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More than a half century of Computational Fluid Dynamics / *Plus d'un demi-siècle de mécanique des fluides numérique*

# Stratified radiative transfer for multidimensional fluids

## *Transfert radiatif stratifié couplé aux equations de la mécanique des fluides*

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**Abstract.** New mathematical and numerical results are given for the coupling of the temperature equation of a fluid with radiative transfer: existence and uniqueness and a convergent monotone numerical scheme. The technique is shown to be feasible for studying the temperature of the Lake Leman heated by the sun and for studying the effects of greenhouse gases on earth's atmosphere.

**Résumé.** Un nouveau résultat d'existence et d'unicité est donné pour le système formé par les équations du transfert radiatif couplées à l'équation de la température d'un fluide. Une méthode numérique convergente et monotone en découle. La technique est appliquée au calcul de la température du lac Leman ainsi qu'à la température de l'atmosphère terrestre pour étudier l'influence des gaz à effet de serre.

**Keywords.** Radiative transfer, Navier–Stokes equations, Integral equations, Numerical method, Convergence, Climate.

**Mots-clés.** Transfert radiatif, Équations de Navier–Stokes, Équations intégrales, Méthode numérique, Convergence, Climat.

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## 1. Introduction

Fifty years ago, the second author was admitted to the prestigious Department of Applied Mathematics and Theoretical Physics at Cambridge, UK, headed then by Sir James Lighthill. Two IBM card punchers connected to the computing center—also one of the best in the world in those days—had been relegated to the basement; to use them was frowned upon as a threat to the speciality of the lab: clever analytic approximations and other multiple scale expansions of special cases of the Navier–Stokes equations.

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It took a decade to prove that computer simulations for fluids were not only possible, but also useful to the industry. A colleague from the wind tunnels in Modane told us then that an airplane could never be designed and validated by a numerical simulation. True to this wrong prediction however, many ad-hoc turbulence models had to be devised, and it was only by a combined theoretical, experimental and computational (TEC) effort that the world's first complete airplane could be simulated at Dassault Aviation in 1979, and airplanes have ever since been flown safely without the difficult certification stamps of wind tunnels.

It was also a success of the top-down approach to CFD. The "JLL" (Lions) School of Applied Mathematics had the luck of being taken seriously by a few French high-tech industry labs. This was not the case in the USA where the head of a national research funding agency had ruled out variational methods (leading to finite volumes and finite elements for fluids) as "incomprehensible by aeronautical engineers", thereafter forcing all numerical schemes to be in the class of body fitted structured meshes, an impossible task for airplanes.

The top-down approach to a problem could be defined by saying that the mathematical model is defined first, then shown to be well posed and then approximated numerically by convergent algorithms. The bottom-up approach is when the problem is made of several modules, studied independently, and patched together at the algorithmic level.

The downside of the top-down approach—from functional analysis to numerical methods—is that it may discard important faster algorithms for which convergence is not known. This was the case for compressible flows in the nineties for which the bottom-up approach pragmatically patched different turbulence and/or numerical models in different zones with the drawback that it was difficult to assert that the computed solution was one of the original problem.

In the numerical simulations which fill the supercomputing centers today, CFD is often only one part of a multi-physics model. Others are the combustion and climate computations. Both need, at least, radiative transfer and chemistry modules.

While the top-down approach is successful in computational chemistry [1], mathematical analysis of climate models is still in progress. The 3D Primitive Equations with hydrostatic and geostrophic approximations have been shown to be well posed (see [2–4] and the bibliography therein) and so are the multi-layered Shallow Water equations for the oceans [5]; but even if the coupled ocean–atmosphere system is mathematically well posed, it is very far from the complete model used in climatology. No doubt when a new numerical climate project is proposed, such as [6], a top-down approach is made [7], but soon overwhelmed by the complexity of the task when more modules are added.

Radiative transfer—one such module that needs to be added—is essential in astrophysics [8] to derive the composition of stars, in nuclear engineering to predict plasma [9], in combustion for engines [10], and many other fields like solar panels [11] and even T-shirts [11]!

In the eighties, at the Centre de l'Energie Atomique, Dautray [9] headed a team of applied mathematicians who used the top-down approach in nuclear engineering. The first author was in close contact with them. But turning his expertise on radiative transfer to climate modeling is not straightforward.

Books on radiative transfer for the atmosphere are numerous, such as [12, 13] and [14]; but to speed up codes, the documentation manual of climate models reveal that many approximations are made. For instance LMDZ refers to a model proposed by Fouquart [15, 16] which suggests that empirical formulas are used in addition to simplified numerical schemes to speed up the computations. The formulas for the absorption, scattering and albedo coefficients are complex and adapted to reproduce the experimental data. In other words the gap is wide between practice and fundamentals as seen in Fowler [17] and Chandrasekhar [8], for instance.

Coupling radiative transfer to the Navier–Stokes system using the top-down approach is the topic of this article. The problem is shown to be well posed in the context of a stratified atmosphere and a numerical method—derived from the mathematical proof of well posedness—is proposed. It is accurate in the sense that there are no singular functions or integrals to approximate. It is fast compared to the fluid solver to which it is coupled, but of course not as fast as the empirical formulas.

