|_2^2$, respectively, in (39) are both positive for $\|\mathbf{f}\|_\star$ and $\|c_d\|_{1,q}$ small enough, with $q > d$, which implies that

$$\|\nabla(c^\star - c^\diamond)\|_2^2 = \|\mathrm{D}(\mathbf{u}^\star - \mathbf{u}^\diamond)\|_2^2 = 0;$$

i.e., the solution is unique. We summarize our findings in the next theorem.

Theorem 6. *Given the assumptions* (AS), (AS$^+$), (AQ), *and* (AQ$^+$). *Then, for small enough data $\|\mathbf{f}\|_\star$ and $\|c_d\|_{1,q}$ with $q > d$, the solution of Problem (W2), and therefore also that of Problem (W1), is unique.*

Remark 7. Some of the assumptions in Theorem 6 can indeed be weakened. For instance, the function φ_L from the assumption (AS$^+$) may also depend (adversely) on the boundary data c_d. Indeed, as before, for small enough $\|c_d\|_{1,q}$, with $q > d$, and $\|\mathbf{f}\|_\star$ we have that

$$\nu_c + 2^{-1}\varphi_L(C_{ub}) = C_{qC} - \frac{1}{\sqrt{2}} C_c^2 C_{cb} - \widetilde{\psi}_L(C_{cb}, C_{ub}) - 2^{-1}\psi_L(C_{cb}) \geq 2^{-1} C_{qC},$$

$$\nu_{\mathbf{u}} + 2^{-1}\varphi_L(C_{ub}) = C_{SM} - C_{ub}C_u - \frac{1}{\sqrt{2}} C_c^2 C_{cb} - 2^{-1}\psi_L(C_{cb}) \geq 2^{-1} C_{SM}.$$

Now let c_d and $\mathbf{f}$ be such that those inequalities are satisfied. We note that, for such a fixed c_d, this inequality remains true if we further decrease $\|\mathbf{f}\|_\star$. Since, moreover, $\varphi_L(C_{ub}) \to 0$ as $\|\mathbf{f}\|_\star \to 0$, we conclude that ν_c and $\nu_{\mathbf{u}}$ are positive for this fixed boundary data c_d and $\|\mathbf{f}\|_\star$ small enough. This yields, as before, the uniqueness of the solution.

Remark 8. Theorem 6 remains valid in the discrete setting, at the level of a finite element approximation of the problem, provided that we consider an inf-sup stable finite element velocity-pressure pair where the discretely divergence-free velocities are in fact pointwise divergence-free.

3.2. *Application to a model of the synovial fluid*

In this section, we want to show that our findings from before can be applied to the model of the synovial fluid to be introduced below, cf. (41), (42). For that purpose, we need to verify that the assumptions of Theorem 6 are satisfied in this context.

For simplicity, we consider a constant diffusion coefficient

$$\mathbf{q}_c(t, \mathbf{g}, \boldsymbol{\kappa}) := K_c \mathbf{g} \tag{41}$$

for some positive real number $K_c > 0$ independent of $t \in \mathbb{R}_{\geq 0}$, $\mathbf{g} \in \mathbb{R}^d$, and $\boldsymbol{\kappa} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$. Consequently, we immediately obtain the following result in the given setting.

Lemma 9. *The properties* (AQ) *and* (AQ$^+$) *are satisfied with $C_{qG} = C_{qC} = K_c$ and $\psi_L \equiv \widetilde{\psi}_L \equiv 0$, respectively.*

Furthermore, the shear stress tensor is defined by

$$\mathbf{S}(c, \boldsymbol{\kappa}) := \mu(c, |\boldsymbol{\kappa}|^2)\boldsymbol{\kappa}, \quad c \in \mathbb{R}_{\geq 0}, \boldsymbol{\kappa} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}, \tag{42}$$

where the viscosity coefficient $\mu : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is given by

$$\mu(c, t) := \mu_0\beta + \mu_0(1-\beta)(1+\lambda t)^{r(c)}; \tag{43}$$

here, $\mu_0 > 0$, $\beta \in (0,1)$, $\lambda > 0$, and the exponent $r : \mathbb{R}_{\geq 0} \to (-0.5, 0]$ satisfies the following properties:

(R1) r is continuously differentiable and monotonically decreasing;

(R2) $r(0) = 0$ and $r(c) < 0$ for $c > 0$;

(R3) the derivative is bounded, i.e., there exists a constant $C_r > 0$ such that

$$|r'(c)| \leq C_r \quad \text{for all } c \in \mathbb{R}_{\geq 0}. \tag{44}$$

We refer to [23, Ch. 5] for the modeling of the viscous response of the synovial fluid: in contrast with that reference, cf. [23, (Model 1)], we introduced an additional term "$\mu_0\beta$", which acts as an infinite shear plateau. Now let us state and prove some useful properties of the viscosity coefficient μ.

Lemma 10. *The viscosity coefficient* $\mu : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *has the following properties:*

(a) μ *is continuous in both arguments, i.e.,* $\mu \in C(\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0})$*;*
(b) *for any given* $c \geq 0$, $\mu(c,\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *is non-increasing, i.e.,* $\partial_t \mu(c,t) \leq 0$ *for all* $t \geq 0$*;*
(c) *we have that*

$$\mu_0\beta(t-s) \leq \mu(c,t^2)t - \mu(c,s^2)s \leq \mu_0(t-s) \quad \textit{for all } c \geq 0 \textit{ and } t \geq s \geq 0; \tag{45}$$

(d) *we have that* $\mu(c,t) \in [\mu_0\beta, \mu_0]$ *for all* $c, t \geq 0$.

Proof. The definition of μ, cf. (43), and the continuity of r, cf. (R1), immediately imply the assertion (a). Moreover, (b) follows from the assumption that $r(c) \in (-0.5, 0]$ for all $c \geq 0$. Concerning (c), let us define, for any fixed $c > 0$ (the case $c = 0$ is trivial), the real-valued function $\xi(t) := \mu(c,t^2)t$. Then, thanks to the mean value theorem we find that

$$\inf_{\tau \geq 0} \xi'(\tau)(t-s) \leq \xi(t) - \xi(s) \leq \sup_{\tau \geq 0} \xi'(\tau)(t-s) \quad \text{for all } t \geq s \geq 0. \tag{46}$$

It can straightforwardly be verified that $\xi''(t) \neq 0$ for all $t > 0$, $\lim_{t\to 0} \xi'(t) = \mu_0$, and $\lim_{t\to\infty} \xi'(t) = \mu_0\beta$; this implies the claim (c). Finally, (d) follows immediately from (c) by setting $s = 0$ and dividing by $t > 0$. □

Furthermore, thanks to Lemma 10, it can be shown that

$$|\mu(c,|\boldsymbol{\kappa}|^2)\boldsymbol{\kappa} - \mu(c,|\boldsymbol{\delta}|^2)\boldsymbol{\delta}| \leq \sqrt{3}\mu_0|\boldsymbol{\kappa} - \boldsymbol{\delta}|, \tag{47}$$

$$(\mu(c,|\boldsymbol{\kappa}|^2)\boldsymbol{\kappa} - \mu(c,|\boldsymbol{\delta}|^2)\boldsymbol{\delta}) : (\boldsymbol{\kappa} - \boldsymbol{\delta}) \geq \mu_0\beta|\boldsymbol{\kappa} - \boldsymbol{\delta}|^2 \tag{48}$$

for all $c \geq 0$ and $\boldsymbol{\kappa}, \boldsymbol{\delta} \in \mathbb{R}^{d\times d}_{\text{sym}}$; we refer to [37, Lemma 2.1] or [38, Lemma 2.1].

Lemma 11. *The shear stress tensor* $\mathbf{S}(c,\boldsymbol{\kappa})$ *from* (42) *satisfies the assumption* (AS) *with*

$$|\mathbf{S}(c,\boldsymbol{\kappa})| \leq \mu_0|\boldsymbol{\kappa}|, \tag{49}$$

$$(\mathbf{S}(c,\boldsymbol{\kappa}) - \mathbf{S}(c,\boldsymbol{\delta})) : (\boldsymbol{\kappa} - \boldsymbol{\delta}) \geq \mu_0\beta|\boldsymbol{\kappa} - \boldsymbol{\delta}|^2, \tag{50}$$

$$\mathbf{S}(c,\boldsymbol{\kappa}) : \boldsymbol{\kappa} \geq \mu_0\beta|\boldsymbol{\kappa}|^2. \tag{51}$$

Proof. The estimates (49), (50), and (51) immediately follow from Lemma 10(d) and (48). □

It remains to verify the assumption (AS$^+$). Let $(c - c_d)$, $(z - c_d) \in W^{1,2}_0(\Omega)$, $\mathbf{u}, \mathbf{v} \in W^{1,2}_0(\Omega)^d$, and consider the integral

$$\int_\Omega (\mu(c,|\mathrm{D}\mathbf{u}|^2) - \mu(z,|\mathrm{D}\mathbf{u}|^2))\mathrm{D}\mathbf{u} : \mathrm{D}\mathbf{v}\,\mathrm{d}\boldsymbol{x}.$$

In order to bound this integral, we will invoke the mean-value theorem, which yields, for almost every $\boldsymbol{x} \in \Omega$,

$$|\mu(c(\boldsymbol{x}),|\mathrm{D}\mathbf{u}(\boldsymbol{x})|^2) - \mu(z(\boldsymbol{x}),|\mathrm{D}\mathbf{u}(\boldsymbol{x})|^2)| = |\partial_c\mu(\xi,|\mathrm{D}\mathbf{u}(\boldsymbol{x})|^2)|\,|c(\boldsymbol{x}) - z(\boldsymbol{x})| \tag{52}$$

for some ξ between $c(\boldsymbol{x})$ and $z(\boldsymbol{x})$. A straightforward calculation reveals that

$$\partial_c\mu(\xi,|\mathrm{D}\mathbf{u}(\boldsymbol{x})|^2) = \mu_0(1-\beta)\log(1+\lambda|\mathrm{D}\mathbf{u}(\boldsymbol{x})|^2)(1+\lambda|\mathrm{D}\mathbf{u}(\boldsymbol{x})|^2)^{r(\xi)}r'(\xi). \tag{53}$$