## 2. Radiative transfer and the temperature equation

Let us begin with a simple problem: the effect of sunlight on a lake $\Omega$. Let $I_\nu(\mathbf{x},\omega,t)$ be the light intensity of frequency $\nu$ at $\mathbf{x}\in\Omega$, in the direction $\omega\in\mathbb{S}^2$, the unit sphere, at time $t\in(0,T)$. Let $T,\rho,\mathbf{u}$ be the temperature, density and velocity in the lake. Energy, momentum and mass conservations (see [17, 18]) yield (1), (2), (3):

### 2.1. *The fundamental equations*

Given $I_\nu, T$ at time zero, find $I_\nu, T$ for all $\{\mathbf{x},\boldsymbol{\omega},t,\nu\}\in\Omega\times\mathbb{S}_2\times(0,T)\times\mathbb{R}^+$ such that

$$\frac{1}{c}\partial_t I_\nu+\boldsymbol{\omega}\cdot\nabla I_\nu+\rho\bar{\kappa}_\nu a_\nu\left[I_\nu-\frac{1}{4\pi}\int_{\mathbb{S}^2}p(\boldsymbol{\omega},\boldsymbol{\omega}')I_\nu(\boldsymbol{\omega}')\,\mathrm{d}\omega'\right]=\rho\bar{\kappa}_\nu(1-a_\nu)[B_\nu(T)-I_\nu],\tag{1}$$

$$\partial_t T+\mathbf{u}\cdot\nabla T-\kappa_T\Delta T=-\nabla\cdot\int_0^\infty\int_{\mathbb{S}^2}I_\nu(\boldsymbol{\omega}')\boldsymbol{\omega}\,\mathrm{d}\omega\,\mathrm{d}\nu.\tag{2}$$

$$\partial_t\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}-\frac{\mu_F}{\rho}\Delta\mathbf{u}+\frac{1}{\rho}\nabla p=\mathbf{g},\quad\nabla\cdot\mathbf{u}=0,\quad\partial_t\rho+\nabla\cdot(\rho\mathbf{u})=0,\tag{3}$$

where $\nabla,\Delta$ are with respect to $\mathbf{x}$, $B_\nu(T)=2\hbar\nu^3/c^2[\mathrm{e}^{\hbar\nu/kT}-1]$, is the Planck function, $\hbar$ is the Planck constant, $c$ is the speed of light in the medium and $k$ is the Boltzmann constant. The absorption coefficient $\kappa_\nu:=\rho\bar{\kappa}_\nu$ is the percentage of light absorbed per unit length, $a_\nu\in(0,1)$ is the scattering albedo, $1/4\pi p(\boldsymbol{\omega},\boldsymbol{\omega}')$ is the probability that a ray in the direction $\boldsymbol{\omega}'$ scatters in the direction $\boldsymbol{\omega}$. The constants $\kappa_T$ and $\mu_F$ are the thermal and molecular diffusions; $\mathbf{g}$ is the gravity.

Existence of solution for (3) has been established by Lions [19].

As $c\gg1$, in a regime where $(1/c)\partial_t I_\nu\ll1$, integrating (1) in $\boldsymbol{\omega}$ leads to an alternative form for (2):

$$\partial_t T+\mathbf{u}\cdot\nabla T-\kappa_T\Delta T=-\int_0^\infty\rho\bar{\kappa}_\nu(1-a_\nu)\left(4\pi B_\nu(T)-\int_{\mathbb{S}^2}I_\nu(\boldsymbol{\omega})\,\mathrm{d}\omega\right)\mathrm{d}\nu.\tag{4}$$

As usual, boundary conditions must be given. Dirichlet or Neumann conditions may be prescribed for $\mathbf{u}$ and $T$ on $\partial\Omega$. For the light intensity equation, $I_\nu$ should be given at all times on $\{(\mathbf{x},\boldsymbol{\omega})\in\partial\Omega\times\mathbb{S}^2:\mathbf{n}(\mathbf{x})\cdot\boldsymbol{\omega}<0\}$, where $\mathbf{n}$ is the outer unit normal of $\partial\Omega$. Finally $\rho$ should be specified on $\partial\Omega$ when $\mathbf{u}\cdot\mathbf{n}<0$.

### 2.2. *Grey medium*

When $\kappa_\nu$ and $a_\nu$ are independent of $\nu$—a so-called *grey medium* (cf. [17], p. 70)—the problem can be written in terms of $I=\int_0^\infty I_\nu\,\mathrm{d}\nu$:

$$\boldsymbol{\omega}\cdot\nabla I+\kappa a\left[I-\frac{1}{4\pi}\int_{\mathbb{S}^2}p(\boldsymbol{\omega},\boldsymbol{\omega}')I(\boldsymbol{\omega}')\,\mathrm{d}\omega'\right]=\kappa(1-a)(B_0T^4-I),\tag{5}$$

$$\partial_t T+\mathbf{u}\cdot\nabla T-\kappa_T\Delta T=-\kappa(1-a)4\pi\left(B_0T^4-\frac{1}{4\pi}\int_{\mathbb{S}^2}I(\boldsymbol{\omega})\,\mathrm{d}\omega\right),\tag{6}$$

where $B_0$ comes from the Boltzmann–Stefan law:

$$\int_0^\infty\frac{2\hbar\nu^3}{c^2[\mathrm{e}^{\frac{\hbar\nu}{kT}}-1]}\mathrm{d}\nu=\left(\frac{\hbar}{kT}\right)^{-4}\frac{2\hbar}{c^2}\int_0^\infty\frac{\left(\frac{\hbar\nu}{kT}\right)^3}{\mathrm{e}^{\frac{\hbar\nu}{kT}}-1}\,\mathrm{d}\frac{\hbar\nu}{kT}=B_0T^4\quad\text{with }B_0:=\frac{2k^4}{\hbar^3c^2}\frac{\pi^4}{15}.$$