We remark that in the proof of Theorem 6 we only required (36) to hold true for solutions of the weak problem (W2). Especially, the concentration fields involved are supposed to satisfy a convection–diffusion equation, which can be cast, in view of (41), into the framework of

[39, Theorem 8.1]. In particular, they satisfy a minimum principle in the sense that, for any solution $(\mathbf{u}^\star, c^\star)$ of (W2) in the given setting, we have that

$$\operatorname*{ess\,inf}_{\boldsymbol{x}\in\Omega} c^\star \geq \min_{\boldsymbol{x}\in\partial\Omega} c_d(\boldsymbol{x}) = c^- > 0;$$

here, we further assume that the boundary function is strictly positive. For that reason, we can assume without loss of generality that

$$c(\boldsymbol{x}), z(\boldsymbol{x}) \geq c^- > 0, \tag{54}$$

and in turn, by (R1)–(R2),

$$r(c(\boldsymbol{x})), r(z(\boldsymbol{x})) \leq r(c^-) < 0 \tag{55}$$

for (almost) all $\boldsymbol{x} \in \Omega$. Since ξ is an element between $c(\boldsymbol{x})$ and $z(\boldsymbol{x})$, we further have that $r(\xi) \leq r(c^-)$. Therefore, recalling (44), we can uniformly bound the term in (53) by

$$|\partial_c \mu(\xi, |\mathrm{D}\mathbf{u}(\boldsymbol{x})|^2)| \leq C_\mu \log(1+\lambda|\mathrm{D}\mathbf{u}(\boldsymbol{x})|^2)(1+\lambda|\mathrm{D}\mathbf{u}(\boldsymbol{x})|^2)^{r(c^-)},$$

where

$$C_\mu := \mu_0(1-\beta)C_r. \tag{56}$$

Consequently, we have that

$$\left|\int_\Omega (\mu(c,|\mathrm{D}\mathbf{u}|^2) - \mu(z,|\mathrm{D}\mathbf{u}|^2))\mathrm{D}\mathbf{u} : \mathrm{D}\mathbf{v}\,\mathrm{d}\boldsymbol{x}\right| \leq C_\mu \int_\Omega \log(1+\lambda|\mathrm{D}\mathbf{u}|^2)(1+\lambda|\mathrm{D}\mathbf{u}|^2)^{r(c^-)}|\mathrm{D}\mathbf{u}|\,|c-z|\,|\mathrm{D}\mathbf{v}|\,\mathrm{d}\boldsymbol{x}.$$

Since $r(c^-) < 0$, there exists, for any given $\varrho \in (0,1)$, a value $M(\varrho, r(c^-)) > 0$ such that

$$\log(1+\lambda t^2) \leq (1+\lambda t^2)^{-\varrho r(c^-)} \quad \text{for all } t \geq M(\varrho, r(c^-)).$$

Furthermore, we note that there exists a constant $C_\varrho > 0$ depending on $r(c^-) < 0$ and $0 < \varrho < 1$ such that

$$\log(1+\lambda t^2)(1+\lambda t^2)^{r(c^-)} t^{-2(1-\varrho)r(c^-)} \leq C_\varrho \quad \text{for all } t \in [0, M(\varrho, r(c^-))]. \tag{57}$$

Hence, upon defining the set $\Omega_M := \{\boldsymbol{x} \in \Omega : |\mathrm{D}\mathbf{u}(\boldsymbol{x})| \leq M(\varrho, r(c^-))\}$, we find that

$$\begin{aligned}\left|\int_\Omega (\mu(c,|\mathrm{D}\mathbf{u}|^2) - \mu(z,|\mathrm{D}\mathbf{u}|^2))\mathrm{D}\mathbf{u} : \mathrm{D}\mathbf{v}\,\mathrm{d}\boldsymbol{x}\right| \;&\leq\; C_\mu C_\varrho \int_{\Omega_M} |\mathrm{D}\mathbf{u}|^{1+2(1-\varrho)r(c^-)}|c-z|\,|\mathrm{D}\mathbf{v}|\,\mathrm{d}\boldsymbol{x} \\ &\quad + C_\mu \int_{\Omega\setminus\Omega_M} (1+\lambda|\mathrm{D}\mathbf{u}|^2)^{(1-\varrho)r(c^-)}|\mathrm{D}\mathbf{u}|\,|c-z|\,|\mathrm{D}\mathbf{v}|\,\mathrm{d}\boldsymbol{x} \\ &=: \text{(I)} + \text{(II)}.\end{aligned}$$

Concerning the integral (II), we first observe that

$$(1+\lambda|\mathrm{D}\mathbf{u}|^2)^{(1-\varrho)r(c^-)} \leq \lambda^{(1-\varrho)r(c^-)}|\mathrm{D}\mathbf{u}|^{2(1-\varrho)r(c^-)},$$

since $(1-\varrho)r(c^-) < 0$, and thus

$$\text{(II)} \leq C_\mu \lambda^{(1-\varrho)r(c^-)} \int_{\Omega\setminus\Omega_M} |\mathrm{D}\mathbf{u}|^{1+2(1-\varrho)r(c^-)}|c-z|\,|\mathrm{D}\mathbf{v}|\,\mathrm{d}\boldsymbol{x}.$$

In particular, we have that

$$\left|\int_\Omega (\mu(c,|\mathrm{D}\mathbf{u}|^2) - \mu(z,|\mathrm{D}\mathbf{u}|^2))\mathrm{D}\mathbf{u} : \mathrm{D}\mathbf{v}\,\mathrm{d}\boldsymbol{x}\right| \leq C_\mu \max\{C_\varrho, \lambda^{(1-\varrho)r(c^-)}\} \underbrace{\int_\Omega |\mathrm{D}\mathbf{u}|^{1+2(1-\varrho)r(c^-)}|c-z|\,|\mathrm{D}\mathbf{v}|\,\mathrm{d}\boldsymbol{x}}_{=:\mathscr{I}}. \tag{58}$$

In the following, we will bound the integral $\mathscr{I}$ from above by applying Hölder's inequality twice. Let $p \in (2,\infty)$ (to be determined later on) and denote by $p' = p/(p-1)$ its Hölder conjugate. Then, Hölder's inequality implies that

$$\mathscr{I} \leq \|c-z\|_p \left(\int_\Omega |\mathrm{D}\mathbf{u}|^{(1+2(1-\varrho)r(c^-))p'}|\mathrm{D}\mathbf{v}|^{p'}\,\mathrm{d}\boldsymbol{x}\right)^{1/p'}.$$

Next, we apply Hölder's inequality for $q = 2/p' > 1$ to obtain

$$\mathscr{I} \le \|c - z\|_p \left(\int_\Omega |\mathrm{D}\mathbf{u}|^{(1+2(1-\varrho)r(c^-))p'q'} \,\mathrm{d}\boldsymbol{x} \right)^{1/p'q'} \|\mathrm{D}\mathbf{v}\|_2.$$

A simple calculation reveals that $p'q' = 2p/(p-2)$, and thus

$$\mathscr{I} \le \|c - z\|_p \left(\int_\Omega |\mathrm{D}\mathbf{u}|^{(1+2(1-\varrho)r(c^-))(2p/(p-2))} \,\mathrm{d}\boldsymbol{x} \right)^{(p-2)/2p} \|\mathrm{D}\mathbf{v}\|_2.$$

Let us recall that $0 < (1+2(1-\varrho)r(c^-)) < 1$. Hence, since $2p/(p-2) \to 2$ as $p \to \infty$ and $2p/(p-2) \to \infty$ as $p \to 2$, we can find a $p > 2$ such that

$$(1 + 2(1-\varrho)r(c^-)) \frac{2p}{p-2} = 2. \tag{59}$$

Indeed, a basic calculation reveals that (59) is satisfied for

$$p = -\frac{1}{(1-\varrho)r(c^-)}; \tag{60}$$

we note that this value is, as required, larger than 2 since $r(c) \in (-0.5, 0)$ for all $c > 0$ by our assumptions on the exponent r. Consequently, it follows for this specific choice of p that

$$\mathscr{I} \le \|c - z\|_p \|\mathrm{D}\mathbf{u}\|_2^{1+2(1-\varrho)r(c^-)} \|\mathrm{D}\mathbf{v}\|_2. \tag{61}$$

In the following, we shall distinguish the two cases $d = 2$ and $d = 3$.

Case $d = 2$: For simplicity we set $\varrho = 1/2$, and in turn $p = -2/r(c^-)$, so that (61) becomes

$$\mathscr{I} \le \|c - z\|_p \|\mathrm{D}\mathbf{u}\|_2^{1+r(c^-)} \|\mathrm{D}\mathbf{v}\|_2. \tag{62}$$

By the Sobolev embedding theorem, cf. Section 2.2.3, there exists a constant $C_p > 0$ such that

$$\|c - z\|_p \le C_p \|c - z\|_{1,2} \le C_p (1 + C_P) \|\nabla(c - z)\|_2, \tag{63}$$

where we employed Poincaré's inequality in the second step. Combining the inequalities (58), (62), and (63), we find that

$$\left| \int_\Omega (\mu(c, |\mathrm{D}\mathbf{u}|^2) - \mu(z, |\mathrm{D}\mathbf{u}|^2)) \mathrm{D}\mathbf{u} : \mathrm{D}\mathbf{v} \,\mathrm{d}\boldsymbol{x} \right| \le C_\mu \max\{C_\varrho, \lambda^{r(c^-)/2}\} C_p (1 + C_P) \|\mathrm{D}\mathbf{u}\|_2^{1+r(c^-)} \|\nabla(c - z)\|_2 \|\mathrm{D}\mathbf{v}\|_2.$$

Case $d = 3$: If $r(c^-) < -1/6$, then we can choose $\varrho \in (0,1)$ such that $p = 6$, cf. (60). Subsequently, if we denote by C_p the positive constant from the continuous embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$, cf. Section 2.2.3, we find similarly as before that

$$\left| \int_\Omega (\mu(c, |\mathrm{D}\mathbf{u}|^2) - \mu(z, |\mathrm{D}\mathbf{u}|^2)) \mathrm{D}\mathbf{u} : \mathrm{D}\mathbf{v} \,\mathrm{d}\boldsymbol{x} \right| \le C_\mu \max\{C_\varrho, \lambda^{-1/6}\} C_p (1 + C_P) \|\mathrm{D}\mathbf{u}\|_2^{2/3} \|\nabla(c - z)\|_2 \|\mathrm{D}\mathbf{v}\|_2.$$

In particular, we have established the following result.

Lemma 12. *For (a) $d = 2$ and $-0.5 < r(c^-) < 0$ and (b) $d = 3$ and $-0.5 < r(c^-) < -1/6$, respectively, we have that*

$$\left| \int_\Omega (\mu(c, |\mathrm{D}\mathbf{u}|^2) - \mu(z, |\mathrm{D}\mathbf{u}|^2)) \mathrm{D}\mathbf{u} : \mathrm{D}\mathbf{v} \,\mathrm{d}\boldsymbol{x} \right| \le \varphi_L(\|\mathrm{D}\mathbf{u}\|) \|\nabla(c - z)\|_2 \|\mathrm{D}\mathbf{v}\|_2$$

for all $(c - c_d)$, $(z - c_d) \in W_0^{1,2}(\Omega)$ satisfying (54) *(for almost every $\boldsymbol{x} \in \Omega$) and $\mathbf{u}, \mathbf{v} \in W_0^{1,2}(\Omega)^d$, where*

$$\text{(a) } \varphi_L(t) := C_\mu \max\{C_\varrho, \lambda^{r(c^-)/2}\} C_p (1 + C_P) t^{1+r(c^-)}, \tag{64}$$

$$\text{(b) } \varphi_L(t) := C_\mu \max\{C_\varrho, \lambda^{-1/6}\} C_p (1 + C_P) t^{2/3}; \tag{65}$$

the constant C_ϱ is defined as in (57)*, and thus depends adversely on $c^- > 0$.*

Together with Theorem 6 and Remark 7 we obtain the following uniqueness result in the setting of Section 3.2.

Corollary 13. *Let the shear stress tensor and the diffusive flux be defined as in* (42) *and* (41)*, respectively. If* $d = 2$ *and* $c_d|_{\partial\Omega} > 0$*, then the solution of (W2) is unique for* $\|\mathbf{f}\|_\star$*,* $\|c_d\|_{1,q}$*, with* $q > d$*, small enough. Moreover, if* $d = 3$ *and the boundary datum* c_d *is such that*

$$K_c - \frac{1}{\sqrt{2}} C_c^2 C_{cb} > 0, \quad \mu_0\beta - C_{ub}C_u - \frac{1}{\sqrt{2}} C_c^2 C_{cb} > 0, \quad \textit{and} \quad r(c^-) < -1/6, \tag{66}$$

then the solution of (W2) is unique for $\|\mathbf{f}\|_\star$ *small enough.*

Remark 14. Let us consider the model exponent

$$r(c) = \tfrac{1}{2}(\mathrm{e}^{-\alpha c} - 1) \in (-0.5, 0], \tag{67}$$

where $\alpha > 0$; cf. [23, p. 32 (Model 2a)]. Then, we have that $r(c^-) < -1/6$ if and only if $c^- > \alpha^{-1}\log(3/2)$. Hence, c^- can be rather small for large values of α. Furthermore, the other two inequalities in (66) are satisfied for small enough $\|\mathbf{f}\|_\star$ and $\|c_d\|_{1,q}$, with $q > d$.

4. A convergent iteration scheme for the synovial fluid flow model

In this section, we shall present a fixed point iteration scheme, which converges to a solution of (W2) in the setting of the synovial fluid flow model introduced in Section 3.2 under suitable assumptions on the data $\mathbf{f}$ and c_d. In the following, we assume without loss of generality that c_d takes its minimum on the boundary $\partial\Omega$, i.e., $c_d(\boldsymbol{x}) \geq c^-$ for all $\boldsymbol{x} \in \Omega$.

Let us define the operator $\mathrm{F}\colon W^{1,2}_{0,\mathrm{div}}(\Omega)^d \times W^{1,2}_0(\Omega) \to \left(W^{1,2}_{0,\mathrm{div}}(\Omega)^d \times W^{1,2}_0(\Omega)\right)^\star$ by

$$\begin{aligned}\langle \mathrm{F}(\mathbf{u}, c), (\mathbf{v}, z)\rangle := &\int_\Omega \mu(c + c_d, |\mathrm{D}\mathbf{u}|^2)\mathrm{D}\mathbf{u} : \mathrm{D}\mathbf{v}\,\mathrm{d}\boldsymbol{x} + B_u[\mathbf{u}, \mathbf{u}, \mathbf{v}] - \langle \mathbf{f}, \mathbf{v}\rangle \\ &+ \int_\Omega K_c \nabla(c + c_d)\cdot\nabla z\,\mathrm{d}\boldsymbol{x} + B_c[c + c_d, \mathbf{u}, z]\end{aligned} \tag{68}$$

for $(\mathbf{u}, c), (\mathbf{v}, z) \in W^{1,2}_{0,\mathrm{div}}(\Omega)^d \times W^{1,2}_0(\Omega)$. Then, problem (W2) is, in particular, equivalent to the operator equation

$$\text{Find } (\mathbf{u}, c) \in W^{1,2}_{0,\mathrm{div}}(\Omega)^d \times W^{1,2}_0(\Omega) \text{ s.t.} \quad \mathrm{F}(\mathbf{u}, c) = 0 \quad \text{in } \left(W^{1,2}_{0,\mathrm{div}}(\Omega)^d \times W^{1,2}_0(\Omega)\right)^\star. \tag{69}$$

We want to apply the Zarantonello iteration scheme, cf. the original work [40] or the monographs [41, Section 3.3] and [42, Section 25.4], to our operator equation (69). For that purpose we further define the operator $\mathrm{T}\colon W^{1,2}_{0,\mathrm{div}}(\Omega)^d \times W^{1,2}_0(\Omega) \to W^{1,2}_{0,\mathrm{div}}(\Omega)^d \times W^{1,2}_0(\Omega)$ by

$$\mathrm{T}(\mathbf{u}, c) := (\mathrm{T}^{\mathbf{u}}(\mathbf{u}, c), \mathrm{T}^c(\mathbf{u}, c)) := (\mathbf{u}, c) - \delta \mathrm{J}^{-1}\mathrm{F}(\mathbf{u}, c), \tag{70}$$

where $\delta > 0$ is a damping parameter and $\mathrm{J} : W^{1,2}_{0,\mathrm{div}}(\Omega)^d \times W^{1,2}_0(\Omega) \to (W^{1,2}_{0,\mathrm{div}}(\Omega)^d \times W^{1,2}_0(\Omega))^\star$ denotes the Riesz isometry with respect to the following inner product on $W^{1,2}_{0,\mathrm{div}}(\Omega)^d \times W^{1,2}_0(\Omega)$:

$$((\mathbf{u}, c), (\mathbf{v}, z))_{\mathrm{J}} := \int_\Omega \mathrm{D}\mathbf{u} : \mathrm{D}\mathbf{v}\,\mathrm{d}\boldsymbol{x} + \int_\Omega \nabla c\cdot\nabla z\,\mathrm{d}\boldsymbol{x}. \tag{71}$$

The norm induced by the inner product (71) will be denoted by $\|\cdot\|_{\mathrm{J}}$; therefore we have that

$$\|(\mathbf{u}, c)\|_{\mathrm{J}}^2 = \|\mathrm{D}\mathbf{u}\|_2^2 + \|\nabla c\|_2^2.$$

Furthermore, we note that $\mathrm{J}^{-1}\mathrm{F}(\mathbf{u}, c) =: (\widetilde{\mathbf{u}}, \widetilde{c}) \in W^{1,2}_{0,\mathrm{div}}(\Omega)^d \times W^{1,2}_0(\Omega)$ is the unique solution of the linear problem

$$\begin{aligned}\int_\Omega \mathrm{D}\widetilde{\mathbf{u}} : \mathrm{D}\mathbf{v}\,\mathrm{d}\boldsymbol{x} + \int_\Omega \nabla\widetilde{c}\cdot\nabla z\,\mathrm{d}\boldsymbol{x} = &\int_\Omega \mu(c + c_d, |\mathrm{D}\mathbf{u}|^2)\mathrm{D}\mathbf{u} : \mathrm{D}\mathbf{v}\,\mathrm{d}\boldsymbol{x} + B_u[\mathbf{u}, \mathbf{u}, \mathbf{v}] - \langle \mathbf{f}, \mathbf{v}\rangle \\ &+ \int_\Omega K_c\nabla(c + c_d)\cdot\nabla z\,\mathrm{d}\boldsymbol{x} + B_c[c + c_d, \mathbf{u}, z]\end{aligned}$$

for all $(\mathbf{v}, z) \in W^{1,2}_{0,\mathrm{div}}(\Omega)^d \times W^{1,2}_0(\Omega)$. We will first show that the operator T maps a closed subset of $W^{1,2}_{0,\mathrm{div}}(\Omega)^d \times W^{1,2}_0(\Omega)$ to itself. Subsequently, we will establish the strong monotonicity and the Lipschitz continuity of F restricted to this closed subset, which, in turn, implies the convergence of the Zarantonello iteration to a solution of (69) for a suitable damping parameter.