## 2.3. *Vertically stratified cases: spatial invariance*

Let $(x, y, z)$ be a Cartesian frame with $z$ the altitude/depth. The sun being very far, the light source on the lake is independent of $x$ and $y$. Then, assuming that $T'$ varies slowly with $x$ and $y$, in the sense that

$$(H) \quad \partial_z I_\nu \gg \partial_x I_\nu, \quad \partial_z I_\nu \gg \partial_y I_\nu, \tag{7}$$

then (1),(2) become [14]

$$\mu\partial_z I_\nu + \kappa_\nu I_\nu = \kappa_\nu(1-a_\nu)B_\nu(T) + \frac{\kappa_\nu a_\nu}{2}\int_{-1}^{1} p(\mu,\mu')I_\nu(z,\mu')\,\mathrm{d}\mu' \tag{8}$$

$$I_\nu(z_M,\mu)|_{\mu<0} = Q^-(\mu)B_\nu(\bar{T}_S),\ I(z_m,\mu)|_{\mu>0} = 0, \tag{9}$$

$$\partial_t T + \mathbf{u}\cdot\nabla T - \kappa_T\Delta T = -4\pi\int_0^\infty \kappa_\nu(1-a_\nu)\left(B_\nu(T) - \frac{1}{2}\int_{-1}^{1} I_\nu\,\mathrm{d}\mu\right)\mathrm{d}\nu, \quad \partial_n T|_{\partial\Omega} = 0, \tag{10}$$

where $z_M(x,y)$ and $z_m(x,y)$ are max and min of $z$ such that $(x,y,z)\in\Omega$, $\mu$ is the cosine of the angle $\boldsymbol{\omega}$ to the vertical axis, $Q^-(\mu) = -\mu Q'\cos\theta$ is the sunlight intensity when $\theta$ is the latitude, and $\bar{T}_S$ is the temperature of the sun; we have assumed that the sun is a black body and that no light comes back from the bottom of the lake. Here $\mathbf{u}$ is given, solenoidal and regular enough for (10) to make sense.

**Remarks 1.**

- Hypothesis (H) will hold if $T$ varies slowly with $x, y$. It will be so if $\mathbf{u}$ is almost horizontal and the vertical cross-sections of $\Omega$ depend slowly on $x, y$. Turbulent flows do not satisfy this criteria.
- According to our definition of top-down analysis, the problem investigated is (8),(9),(10), not (1),(2),(3), justifying the restriction “stratified” in the title.
- All terms of (10) must be kept, except maybe, $\kappa_T\partial_{xx}T$ and $\kappa_T\partial_{yy}T$, but neglecting them renders the boundary conditions mathematically difficult.
- We shall ignore the mathematical difficulty induced by the boundary condition $\partial_n T|_{\partial\Omega} = 0$ when the intersection of the side of the lake with the water surface is not at right angle.

## 2.4. *The vertically stratified grey problem*

For a grey medium (8),(10) become

$$(P^1)\begin{cases} \mu\partial_z I + \kappa I = \kappa(1-a)B_0T^4 + \dfrac{\kappa a}{2}\displaystyle\int_{-1}^{1} pI\,\mathrm{d}\mu', \quad I|_{z_M,\mu<0} = -\mu Q B_0\bar{T}_S^4,\ I|_{z_m,\mu>0} = 0,\\ \partial_t T + \mathbf{u}\cdot\nabla T - \kappa_T\Delta T = -4\pi\kappa(1-a)\left(B_0T^4 - \dfrac{1}{2}\displaystyle\int_{-1}^{1} I\,\mathrm{d}\mu\right), \quad \partial_n T|_{\partial\Omega} = 0. \end{cases} \tag{11}$$

## 2.5. *Elimination of $I$ when the scattering is isotropic*

Denote the exponential integral and the mean light intensity respectively by

$$E_m(x) := \int_0^1 \mu^{m-2}\mathrm{e}^{-\frac{x}{\mu}}\,\mathrm{d}\mu, \quad J(z) := \frac{1}{2}\int_{-1}^{1} I(z,\mu)\,\mathrm{d}\mu.$$

Then the method of characteristics applied to (11) gives

$$(P^2)\begin{cases} J(z) = \dfrac{1}{2}QB_0\bar{T}_S^4E_3(\kappa(z_M-z)) + \dfrac{1}{2}\displaystyle\int_{z_m}^{z_M}\kappa E_1(\kappa|s-z|)\left((1-a)B_0T_s^4 + aJ(s)\right)\mathrm{d}s,\\ \partial_t T + \mathbf{u}\cdot\nabla T - \kappa_T\Delta T = -4\pi\kappa(1-a)\left(B_0T^4(z) - J(z)\right). \end{cases} \tag{12}$$

Note that to improve readability, we write indifferently $T(z)$ or $T_z$.