For the sake of deriving the self-mapping property, we consider a closed ball of the form

$$\mathbf{B}_R := \{(\mathbf{u}, c) \in W^{1,2}_{0,\mathrm{div}}(\Omega)^d \times W^{1,2}_0(\Omega) : \|(\mathbf{u}, c)\|_{\mathrm{J}} \le R\}.$$

By the definitions of the operator T and the inner product $(\cdot,\cdot)_{\mathrm{J}}$ we have that

$$\begin{aligned}
\|\mathrm{T}(\mathbf{u},c)\|_{\mathrm{J}}^2 &= ((\mathbf{u},c) - \delta \mathrm{J}^{-1}\mathrm{F}(\mathbf{u},c), (\mathrm{T}^{\mathbf{u}}(\mathbf{u},c), \mathrm{T}^c(\mathbf{u},c)))_{\mathrm{J}} \\
&= \int_\Omega \mathrm{D}\mathbf{u} : \mathrm{D}\mathrm{T}^{\mathbf{u}}(\mathbf{u},c)\,\mathrm{d}\boldsymbol{x} + \int_\Omega \nabla c \cdot \nabla \mathrm{T}^c(\mathbf{u},c)\,\mathrm{d}\boldsymbol{x} \\
&\quad - \delta \int_\Omega \mu(c + c_d, |\mathrm{D}\mathbf{u}|^2)\mathrm{D}\mathbf{u} : \mathrm{D}\mathrm{T}^{\mathbf{u}}(\mathbf{u},c)\,\mathrm{d}\boldsymbol{x} - \delta B_u[\mathbf{u},\mathbf{u},\mathrm{T}^{\mathbf{u}}(\mathbf{u},c)] + \delta\langle \mathbf{f}, \mathrm{T}^{\mathbf{u}}(\mathbf{u},c)\rangle \\
&\quad - \delta \int_\Omega K_c \nabla(c + c_d)\cdot \nabla \mathrm{T}^c(\mathbf{u},c)\,\mathrm{d}\boldsymbol{x} - \delta B_c[c + c_d, \mathbf{u}, \mathrm{T}^c(\mathbf{u},c)].
\end{aligned}$$

Invoking Lemmas 2, 4, Korn's inequality (3), and the Cauchy–Schwarz inequality leads to

$$\begin{aligned}
\|\mathrm{T}(\mathbf{u},c)\|_{\mathrm{J}}^2 &\le \int_\Omega (1 - \delta\mu(c + c_d, |\mathrm{D}\mathbf{u}|^2))\mathrm{D}\mathbf{u} : \mathrm{D}\mathrm{T}^{\mathbf{u}}(\mathbf{u},c)\,\mathrm{d}\boldsymbol{x} + \int_\Omega (1 - \delta K_c)\nabla c \cdot \nabla \mathrm{T}^c(\mathbf{u},c)\,\mathrm{d}\boldsymbol{x} \\
&\quad + \delta K_c \|\nabla c_d\|_2 \|\nabla \mathrm{T}^c(\mathbf{u},c)\|_2 + \delta C_u \|\mathrm{D}\mathbf{u}\|_2^2 \|\mathrm{D}\mathrm{T}^{\mathbf{u}}(\mathbf{u},c)\|_2 + \delta \|\mathbf{f}\|_\star \|\mathrm{D}\mathrm{T}^{\mathbf{u}}(\mathbf{u},c)\|_2 \\
&\quad + \delta\sqrt{2}C_c^2 \|\mathrm{D}\mathbf{u}\|_2 \|\nabla \mathrm{T}^c(\mathbf{u},c)\|_2 (\|\nabla c\|_2 + \|\nabla c_d\|_2).
\end{aligned}$$

Recall that $\beta\mu_0 \le \mu(c,t) \le \mu_0$; here, we assume without loss of generality that this remains true for negative values of c. Hence, we further find that, for $\delta \le \min\{1/K_c, 2/(\mu_0(1+\beta))\}$,

$$\begin{aligned}
\|\mathrm{T}(\mathbf{u},c)\|_{\mathrm{J}}^2 &\le (1 - \delta\beta\mu_0)\|\mathrm{D}\mathbf{u}\|_2 \|\mathrm{D}\mathrm{T}^{\mathbf{u}}(\mathbf{u},c)\|_2 + (1 - \delta K_c)\|\nabla c\|_2 \|\nabla \mathrm{T}^c(\mathbf{u},c)\|_2 \\
&\quad + \delta K_c \|\nabla c_d\|_2 \|\nabla \mathrm{T}^c(\mathbf{u},c)\|_2 + \delta C_u \|\mathrm{D}\mathbf{u}\|_2^2 \|\mathrm{D}\mathrm{T}^{\mathbf{u}}(\mathbf{u},c)\|_2 + \delta \|\mathbf{f}\|_\star \|\mathrm{D}\mathrm{T}^{\mathbf{u}}(\mathbf{u},c)\|_2 \\
&\quad + \delta\sqrt{2}C_c^2 \|\mathrm{D}\mathbf{u}\|_2 \|\nabla \mathrm{T}^c(\mathbf{u},c)\|_2 \|\nabla c\|_2 + \delta\sqrt{2}C_c^2 \|\mathrm{D}\mathbf{u}\|_2 \|\nabla \mathrm{T}^c(\mathbf{u},c)\|_2 \|\nabla c_d\|_2 \\
&=: (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) + (\mathrm{IV}) + (\mathrm{V}) + (\mathrm{VI}) + (\mathrm{VII}).
\end{aligned}$$

The Cauchy–Schwarz inequality implies that

$$(\mathrm{I}) + (\mathrm{II}) \le (1 - \delta\min\{\beta\mu_0, K_c\})\|(\mathbf{u},c)\|_{\mathrm{J}} \|\mathrm{T}(\mathbf{u},c)\|_{\mathrm{J}}.$$

Now let us assume that

$$\alpha_{\min} := \min\{\beta\mu_0, K_c\} - \sqrt{2}C_c^2 \|\nabla c_d\|_2 > 0, \tag{72}$$

which is the case for $\|c_d\|_{1,q}$ small enough, with $q > d$. Then, we have that

$$(\mathrm{I}) + (\mathrm{II}) + (\mathrm{VII}) \le (1 - \delta\alpha_{\min})\|(\mathbf{u},c)\|_{\mathrm{J}} \|\mathrm{T}(\mathbf{u},c)\|_{\mathrm{J}}. \tag{73}$$

Furthermore, again by employing the Cauchy–Schwarz inequality, we obtain the bounds

$$(\mathrm{III}) + (\mathrm{V}) \le \delta(K_c^2 \|\nabla c_d\|_2^2 + \|\mathbf{f}\|_\star^2)^{1/2} \|\mathrm{T}(\mathbf{u},c)\|_{\mathrm{J}}, \tag{74}$$

and

$$(\mathrm{IV}) + (\mathrm{VI}) \le \delta(C_u^2 + 2^{-1}C_c^4)^{1/2} \|(\mathbf{u},c)\|_{\mathrm{J}}^2 \|\mathrm{T}(\mathbf{u},c)\|_{\mathrm{J}}. \tag{75}$$

Combining the bounds (73), (74), and (75) yields that

$$\|\mathrm{T}(\mathbf{u},c)\|_{\mathrm{J}} \le (1 - \delta\alpha_{\min})\|(\mathbf{u},c)\|_{\mathrm{J}} + \delta(K_c^2 \|\nabla c_d\|_2^2 + \|\mathbf{f}\|_\star^2)^{1/2} + \delta(C_u^2 + 2^{-1}C_c^4)^{1/2} \|(\mathbf{u},c)\|_{\mathrm{J}}^2.$$

Consequently, in order to obtain a self-mapping $\mathrm{T}\colon \mathbf{B}_R \to \mathbf{B}_R$, we require that

$$(1 - \delta\alpha_{\min})t + \delta(K_c^2 \|\nabla c_d\|_2^2 + \|\mathbf{f}\|_\star^2)^{1/2} + \delta(C_u^2 + 2^{-1}C_c^4)^{1/2} t^2 \le R$$

for all $t \le R$. Since $\delta\alpha_{\min} < 1$, it is sufficient to find an $R > 0$ such that

$$(C_u^2 + 2^{-1}C_c^4)^{1/2} R^2 - \alpha_{\min} R + (K_c^2 \|\nabla c_d\|_2^2 + \|\mathbf{f}\|_\star^2)^{1/2} \le 0. \tag{76}$$

We remark that the corresponding quadratic equation has the solutions

$$R_\pm := \frac{\alpha_{\min} \pm \sqrt{\alpha_{\min}^2 - 4(C_u^2 + 2^{-1}C_c^4)^{1/2}(K_c^2\|\nabla c_d\|_2^2 + \|\mathbf{f}\|_\star^2)^{1/2}}}{2(C_u^2 + 2^{-1}C_c^4)^{1/2}} \geq 0 \tag{77}$$

provided that the expression under the square root is nonnegative. In particular, we have established the following result.

Proposition 15. *Assume the small data properties* (72) *and*

$$(K_c^2\|\nabla c_d\|_2^2 + \|\mathbf{f}\|_\star^2)^{1/2} \leq \frac{\alpha_{\min}^2}{4(C_u^2 + 2^{-1}C_c^4)^{1/2}}, \tag{78}$$

where $\alpha_{\min}$ *is defined as in* (72), *and that the damping parameter satisfies* $0 < \delta \leq \min\{1/K_c, 2/(\mu_0(1+\beta))\}$. *Then, for any* $R \in [R_-, R_+]$, *cf.* (77), *we have that* $\mathrm{T} : \mathbf{B}_R \to \mathbf{B}_R$ *is a self-mapping.*

Next, we will show that $\mathrm{F}|_{\mathbf{B}_R}$ is strongly monotone for sufficiently small R. To that end, let $(\mathbf{u}, c), (\mathbf{v}, z) \in \mathbf{B}_R$ and consider

$$\begin{aligned}
\langle \mathrm{F}(\mathbf{u}, c) - \mathrm{F}(\mathbf{v}, z), (\mathbf{u}, c) - (\mathbf{v}, z)\rangle &= \int_\Omega (\mu(c + c_d, |\mathrm{D}\mathbf{u}|^2)\mathrm{D}\mathbf{u} - \mu(z + c_d, |\mathrm{D}\mathbf{v}|^2)\mathrm{D}\mathbf{v}) : \mathrm{D}(\mathbf{u} - \mathbf{v})\,\mathrm{d}\boldsymbol{x} \\
&\quad + B_u[\mathbf{u}, \mathbf{u}, \mathbf{u} - \mathbf{v}] - B_u[\mathbf{v}, \mathbf{v}, \mathbf{u} - \mathbf{v}] \\
&\quad + B_c[c + c_d, \mathbf{u}, c - z] - B_c[z + c_d, \mathbf{v}, c - z] \\
&\quad + \int_\Omega K_c \nabla(c - z) \cdot \nabla(c - z)\,\mathrm{d}\boldsymbol{x} \\
&=: (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) + (\mathrm{IV});
\end{aligned}$$

cf. (35). Using similar arguments as in the analysis of Section 3.1, we obtain the bounds

$$|(\mathrm{II})| \leq C_u\|\mathrm{D}\mathbf{u}\|_2\|\mathrm{D}(\mathbf{u} - \mathbf{v})\|_2^2$$

and

$$|(\mathrm{III})| \leq \sqrt{2}C_c^2\|\mathrm{D}(\mathbf{u} - \mathbf{v})\|_2\|\nabla(c - z)\|_2(\|\nabla c\|_2 + \|\nabla c_d\|_2);$$

we note that only the estimates (22), (23) are required to obtain the latter bound, and therefore this remains valid in the setting of merely discretely divergence-free finite element approximations to the velocity field. Moreover, it is evident that

$$(\mathrm{IV}) = K_c\|\nabla(c - z)\|_2^2,$$

and thus it remains to bound (I). As in the analysis of Section 3.1, we will consider the following decomposition:

$$\begin{aligned}
(\mathrm{I}) &= \int_\Omega (\mu(c + c_d, |\mathrm{D}\mathbf{u}|^2)\mathrm{D}\mathbf{u} - \mu(c + c_d, |\mathrm{D}\mathbf{v}|^2)\mathrm{D}\mathbf{v}) : \mathrm{D}(\mathbf{u} - \mathbf{v})\,\mathrm{d}\boldsymbol{x} \\
&\quad + \int_\Omega (\mu(c + c_d, |\mathrm{D}\mathbf{v}|^2)\mathrm{D}\mathbf{v} - \mu(z + c_d, |\mathrm{D}\mathbf{v}|^2)\mathrm{D}\mathbf{v}) : \mathrm{D}(\mathbf{u} - \mathbf{v})\,\mathrm{d}\boldsymbol{x} \\
&=: (\mathrm{Ia}) + (\mathrm{Ib}).
\end{aligned}$$

Thanks to (48) we have that

$$(\mathrm{Ia}) \geq \mu_0\beta\|\mathrm{D}(\mathbf{u} - \mathbf{v})\|_2^2.$$

Next, we want to bound (Ib) in the same manner as in Section 3.2. We note that in the analysis of Section 3.2 we used that $c, z \geq c^- > 0$, and in turn $r(c), r(z) \leq r(c^-) < 0$, which is not guaranteed in the context of the Zarantonello iteration. However, this issue can easily be circumvented by redefining the exponent function $r(c) := \min\{r(c), r(c^-)\}$. Indeed, this does not interfere with any solution $(\mathbf{u}^\star, c^\star)$ of (W2), since r is monotonically decreasing and $c^\star(\boldsymbol{x}) \geq c^-$ for every $\boldsymbol{x} \in \Omega$. Moreover, even though $r : \mathbb{R} \to (-0.5, r(c^-)]$ might no longer be continuously differentiable, all of the results from the previous section remain valid. Therefore, we may apply Lemma 12, which states that, under suitable assumptions,

$$|(\mathrm{Ib})| \le \varphi_L(\|\mathrm{D}\mathbf{v}\|_2)\|\nabla(c-z)\|_2\|\mathrm{D}(\mathbf{u}-\mathbf{v})\|_2,$$

where φ_L is defined as in (64) and (65) for $d=2$ and $d=3$, respectively. Combining all of the inequalities established above and recalling that $(\mathbf{u},c),(\mathbf{v},z)\in\mathbf{B}_R$ leads to

$$\begin{aligned}\langle \mathrm{F}(\mathbf{u},c)-\mathrm{F}(\mathbf{v},z),(\mathbf{u},c)-(\mathbf{v},z)\rangle &\ge (\mu_0\beta - C_u R)\|\mathrm{D}(\mathbf{u}-\mathbf{v})\|_2^2 + K_c\|\nabla(c-z)\|_2^2\\ &\quad -\left(\sqrt{2}C_c^2(R+\|\nabla c_d\|_2)+\varphi_L(R)\right)\|\mathrm{D}(\mathbf{u}-\mathbf{v})\|_2\|\nabla(c-z)\|_2\\ &\ge \nu_{\mathrm{F}}\|(\mathbf{u},c)\|_{\mathrm{J}}^2,\end{aligned}$$

where

$$\nu_{\mathrm{F}} := \min\{\mu_0\beta - C_u R, K_c\} - 2^{-1/2}C_c^2(R+\|\nabla c_d\|_2) - 2^{-1}\varphi_L(R).$$

Remark 16. We note that $\nu_{\mathrm{F}}>0$ for R and $\|c_d\|_{1,q}$ small enough, with $q>d$. Moreover, we have that $R_-\to 0$ in (77) as $\|\mathbf{f}\|_\star$, $\|c_d\|_{1,q}\to 0$, with $q>d$. In particular, for small enough data $\|c_d\|_{1,q}$ and $\|\mathbf{f}\|_\star$, with $q>d$, there exists a radius $R>0$ such that $\mathrm{T}:\mathbf{B}_R\to\mathbf{B}_R$ is a self-mapping, cf. Proposition 15, and $\mathrm{F}|_{\mathbf{B}_R}$ is strongly monotone.

It remains to verify the Lipschitz continuity of $\mathrm{F}|_{\mathbf{B}_R}$, where R is chosen according to Remark 16. For $(\mathbf{u},c)$, $(\mathbf{v},z)$, $(\mathbf{w},h)\in\mathbf{B}_R$ we have that

$$\begin{aligned}\langle \mathrm{F}(\mathbf{u},c)-\mathrm{F}(\mathbf{v},z),(\mathbf{w},h)\rangle &= \int_\Omega (\mu(c+c_d,|\mathrm{D}\mathbf{u}|^2)\mathrm{D}\mathbf{u} - \mu(z+c_d,|\mathrm{D}\mathbf{v}|^2)\mathrm{D}\mathbf{v}):\mathrm{D}\mathbf{w}\,\mathrm{d}\boldsymbol{x}\\ &\quad + B_u[\mathbf{u},\mathbf{u},\mathbf{w}] - B_u[\mathbf{v},\mathbf{v},\mathbf{w}]\\ &\quad + B_c[c+c_d,\mathbf{u},h] - B_c[z+c_d,\mathbf{v},h]\\ &\quad + \int_\Omega K_c\nabla(c-z)\cdot\nabla h\,\mathrm{d}\boldsymbol{x}\\ &=: (\mathrm{I})+(\mathrm{II})+(\mathrm{III})+(\mathrm{IV}).\end{aligned}$$

By similar arguments as before, and since all of the elements considered are contained in the closed ball $\mathbf{B}_R$, we find that

$$|(\mathrm{II})| \le 2C_u R\|\mathrm{D}(\mathbf{u}-\mathbf{v})\|_2\|\mathrm{D}\mathbf{w}\|_2 \tag{79}$$

and

$$\begin{aligned}|(\mathrm{III})| &\le \sqrt{2}C_c^2 R\|\nabla(c-z)\|_2\|\nabla h\|_2 + \sqrt{2}C_c^2(R+\|\nabla c_d\|_2)\|\mathrm{D}(\mathbf{u}-\mathbf{v})\|_2\|\nabla h\|_2\\ &\le 2C_c^2(R+\|\nabla c_d\|_2)\|(\mathbf{u},c)-(\mathbf{v},z)\|_{\mathrm{J}}\|\nabla h\|_2;\end{aligned}$$

again, this bound further on holds true in the discrete setting without requiring that the discretely divergence-free finite element velocity fields are also pointwise divergence-free. Moreover, the Cauchy–Schwarz inequality implies that

$$|(\mathrm{IV})| \le K_c\|\nabla(c-z)\|_2\|\nabla h\|_2. \tag{80}$$

We can bound (I) in the same manner as in the derivation of the strong monotonicity, whereby we have to replace the lower bound (48) with the upper bound (47), which yields

$$\begin{aligned}|(\mathrm{I})| &\le \sqrt{3}\mu_0\|\mathrm{D}(\mathbf{u}-\mathbf{v})\|_2\|\mathrm{D}\mathbf{w}\|_2 + \varphi_L(R)\|\nabla(c-z)\|_2\|\mathrm{D}\mathbf{w}\|_2\\ &\le (3\mu_0^2+\varphi_L(R)^2)^{1/2}\|(\mathbf{u},c)-(\mathbf{v},z)\|_{\mathrm{J}}\|\mathrm{D}\mathbf{w}\|_2.\end{aligned}$$