## 2.6. *No scattering*

Let $T_e(z) = ((1/2)QE_3(\kappa|z_M - z|))^{\frac{1}{4}}\ \bar{T}_S$ and assume that $a = 0$, then

$$(P^3)\left\{(4\pi\kappa B_0)^{-1}(\partial_t T + \mathbf{u}\cdot\nabla T - \kappa_T \Delta T) + T^4 = T_e^4 + \frac{1}{2}\int_{z_m}^{z_M} \kappa E_1(\kappa|s-z|)T_s^4\,\mathrm{d}s, \quad \partial_n T|_{\partial\Omega} = 0.\right. \tag{13}$$

## 2.7. *Algorithm for $(P^3)$ in the stationary static case*

Assume $T$ stationary and $\mathbf{u} = 0$. Let $\bar{\kappa}_T = (4\pi\kappa B_0)^{-1}\kappa_T$.

Generate $\{T^n\}_{n\ge 0}$ from $T^0 = 0$ by,

$$\left|\begin{array}{l} \left(T^{n+\frac{1}{2}}\right)^4 := T_e^4 + \frac{1}{2}\int_{z_m}^{z_M} \kappa E_1(\kappa|s-z|)T_s^{n4}\,\mathrm{d}s, \quad T^{n+\frac{1}{2}} \ge 0 \\ -\bar{\kappa}_T \Delta T^{n+1} + (T_+^{n+1})^4 = \left(T^{n+\frac{1}{2}}\right)^4, \quad \partial_n T^{n+1}|_{\partial\Omega} = 0, \end{array}\right. \tag{14}$$

where $T_+ = \max(T, 0)$. Note that $T \mapsto -\bar{\kappa}_T \Delta T + T_+^4$ is a monotone operator for which Newton or fixed point iterations can be applied to solve the PDE. To prove monotone convergence, the following result is needed.

**Lemma 1.** $C_1(\kappa) := (1/2)\max_z \int_0^Z \kappa E_1(\kappa|s-z|)\,\mathrm{d}s < 1.$

**Proof.**

$$\int_0^X E_1(x)\,\mathrm{d}x = \int_1^\infty \int_0^X \frac{\mathrm{e}^{-xt}}{t}\,\mathrm{d}x\,\mathrm{d}t = \int_1^\infty \frac{1-\mathrm{e}^{-Xt}}{t^2}\,\mathrm{d}t < \int_1^\infty \frac{1}{t^2}\,\mathrm{d}t = 1.$$

$$\Rightarrow \kappa\int_0^Z E_1(\kappa|\tau - t|)\,\mathrm{d}t = \int_0^{\kappa Z} E_1(|s-\kappa\tau|)\,\mathrm{d}s = \int_0^{\kappa\tau} E_1(\kappa\tau - s)\,\mathrm{d}s + \int_{\kappa\tau}^{\kappa Z} E_1(s-\kappa\tau)\,\mathrm{d}s$$

$$= \int_0^{\kappa\tau} E_1(\theta)\,\mathrm{d}\theta + \int_0^{\kappa(Z-\tau)} E_1(\theta)\,\mathrm{d}\theta < 2. \tag{15}$$

□

**Theorem 1.** $\{T^n\}_{n\ge 0}$ *generated by Algorithm (14) converges to a solution of (13) and the convergence is monotone:* $T^{n+1}(\mathbf{x}) > T^n(\mathbf{x})$ *for all* $\mathbf{x}$ *and all* $n$.

**Proof.** From (14)

$$(T^{n+\frac{1}{2}})^4 \le |T_e^4|_\infty + C_1(\kappa)|T^n|_\infty^4.$$

By the maximum principle for the PDE in (14), $T^{n+1} \ge 0$ and $|T^{n+1}|_\infty \le |T^{n+\frac{1}{2}}|_\infty$, therefore

$$|T^{n+1}|^4 \le |T_e|_\infty^4 + C_1(\kappa)|T^n|_\infty^4.$$

Hence $|T^{n+1}|_\infty$ is bounded. Assume that $T^n \ge T^{n-1}$. The convergence is monotone because

$$(T^{n+\frac{1}{2}})^4 - (T^{n-\frac{1}{2}})^4 = \frac{1}{2}\int_{z_m}^{z_M} \kappa E_1(\kappa|s-z|)\left[(T_s^n)^4 - (T_s^{n-1})^4\right] \ge 0,$$

and as

$$-\bar{\kappa}_T \Delta(T^{n+1} - T^n) + b(T^{n+1} - T^n) = (T^{n+\frac{1}{2}})^4 - (T^{n-\frac{1}{2}})^4 \tag{16}$$

with $b = ((T^{n+1})^2 + (T^n)^2)(T^{n+1} + T^n) \ge 0$, the maximum principle implies that $T^{n+1} - T^n \ge 0$. □