Next, we may combine the bounds (79) and (80) to obtain

$$|(\mathrm{II})|+|(\mathrm{IV})| \le \max\{K_c, 2C_u R\}\|(\mathbf{u},c)-(\mathbf{v},z)\|_{\mathrm{J}}\|(\mathbf{w},h)\|_{\mathrm{J}}. \tag{81}$$

Together with the bounds for (I) and (III), we find that

$$\langle \mathrm{F}(\mathbf{u},c)-\mathrm{F}(\mathbf{v},z),(\mathbf{w},h)\rangle \le L_{\mathrm{F}}\|(\mathbf{u},c)-(\mathbf{v},z)\|_{\mathrm{J}}\|(\mathbf{w},h)\|_{\mathrm{J}},$$

where

$$L_{\mathrm{F}} := (3\mu_0^2 + \varphi_L(R)^2 + 4C_c^4(R + \|\nabla c_d\|_2)^2)^{1/2} + \max\{K_c, 2C_u R\}. \tag{82}$$

Theorem 17. *Assume that the data $\|\mathbf{f}\|_\star$ and $\|c_d\|_{1,q}$, with $q > d$, $c_d \geq c^-$ and (a) $c^- > 0$ for $d = 2$ and (b) $r(c^-) < -1/6$ for $d = 3$, respectively, are small enough such that (72) and (78) are both satisfied and, in addition, $R \in [R_-, R_+]$ can be chosen so that $\nu_{\mathrm{F}} > 0$, cf. Remark 16. Moreover, consider the modified exponent function $r(c) := \min\{r(c), r(c^-)\}$. Then, the Zarantonello iteration,*

$$(\mathbf{u}^{n+1}, c^{n+1}) := \mathrm{T}(\mathbf{u}^n, c^n), \tag{83}$$

where $\mathrm{T} : W^{1,2}_{0,\mathrm{div}}(\Omega)^d \times W^{1,2}_0(\Omega) \to W^{1,2}_{0,\mathrm{div}}(\Omega)^d \times W^{1,2}_0(\Omega)$ is defined as in (70), converges to the unique solution of (69) contained in $\mathbf{B}_R$ for any initial guess $(\mathbf{u}^0, c^0) \in \mathbf{B}_R$ and damping parameter

$$0 < \delta < \min\{2\nu_{\mathrm{F}}/L_{\mathrm{F}}^2, 1/K_c, 2/(\mu_0(1+\beta))\}.$$

This statement remains valid for the iteration on conforming, discretely divergence-free finite element spaces, cf. Section 5.1.

Proof. The claim follows from the Lipschitz continuity and the strong monotonicity, see, e.g., [41, Theorem 3.3.23]. □

5. Numerical experiments

In this section, we will perform numerical experiments in two space-dimensions, i.e. $d = 2$, to empirically examine the convergence of the Zarantonello iteration (83) in the context of the synovial fluid model (41), (42) with the exponent given as in (67). For that purpose, we will consider appropriate mixed finite element methods to discretize the problem.

5.1. *Finite element spaces*

Let $\mathscr{T}$ be an admissible triangulation, cf. [43, Definition 5.1], of $\Omega \subset \mathbb{R}^2$. Then, we consider the following conforming finite element spaces of the velocity, pressure, and concentration fields, respectively:

$$\begin{aligned}
\mathbb{V}(\mathscr{T}) &:= \{\mathbf{V} \in W^{1,2}_0(\Omega)^2 : \mathbf{V}|_T \in \mathbb{P}_2(T)^2 \text{ for all } T \in \mathscr{T}\},\\
\mathbb{Q}(\mathscr{T}) &:= \{Q \in C(\Omega) : Q|_T \in \mathbb{P}_1(T) \text{ for all } T \in \mathscr{T}\} \cap L^2_0(\Omega),\\
\mathbb{Z}(\mathscr{T}) &:= \{Z \in W^{1,2}_0(\Omega) : Z|_T \in \mathbb{P}_2(T) \text{ for all } T \in \mathscr{T}\},
\end{aligned}$$

where $\mathbb{P}_k$ denotes the set of polynomials of total degree at most $k \in \mathbb{N}$. In particular, we consider the lowest order Taylor–Hood element for the velocity-pressure pair, see, e.g., [32, Section 3.6.2]; it is well-known that this pair satisfies the discrete inf-sup condition. Moreover, the space of discretely divergence-free velocity vectors is given by

$$\mathbb{V}_0(\mathscr{T}) := \left\{\mathbf{V} \in \mathbb{V}(\mathscr{T}) : \int_\Omega Q \operatorname{div}\mathbf{V} \,\mathrm{d}\boldsymbol{x} = 0 \text{ for all } Q \in \mathbb{Q}(\mathscr{T})\right\}.$$

5.2. *Discrete iteration*

For our numerical experiment, we will include the pressure field into the iteration scheme (83), and, in turn, do not require the velocity vectors to be (discretely) divergence-free. To that end, we redefine the iteration scheme as follows: find $(\mathbf{U}^{n+1}, P^{n+1}, C^{n+1}) \in \mathbb{V}(\mathscr{T}) \times \mathbb{Q}(\mathscr{T}) \times \mathbb{Z}(\mathscr{T})$ such that

$$\begin{aligned}
((\mathbf{U}^{n+1}, C^{n+1}), (\mathbf{V}, Z))_{\mathrm{J}} - \delta \int_\Omega P^{n+1} \operatorname{div}\mathbf{V} \,\mathrm{d}\boldsymbol{x} + \int_\Omega Q \operatorname{div}\mathbf{U}^{n+1} \,\mathrm{d}\boldsymbol{x}&\\
= ((\mathbf{U}^n, C^n), (\mathbf{V}, Z))_{\mathrm{J}} - \delta \langle \mathrm{F}(\mathbf{U}^n, C^n), (\mathbf{V}, Z)\rangle& \quad (84)
\end{aligned}$$

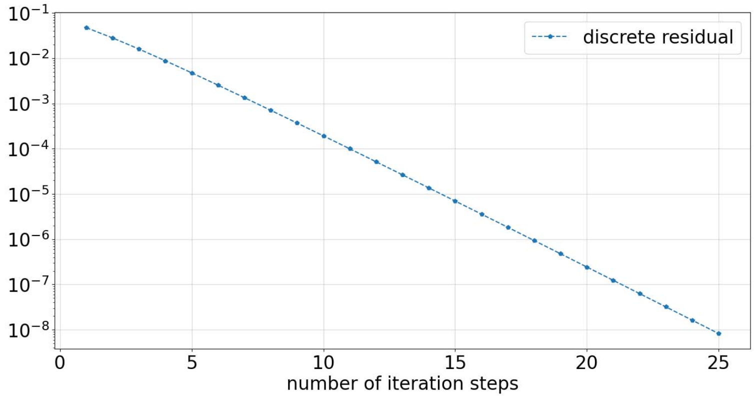

Figure 1. Experiment 1: Plot of the discrete residual against the number of iteration steps.

for all $(\mathbf{V}, Q, Z) \in \mathbb{V}(\mathscr{T}) \times \mathbb{Q}(\mathscr{T}) \times \mathbb{Z}(\mathscr{T})$, where $\mathrm{F} : \mathbb{V}(\mathscr{T}) \times \mathbb{Z}(\mathscr{T}) \to (\mathbb{V}(\mathscr{T}) \times \mathbb{Z}(\mathscr{T}))^\star$ is similarly defined as in (68). In particular,

$$\langle \mathrm{F}(\mathbf{U}, C), (\mathbf{V}, Z)\rangle := \int_\Omega \mu(C + c_d, |\mathrm{D}\mathbf{U}|^2)\mathrm{D}\mathbf{U} : \mathrm{D}\mathbf{V}\,\mathrm{d}\boldsymbol{x} + B_u[\mathbf{U}, \mathbf{U}, \mathbf{V}] - \langle \mathbf{f}, \mathbf{V}\rangle$$
$$+ \int_\Omega K_c \nabla(C + c_d) \cdot \nabla Z\,\mathrm{d}\boldsymbol{x} + B_c[C + c_d, \mathbf{U}, Z]$$

for $(\mathbf{U}, C), (\mathbf{V}, Z) \in \mathbb{V}(\mathscr{T}) \times \mathbb{Z}(\mathscr{T})$. It can be shown that, thanks to the discrete inf-sup condition, the convergence of the divergence-free iteration scheme (83) implies the convergence of the iteration procedure (84); we refer to the analysis from [44, Section 4.2].

5.3. *Numerical tests*

We will now provide some numerical results. For the purpose of our experiments, the algorithm has been implemented in Python using the FEniCS software [45,46]. In all of our numerical tests, we will consider no-slip boundary conditions for the velocity vector on the rectangular domain $\Omega := (0, 10) \times (0, 1) \subset \mathbb{R}^2$. Furthermore, the source function is defined by

$$\mathbf{f}(x, y) := ((x + 0.1)^{-1/4}(y + 0.1)^{-1/4}, (10.1 - x)^{-1/2})^{\mathrm{T}}, \tag{85}$$

where $(x, y) \in \Omega$ denote the Euclidean coordinates in $\mathbb{R}^2$, and the boundary function is given by

$$c_d(x, y) = x + y + xy + 1. \tag{86}$$

For our initial guess $(\mathbf{U}^0, C^0) \in \mathbb{V}(\mathscr{T}) \times \mathbb{Z}(\mathscr{T})$, we will always use the constant null function.

Experiment 1. In our first experiment, we will consider the fixed parameters $K_c = 1$, $\mu_0 = 1$, $\beta = 0.01$, $\lambda = 10$, $\alpha = 3$, and a uniform mesh consisting of 2000 elements. In this setting, the Zarantonello iteration with damping parameter $\delta = 1.5$ required 25 steps to generate an approximation whose discrete residual has a norm smaller than $\varepsilon_{\mathrm{tol}} := 10^{-8}$; the corresponding convergence plot is shown in Figure 1.

We performed this experiment for different mesh sizes. In each case, the same number of iteration steps were required to obtain the given tolerance for the discrete residual. This indicates that the Zarantonello iteration is robust with respect to the mesh size. This is one of the reasons we endeavoured to carry out the convergence analysis at the function space level; for instance, a failure to note that a map such as J maps between a function space and its dual space can lead to mesh-dependence in the resulting iterative algorithm (cf. [47]).

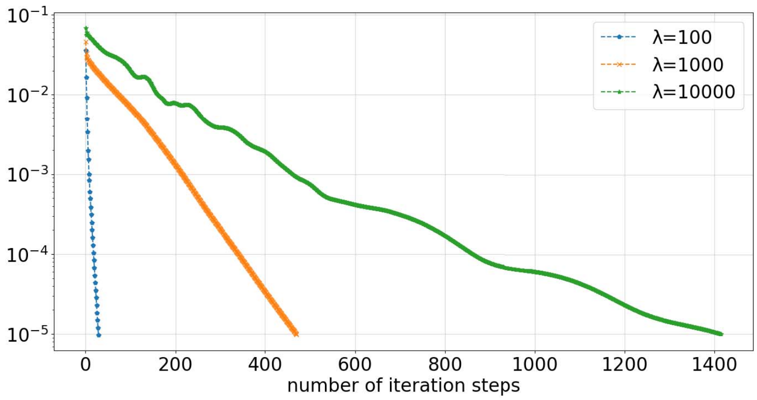

Figure 2. Experiment 2: Plot of the discrete residual against the number of iteration steps for different choices of the relaxation parameter λ.

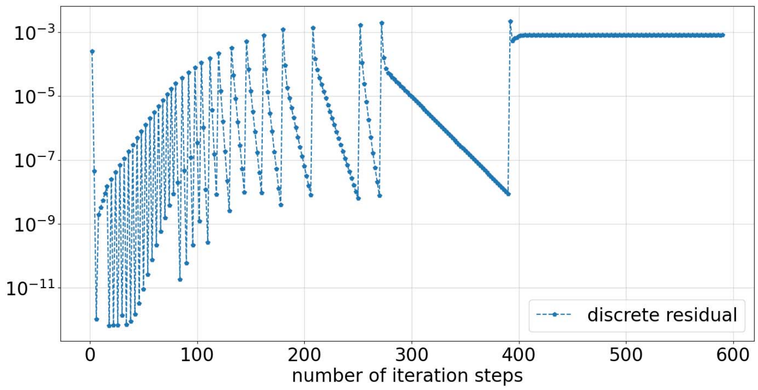

Figure 3. Experiment 3: Plot of the discrete residual against the number of iteration steps for the Newton method with a continuation with respect to λ.

Experiment 2 (Influence of the relaxation time λ). Next, we want to examine how different choices of the relaxation parameter λ affect the convergence rate of the Zarantonello iteration scheme. Except for λ, we choose the same parameters as in the previous experiment. As can be seen from Figure 2, the convergence rate deteriorates for an increasing value of λ; however, the Zarantonello iteration still generates a highly accurate approximation even for exceptionally large λ. In contrast, the classical Newton method already fails to converge for a much smaller value of the parameter λ, as shall be shown in the next experiment.

Experiment 3 (Performance of the classical Newton method for large λ). Once more, we consider the setting from the Experiments 1 and 2, but now starting with the value $\lambda = 1$, and employing the classical Newton method. For a given relaxation time λ, we iterate until the norm of the discrete residual drops below $\varepsilon_{\text{tol}} := 10^{-8}$. We then update the relaxation parameter by the rule $\lambda \leftarrow 2^{1/4}\lambda$, and repeat the process until Newton's method fails to converge within 200 steps for a given $\lambda > 0$. In particular, we employ a very generous continuation method with respect to the parameter λ. Nonetheless, Newton's method already fails to converge for $\lambda \approx 430$ in the given setting. The corresponding convergence plot is depicted in Figure 3.

6. Conclusion

We proved that, under suitable assumptions on the shear stress tensor and diffusive flux, the solution to our chemically reacting incompressible fluid flow problem is unique for small enough data. It was further shown that this abstract analysis can be applied to a model of the synovial fluid. In this context, we have, in addition, introduced a fixed point iteration scheme, which generates a sequence converging to a solution of the given problem. Finally, our numerical experiments indicate that the proposed iteration scheme converges in certain situations in which the classical Newton method fails. Thanks to the simplicity of the proposed iterative method (it amounts to solving Stokes–Laplace systems), it is likely that it could be successfully employed as a relaxation scheme within a nonlinear multigrid method such as the Full Approximation Scheme (FAS), in order to accelerate convergence. In addition, the method would be suitable for computation at scale, since efficient linear solvers and preconditioners are already available for the Stokes system [48]; in contrast, developing preconditioners for the Newton linearization of Problem (W1) would most certainly prove to be very challenging.

Conflicts of interest

The authors declare no competing financial interest.

Dedication

The manuscript was written through contributions of all authors. All authors have given approval to the final version of the manuscript.

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Appendix A. Existence of a finite element solution

In this appendix, we shall provide a sketch of the existence of a finite element solution. For that purpose we consider the finite element spaces from Section 5.1 and the corresponding discretized problem: Find $(\mathbf{U}, C) \in \mathbb{V}_0(\mathscr{T}) \times (\mathbb{Z}(\mathscr{T}) + c_d)$ such that

$$\int_\Omega \mathbf{S}(C, \mathrm{D}\mathbf{U}) : \mathrm{D}\mathbf{V} \,\mathrm{d}\boldsymbol{x} + B_u[\mathbf{U}, \mathbf{U}, \mathbf{V}] = \langle \mathbf{f}, \mathbf{V} \rangle \quad \forall\, \mathbf{V} \in \mathbb{V}_0(\mathscr{T}), \tag{A 1}$$

$$\int_\Omega \mathbf{q}_c(C, \nabla C, \mathrm{D}\mathbf{U}) \cdot \nabla Z \,\mathrm{d}\boldsymbol{x} + B_c[C, \mathbf{U}, Z] = 0 \quad \forall\, Z \in \mathbb{Z}(\mathscr{T}); \tag{A 2}$$

here, we assume without loss of generality that $c_d \in W^{1,\infty}(\Omega)$. We further assume that the assumptions (AS) and (AQ) are satisfied and that the diffusive flux, in addition, fulfils the following strong monotonicity assumption: there exists a constant $C_{qM} > 0$ such that

$$(\mathbf{q}_c(c, \nabla c, \boldsymbol{\kappa}) - \mathbf{q}_c(z, \nabla z, \boldsymbol{\kappa})) \cdot \nabla(c - z) \geq C_{qM} |\nabla(c - z)|^2 \tag{A 3}$$

for all c, $z \in (W_0^{1,2}(\Omega) + c_d)$ and $\boldsymbol{\kappa} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$. We note that (A 3) is indeed satisfied for our model of the synovial fluid, cf. (41).

First of all, for given $\widehat{\mathbf{U}} \in \mathbb{V}_0(\mathscr{T})$ and $\widehat{C} \in (\mathbb{Z}(\mathscr{T}) + c_d)$, it can be shown in the same manner as in [27] that the problem

$$\text{Find } \mathbf{U} \in \mathbb{V}_0(\mathscr{T}) \text{ s.t. } \int_\Omega \mathbf{S}(\widehat{C}, \mathrm{D}\mathbf{U}) : \mathrm{D}\mathbf{V}\,\mathrm{d}\boldsymbol{x} + B_u[\widehat{\mathbf{U}}, \mathbf{U}, \mathbf{V}] = \langle \mathbf{f}, \mathbf{V} \rangle \quad \forall \mathbf{V} \in \mathbb{V}_0(\mathscr{T}), \tag{A 4}$$

has a unique solution with $\|\mathbf{U}\|_{1,2} \leq K_u$ for some constant K_u independent of $\widehat{\mathbf{U}}$ and $\widehat{C}$. In the following, we will denote by $\mathrm{G} : \mathbb{V}_0(\mathscr{T}) \times (\mathbb{Z}(\mathscr{T}) + c_d) \to \mathbb{V}_0(\mathscr{T})$ the corresponding solution mapping. In [27] it was further shown that, for any fixed $\mathbf{U} \in \mathbf{B}^{\mathbb{V}}_{K_u} := \{\mathbf{V} \in \mathbb{V}_0(\mathscr{T}) : \|\mathbf{V}\|_{1,2} \leq K_u\}$, the convection–diffusion equation (A 2) has a unique solution $C \in (\mathbb{Z}(\mathscr{T}) + c_d)$ with $\|C\|_{1,2} \leq K_c$ for some constant K_c independent of $\mathbf{U}$; let $\mathrm{N} : \mathbf{B}^{\mathbb{V}}_{K_u} \to (\mathbb{Z}(\mathscr{T}) + c_d)$ denote the corresponding solution mapping. Then, we have that $\mathrm{H} := \mathrm{G}(\cdot, \mathrm{N}(\cdot)) : \mathbf{B}^{\mathbb{V}}_{K_u} \to \mathbf{B}^{\mathbb{V}}_{K_u}$, and any fixed point $\mathbf{U}$ of H provides a solution $(\mathbf{U}, \mathrm{N}(\mathbf{U})) \in \mathbb{V}_0(\mathscr{T}) \times (\mathbb{Z}(\mathscr{T}) + c_d)$ of (A 1), (A 2). Hence, if we can show that $\mathrm{H} : \mathbf{B}^{\mathbb{V}}_{K_u} \to \mathbf{B}^{\mathbb{V}}_{K_u}$ is a continuous operator, then Brouwer's fixed point theorem yields the existence of a fixed point, and, in turn, of a solution of the system of equations (A 1), (A 2).