**Remark 1.** Generalization of the above result to $(P^3)$ is straightforward because the maximum principle holds also for the temperature equation with convection. Consequently it seems doable to extend the above to the system (2), (3). When the density variations with the temperature are small the Boussineq approximation can be used in conjunction with (13):

$$(P^4)\begin{cases}(4\pi\kappa B_0)^{-1}(\partial_t T+\mathbf{u}\cdot\nabla T-\kappa_T\Delta T)+T^4=T_e^4+\dfrac{1}{2}\displaystyle\int_{z_m}^{z_M}\kappa E_1(\kappa|s-z|)T_s^4\,\mathrm{d}s,\\ \partial_t\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}-\nu_F\Delta\mathbf{u}+\nabla p=-b(T-T_0)\mathbf{g},\quad \nabla\cdot\mathbf{u}=0,\end{cases}\tag{17}$$

with $\mathbf{u}$, $T$ given at $t=0$ and $\mathbf{u}$ or $\partial_n\mathbf{u}$ or $p\mathbf{n}+\nu_T\partial_{\mathbf{n}}\mathbf{u}$ and $\partial_n T=0$ or $T$ given on $\partial\Omega$. The kinematic viscosity $\nu_F=\mu_F/\rho$ is taken constant; $b$ is a measure of $\partial_T\rho$ and $T_0$ is the average temperature. See [20], for instance, for the mathematical analysis of the Boussinesq–Stefan problem (similar to $(P^4)$ without the $T^4$ terms).

## 3. Numerical tests

The physical constants are given in Table 1. Earth sees the sun as a black body at temperature $\bar{T}_S=5800$ K radiating with an intensity $Q'=1370\ \mathrm{W/m^2}$ of which 70% reach the ground, giving at noon in Milano $Q=1370\times0.7\cos(\pi/4)=678$.

For water $\rho=1000\ \mathrm{kg/m^3}$; light absorption is $\kappa=0.1$ for one meter and thermal diffusivity of water is $\kappa_T=1.5\times10^{-7}\ \mathrm{m^2/s}$ giving $\bar{\kappa}_T=0.66\times10^{11}$.

To avoid those large numbers we scale $T$ by $10^{-3}$. Then $\bar{T}_S=5.8$, $(Q/2)^{1/4}\bar{T}_S=24.9$, $\tilde{\kappa}_T=10^{-9}\bar{\kappa}_T=66$.

### 3.1. *A 1D test*

If $\Omega=(0,10)$, we need to solve with Algorithm (14) the integro-differential equation in $z$:

$$-66T''+T^4=12.5E_3(0.1|10-z|)+0.05\int_0^{10}E_1(0.1|s-z|)T^4(s)\,\mathrm{d}s,\quad T(0)=(12.5E_3(0))^{\frac{1}{4}},\ T'(10)=0.\tag{18}$$

To solve $-66T''+T^4=f$, three iterations of a fixed point loop are used: $-830T''^{m+1}+T^{m3}T^{m+1}=f$.

The results are shown in Figure 1. The convergence is monotone as expected, even though Theorem 1 has not been proved when a Dirichlet condition is applied to $T$ on part of $\partial\Omega$. Notice that in the absence of sunlight the temperature would be $T(0)$ everywhere.

### 3.2. *A 2D test for a lake*

Now $\Omega$ is half of the vertical cross-section of a symmetric lake. The lower right quarter side of the unit circle is stretched by $x,z\mapsto 30x,10z$. The bottom boundary has an equation named $z=z_m(x)$. The same problem is solved in 2D:

$$-66\Delta T+T^4=12.5E_3(0.1|z_m(x)-z|)+0.05\int_{z_m(x)}^{10}E_1(0.1|s-z|)T^4(s)\,\mathrm{d}s,$$
$$T(x,z_m(x))=(12.5E_3(0.1z_m(x)))^{\frac{1}{4}},\ \partial_z T(10)=0.\tag{19}$$

The same $3\times10$ double iteration loop is used; the results are shown in Figure 2.

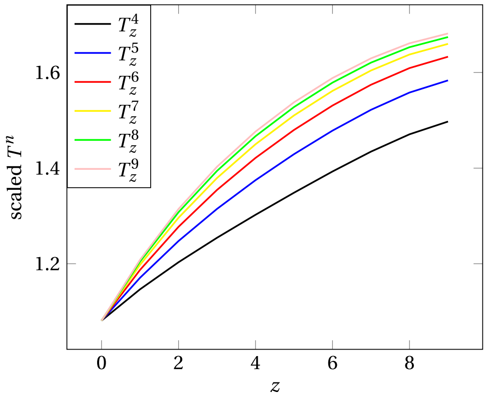


**Figure 1.** Convergence of $T^n$ solution of (18).

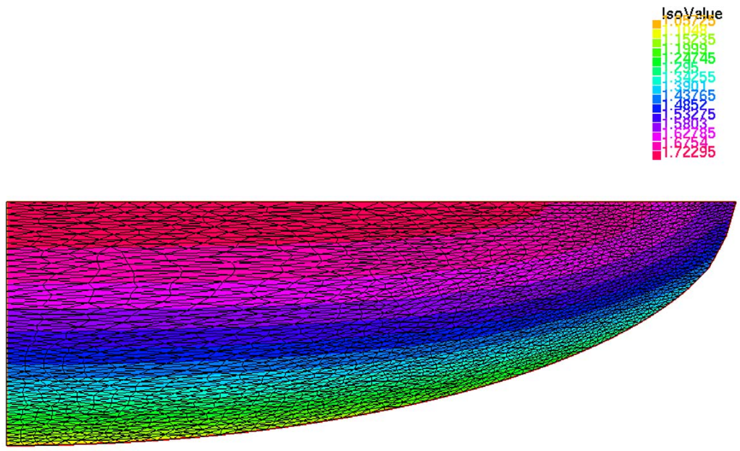


**Figure 2.** Color map of $T(x,z)$ at iteration 10. The triangulation is also shown, adapted from $T$ computed at iteration 5.