Let us first establish the continuity of the mapping $\mathrm{N} : \mathbf{B}^{\mathbb{V}}_{K_u} \to (\mathbb{Z}(\mathscr{T}) + c_d)$. For any $\mathbf{U}, \mathbf{V} \in \mathbf{B}^{\mathbb{V}}_{K_u}$, the strong monotonicity (A 3) yields that

$$\begin{aligned} C_{qM} \|\nabla(\mathrm{N}(\mathbf{U}) - \mathrm{N}(\mathbf{V}))\|_2^2 &\leq \int_\Omega (\mathbf{q}_c(\mathrm{N}(\mathbf{U}), \nabla \mathrm{N}(\mathbf{U}), \mathrm{D}\mathbf{U}) - \mathbf{q}_c(\mathrm{N}(\mathbf{V}), \nabla \mathrm{N}(\mathbf{V}), \mathrm{D}\mathbf{U})) \cdot \nabla(\mathrm{N}(\mathbf{U}) - \mathrm{N}(\mathbf{V}))\,\mathrm{d}\boldsymbol{x} \\ &= \int_\Omega (\mathbf{q}_c(\mathrm{N}(\mathbf{U}), \nabla \mathrm{N}(\mathbf{U}), \mathrm{D}\mathbf{U}) - \mathbf{q}_c(\mathrm{N}(\mathbf{V}), \nabla \mathrm{N}(\mathbf{V}), \mathrm{D}\mathbf{V})) \cdot \nabla(\mathrm{N}(\mathbf{U}) - \mathrm{N}(\mathbf{V}))\,\mathrm{d}\boldsymbol{x} \\ &\quad + \int_\Omega (\mathbf{q}_c(\mathrm{N}(\mathbf{V}), \nabla \mathrm{N}(\mathbf{V}), \mathrm{D}\mathbf{V}) - \mathbf{q}_c(\mathrm{N}(\mathbf{V}), \nabla \mathrm{N}(\mathbf{V}), \mathrm{D}\mathbf{U})) \cdot \nabla(\mathrm{N}(\mathbf{U}) - \mathrm{N}(\mathbf{V}))\,\mathrm{d}\boldsymbol{x} \\ &=: \text{(I)} + \text{(II)}. \end{aligned}$$

Since $\mathrm{N}(\mathbf{V})$ and $\mathrm{N}(\mathbf{U})$ are solutions of the convection–diffusion equation (A 2) for fixed velocity vectors $\mathbf{V}$ and $\mathbf{U}$, respectively, we find that

$$\text{(I)} = B_c[\mathrm{N}(\mathbf{V}), \mathbf{V}, \mathrm{N}(\mathbf{U}) - \mathrm{N}(\mathbf{V})] - B_c[\mathrm{N}(\mathbf{U}), \mathbf{U}, \mathrm{N}(\mathbf{U}) - \mathrm{N}(\mathbf{V})] = B_c[\mathrm{N}(\mathbf{V}), \mathbf{V} - \mathbf{U}, \mathrm{N}(\mathbf{U})],$$

where the latter follows from the anti-symmetry of the trilinear form. We recall that, for any $\mathbf{W} \in \mathbf{B}^{\mathbb{V}}_{K_u}$, $\|\mathrm{N}(\mathbf{W})\|_{1,2} \leq K_c$. In turn, $\|\xi_{\mathbf{W}}\|_{1,2} \leq K_c + \|c_d\|_{1,2} =: \widetilde{K}_c$, where $\xi_{\mathbf{W}} = \mathrm{N}(\mathbf{W}) - c_d \in \mathbb{Z}(\mathscr{T})$. Hence, by invoking (22), (23) and the anti-symmetry of the trilinear form, we find that

$$|\text{(I)}| = |B_c[\xi_{\mathbf{V}} + c_d, \mathbf{V} - \mathbf{U}, \xi_{\mathbf{U}} + c_d]| \leq (C_c^2 \widetilde{K}_c^2 + 2C_B \widetilde{K}_c) \|\nabla(\mathbf{U} - \mathbf{V})\|_2,$$

which vanishes for $\mathbf{V} \to \mathbf{U}$. In order to bound the second summand, we first observe that by the Cauchy–Schwarz inequality

$$\begin{aligned} |\text{(II)}| &\leq \left(\int_\Omega |\mathbf{q}_c(\mathrm{N}(\mathbf{V}), \nabla \mathrm{N}(\mathbf{V}), \mathrm{D}\mathbf{V}) - \mathbf{q}_c(\mathrm{N}(\mathbf{V}), \nabla \mathrm{N}(\mathbf{V}), \mathrm{D}\mathbf{U})|^2\,\mathrm{d}\boldsymbol{x} \right)^{1/2} \|\nabla(\mathrm{N}(\mathbf{U}) - \mathrm{N}(\mathbf{V}))\|_2 \\ &\leq 2K_c \left(\int_\Omega |\mathbf{q}_c(\mathrm{N}(\mathbf{V}), \nabla \mathrm{N}(\mathbf{V}), \mathrm{D}\mathbf{V}) - \mathbf{q}_c(\mathrm{N}(\mathbf{V}), \nabla \mathrm{N}(\mathbf{V}), \mathrm{D}\mathbf{U})|^2\,\mathrm{d}\boldsymbol{x} \right)^{1/2}, \end{aligned}$$

where we employed the uniform bound on the mapping $\mathrm{N} : B^{\mathbb{V}}_{K_u} \to (\mathbb{Z}(\mathscr{T}) + c_d)$ in the second inequality. Since the concentration flux is continuous and all of the arguments involved are essentially bounded thanks to the equivalence of norms on finite-dimensional spaces, we find that the integrand in the second inequality above is essentially bounded as well. Moreover, if $\mathbf{V} \to \mathbf{U}$ with respect to the $\|\cdot\|_{1,2}$-norm, then this is also true with respect to the $\|\cdot\|_{1,\infty}$-norm, which, in turn, implies the almost everywhere convergence. Consequently, we can employ the dominated convergence theorem to obtain that the integral vanishes for $\mathbf{V} \to \mathbf{U}$; i.e. $|\text{(II)}| \to 0$ as $\mathbf{V} \to \mathbf{U}$. By combining the above observations we find that $\mathrm{N}(\mathbf{V}) \to \mathrm{N}(\mathbf{U})$ as $\mathbf{V} \to \mathbf{U}$, which is the desired continuity property.

Finally, we will show that $\mathrm{H} := \mathrm{G}(\cdot,\mathrm{N}(\cdot)) : \mathbf{B}^{\mathbb{V}}_{K_u} \to \mathbf{B}^{\mathbb{V}}_{K_u}$ is also continuous. From the strong monotonicity assumption (10) we obtain that

$$\begin{aligned}
C_{SM}\|\mathrm{D}(\mathrm{H}(\mathbf{U})-\mathrm{H}(\mathbf{V}))\|_2^2 &\le \int_\Omega (\mathbf{S}(\mathrm{N}(\mathbf{U}),\mathrm{D}\mathrm{H}(\mathbf{U})) - \mathbf{S}(\mathrm{N}(\mathbf{U}),\mathrm{D}\mathrm{H}(\mathbf{V}))) : \mathrm{D}(\mathrm{H}(\mathbf{U})-\mathrm{H}(\mathbf{V}))\,\mathrm{d}\boldsymbol{x}\\
&= \int_\Omega (\mathbf{S}(\mathrm{N}(\mathbf{U}),\mathrm{D}\mathrm{H}(\mathbf{U})) - \mathbf{S}(\mathrm{N}(\mathbf{V}),\mathrm{D}\mathrm{H}(\mathbf{V}))) : \mathrm{D}(\mathrm{H}(\mathbf{U})-\mathrm{H}(\mathbf{V}))\,\mathrm{d}\boldsymbol{x}\\
&\quad + \int_\Omega (\mathbf{S}(\mathrm{N}(\mathbf{V}),\mathrm{D}\mathrm{H}(\mathbf{V})) - \mathbf{S}(\mathrm{N}(\mathbf{U}),\mathrm{D}\mathrm{H}(\mathbf{V}))) : \mathrm{D}(\mathrm{H}(\mathbf{U})-\mathrm{H}(\mathbf{V}))\,\mathrm{d}\boldsymbol{x}\\
&=: (\mathrm{I}) + (\mathrm{II}).
\end{aligned}$$

We can show that $|(\mathrm{II})| \to 0$ for $\mathbf{V}\to\mathbf{U}$ in an analogous way as before by using the continuity of $\mathbf{S}$ and N. In order to show that (I) vanishes for $\mathbf{V}\to\mathbf{U}$, we first exploit that $\mathrm{H}(\mathbf{U})$ is a solution of (A 4) for $\widehat{C} = \mathrm{N}(\mathbf{U})$ and $\widehat{\mathbf{U}} = \mathbf{U}$, which implies that

$$(\mathrm{I}) = -B_u[\mathbf{U},\mathrm{H}(\mathbf{U}),\mathrm{H}(\mathbf{U})-\mathrm{H}(\mathbf{V})] + B_u[\mathbf{V},\mathrm{H}(\mathbf{V}),\mathrm{H}(\mathbf{U})-\mathrm{H}(\mathbf{V})] = B_u[\mathbf{U}-\mathbf{V},\mathrm{H}(\mathbf{U}),\mathrm{H}(\mathbf{V})].$$

Then, by Lemma 4 and the uniform boundedness of the operator H, this leads to $|(\mathrm{I})| \to 0$ for $\mathbf{V}\to\mathbf{U}$. Combining those observations proves the continuity of the operator $\mathrm{H} : \mathbf{B}^{\mathbb{V}}_{K_u} \to \mathbf{B}^{\mathbb{V}}_{K_u}$, which concludes the proof.

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