**Table 1.** The physical constants

| $c$ | $\hbar$ | $k$ | $B_0$ |
|---|---|---|---|
| $2.998\times10^8$ | $6.6261\times10^{-34}$ | $1.381\times10^{-23}$ | $1.806657\times10^{-19}$ |

### 3.3. *A 3D case with convection in Lake Leman*

Lake Leman is discretized into 33,810 tetrahedra. The surface has 1287 triangles. The Finite Element method of degree 1 is used. This is too coarse for a Navier–Stokes simulation but appropriate for a potential flow. Pressure is imposed on the left and right tips to simulate the depth of the Rhône. The pressure $p$ solves $-\Delta p = 0$ with $\partial_n p = 0$ on the remaining boundaries; the velocity is $\mathbf{u} = \nabla p$. The top plot in Figure 3 shows $p$ and $\mathbf{u}$.

The full temperature equation of Problem $(P^4)$ is solved with the same physical constant as above. The temperature is set at $T_e$ initially and on the bottom and side boundaries of the lake. The time step is $t = 0.1$; the method is fully implicit for the temperature. At each time step three

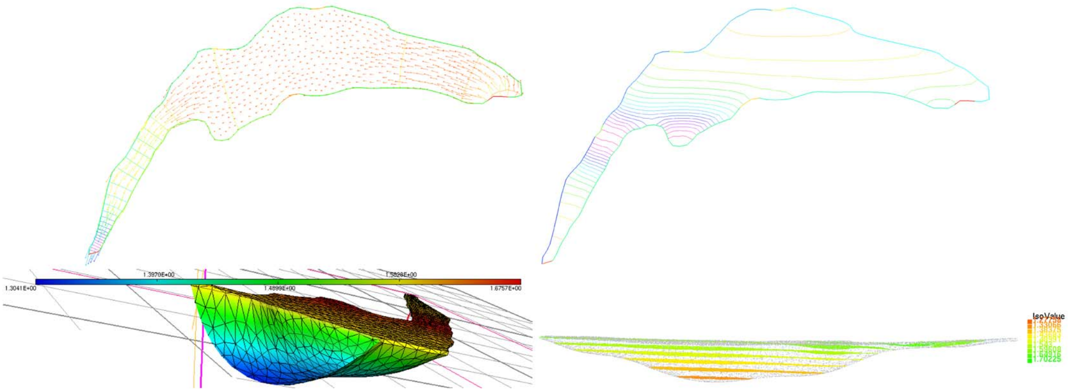

**Figure 3.** Top left: velocity vectors and pressure isolines at the surface of the lake. Top right: isolines of the surface temperature. Bottom left: perspective view of a 3D color map of the temperature on the side of the lake past a middle vertical plane. Bottom right: perspective view showing some temperature level surfaces inside the lake.

iterations are needed to handle the $T^4$ term. Figure 3 shows the temperature after 15 time steps; it appears to have reached a steady state. The top right view of Figure 3 shows a region in red where the water at the surface is the hottest.

This computation is merely a feasibility study to prove that the implementation of the RT module in a standard CFD code is easy and fast. Computing time on an Intel core i9 takes less than a minute.

## 4. The general case, $\kappa_\nu, a_\nu$ non constant

Photons interact with the atomic structure of the medium which implies that $\kappa_\nu$ depends strongly on $\nu$ but also on the temperature and pressure. For the earth's atmosphere, pressure and temperature are approximately decaying exponentially with altitude.

Assume that variations with altitude are known: $\rho\bar{\kappa}_\nu = \varphi(z)\kappa_\nu$ with $z = z_m = 0$ on the ground. Let $\tau = \int_0^z \varphi(s)\,\mathrm{d}s$; for instance $\tau = 1 - \mathrm{e}^{-z}$ when $\varphi(z) = \mathrm{e}^{-z}$. Now (8),(10) hold with $0 < \tau < Z := 1 - \mathrm{e}^{-z_M}$ instead of $z_m < z < z_M$.

Consider two types of scattering kernels: a Rayleigh scattering kernel $p^r(\mu,\mu') = (3/8)[3 - \mu^2 + 3(\mu^2 - 1)\mu'^2]$ and an isotropic scattering kernel $p = 1$. Let $a_\nu^r$ and $a_\nu^i := a_\nu - a_\nu^r$ be the scattering coefficients for both. The problem is

$$\mu\partial_\tau I_\nu + \kappa_\nu I_\nu = \kappa_\nu(1 - a_\nu)B_\nu(T) + \frac{1}{2}\kappa_\nu \int_{-1}^{1} (a_\nu^r p^r + a_\nu^i) I_\nu \,\mathrm{d}\mu'$$
$$I(0,\mu)|_{\mu>0} = \alpha I(0,-\mu) + Q_\nu^+(\mu), \quad I(Z,\mu)|_{\mu<0} = Q_\nu^-(\mu). \tag{20}$$

The boundary condition at $\tau = 0$ is a simplified Lambert condition which says that a portion $\alpha$ of the incoming light is reflected back (Earth albedo) and adds to the prescribed upgoing light $Q_\nu^+$. Sun light is prescribed at high altitude, $Z$, to be $Q^-(\mu)$.

Let

$$J_\nu(\tau) = \frac{1}{2}\int_{-1}^{1} I_\nu(\tau,\mu)\,\mathrm{d}\mu, \quad K_\nu(\tau) = \frac{1}{2}\int_{-1}^{1} \mu^2 I_\nu(\tau,\mu)\,\mathrm{d}\mu.$$

An integral formulation can be derived from (20) as in [8], Section 11.2:

$$(\mu\partial_\tau+\kappa_\nu)I_\nu=\kappa_\nu H_\nu(\tau,\mu):=\kappa_\nu\Big((1-a_\nu)B_\nu(T_\tau)+[a_\nu^i+\tfrac{3}{8}a_\nu^r(3-\mu^2)]J_\nu(\tau)+\tfrac{9}{8}a_\nu^r(\mu^2-1)K_\nu(\tau)\Big) \tag{21}$$

$$\Rightarrow I_\nu(\tau,\mu)=\mathbf{1}_{\mu>\mathbf{0}}\left[R_\nu^+(\mu)\mathrm{e}^{-\kappa_\nu\frac{\tau}{\mu}}+\int_0^\tau\frac{\mathrm{e}^{\kappa_\nu\frac{t-\tau}{\mu}}}{\mu}\kappa_\nu H_\nu(t,\mu)\,\mathrm{d}t\right]$$
$$+\mathbf{1}_{\mu<\mathbf{0}}\left[Q_\nu^-(\mu)\mathrm{e}^{\kappa_\nu\frac{Z-\tau}{\mu}}-\int_\tau^Z\frac{\mathrm{e}^{\kappa_\nu\frac{t-\tau}{\mu}}}{\mu}\kappa_\nu H_\nu(t,\mu)\,\mathrm{d}t\right], \tag{22}$$

where $R^+(\mu)=Q_\nu^+(\mu)+\alpha I(0,-\mu)$, i.e.,

$$R_\nu^+(\mu)|_{\mu>0}=Q_\nu^+(\mu)+\alpha\left[Q_\nu^-(-\mu)\mathrm{e}^{-\kappa_\nu\frac{Z}{\mu}}+\int_0^Z\frac{\mathrm{e}^{-\kappa_\nu\frac{t}{\mu}}}{\mu}\kappa_\nu H_\nu(t,-\mu)\,\mathrm{d}t\right]. \tag{23}$$

From (22), since $H_\nu=H_\nu^0+\mu^2H_\nu^2$, with $H_\nu^0,H_\nu^2$ independent of $\mu$, linear functions of $J_\nu$ and $K_\nu$:

$$H_\nu^0(\tau)\;=\;\kappa_\nu(1-a_\nu)B_\nu(T)+\kappa_\nu\left(\left(a_\nu^i+\frac{9a_\nu^r}{8}\right)J_\nu-\frac{9a_\nu^r}{8}K_\nu\right),\quad H_\nu^2(\tau)=-\kappa_\nu\frac{3a_\nu^r}{8}[J_\nu-3K_\nu]. \tag{24}$$

$$J_\nu(\tau)\;=\;\frac{1}{2}\int_0^1\left(\mathrm{e}^{-\kappa_\nu\frac{\tau}{\mu}}Q_\nu^+(\mu)+\left[\mathrm{e}^{-\kappa_\nu\frac{(Z-\tau)}{\mu}}+\alpha\mathrm{e}^{-\kappa_\nu\frac{(Z+\tau)}{\mu}}\right]Q_\nu^-(-\mu)\right)\mathrm{d}\mu$$
$$+\frac{1}{2}\int_0^Z\big([E_1(\kappa_\nu|\tau-t|)+\alpha E_1(\kappa_\nu(\tau+t))]H_\nu^0(\tau)+[E_3(\kappa_\nu|\tau-t|)+\alpha E_3(\kappa_\nu(\tau+t))]H_\nu^2(\tau)\big)\,\mathrm{d}t \tag{25}$$

$$K_\nu(\tau)\;=\;\frac{1}{2}\int_0^1\mu^2\left(\mathrm{e}^{-\kappa_\nu\frac{\tau}{\mu}}Q_\nu^+(\mu)+\left[\mathrm{e}^{-\kappa_\nu\frac{(Z-\tau)}{\mu}}+\alpha\mathrm{e}^{-\kappa_\nu\frac{(Z+\tau)}{\mu}}\right]Q_\nu^-(-\mu)\right)\mathrm{d}\mu$$
$$+\frac{1}{2}\int_0^Z\big([E_3(\kappa_\nu|\tau-t|)+\alpha E_3(\kappa_\nu(\tau+t))]H_\nu^0(\tau)+[E_5(\kappa_\nu|\tau-t|)+\alpha E_5(\kappa_\nu(\tau+t))]H_\nu^2(\tau)\big)\,\mathrm{d}t. \tag{26}$$

The system is coupled to

$$\partial_tT+\mathbf{u}\cdot\nabla T-\kappa_T\Delta_{x,y,z}T+4\pi\int_0^\infty\kappa_\nu(1-a_\nu)B_\nu(T_\tau)\,\mathrm{d}\nu=4\pi\int_0^\infty\kappa_\nu(1-a_\nu)\,J_\nu(\tau)\,\mathrm{d}\nu. \tag{27}$$

## 4.1. *Iterative method for the general case*

In the spirit of (14), consider

## 4.2. *Algorithm 2*

1. Starting from $T^0=0$, $J_\nu^0=0$, $K_\nu^0=0$.
2. Compute $J_\nu^{n+1}(\tau),K_\nu^{n+1}(\tau)$ by (25), (26) with $T^n$, $J^n$, $K^n$ in place of $T,J,K$.
3. Compute $T^{n+1}$ by solving (27) with $J_\nu^{n+1}(\tau)$ in the r.h.s.

Note that for isotropic scattering $K_\nu$ is not needed. Then the following convergence results hold when thermal diffusion is neglected.

**Theorem 2.** *Assume $\alpha=0$, $\mathbf{u}=0$, $\kappa_T=0$, $\partial_tT=0$. Assume $\kappa_\nu$ is strictly positive and uniformly bounded, and $0\le a_\nu<1$ for all $\nu>0$. Let $Q_\nu^\pm\ge0$ satisfy, for some $T_M$ and some $Q$*

$$0\le Q_\nu^\pm(\mu)\le QB_\nu(T_M)\quad\forall\mu,\nu\in(-1,1)\times\mathbb{R}^+. \tag{28}$$

*Then Algorithm 4.2 defines a sequence of radiative intensities $I_\nu^n$ and temperatures $T^n$ converging pointwise to $I_\nu$ and $T$ respectively, which is a solution of (20),(27) and the convergence is uniformly increasing.*

**Remarks 2.**

(1) Starting with $T^0 = 0$ is a sure way to initialise the recurrence and have $T^1 > T^0$.

(2) Most likely, monotone convergence holds also in the general case $\alpha > 0$, $\mathbf{u}$, $\kappa_T$ and $\partial_t T$ non-zero because, just like $T \mapsto T^4$, the function $T \mapsto \int_0^\infty \kappa_\nu (1-a_\nu) B_\nu(T)\,\mathrm{d}\nu$ is monotone increasing (its derivative is strictly positive).

(3) In the special case $a_\nu^r = 0$, and $Q_\nu^\pm(\mu) = |\mu| Q_\nu^\pm$ the problem is

$$
\begin{aligned}
&(\mu\partial_\tau + \kappa_\nu) I_\nu(\tau,\mu) = \kappa_\nu a_\nu J_\nu(\tau) + \kappa_\nu (1-a_\nu) B_\nu(T_\tau), \quad J_\nu(\tau) = \frac{1}{2}\int_{-1}^{1} I_\nu(\tau,\mu)\,\mathrm{d}\mu,\\
&I_\nu(0,\mu) = Q_\nu^+ \mu, \quad I_\nu(Z,-\mu) = Q_\nu^- \mu, \qquad 0 < \mu < 1,\\
&\int_0^\infty \kappa_\nu (1-a_\nu) B_\nu(T_\tau)\,\mathrm{d}\nu = \int_0^\infty \kappa_\nu (1-a_\nu) J_\nu(\tau)\,\mathrm{d}\nu.
\end{aligned}
\tag{29}
$$

The iterative process is then to start with $T^0 = 0$, and compute $T^{n+1}$ from $T^n$ by

$$
\begin{aligned}
J_\nu^{n+1}(\tau) = \tfrac{1}{2} Q_\nu^+ E_3(\kappa_\nu \tau) + \tfrac{1}{2} Q_\nu^- E_3(\kappa_\nu (Z-\tau))&\\
+ \kappa_\nu \int_0^Z E_1(\kappa_\nu |\tau - t|) \left( a_\nu J_\nu^n(t) + (1-a_\nu) B_\nu(T_t^n) \right) \mathrm{d}t&,
\end{aligned}
\tag{30}
$$

$$
\int_0^\infty \kappa_\nu (1-a_\nu) B_\nu(T_\tau^{n+1})\,\mathrm{d}\nu = \int_0^\infty \kappa_\nu (1-a_\nu) J_\nu^{n+1}(\tau)\,\mathrm{d}\nu. \tag{31}
$$

(4) Note that $T \mapsto \int_0^\infty \kappa_\nu (1-a_\nu) B_\nu(T)\,\mathrm{d}\nu$ is continuous, strictly increasing, hence invertible. Thus (31) defines $T_\tau^{n+1}$ uniquely.

(5) One may recover the light intensity by

$$
\begin{aligned}
I_\nu^{n+1}(\tau,\mu) \;=\; &\mathrm{e}^{-\kappa_\nu \frac{\tau}{\mu}} Q_\nu^+(\mu) \mathbf{1}_{\mu>0} + \mathrm{e}^{-\kappa_\nu \frac{(Z-\tau)}{|\mu|}} Q_\nu^-(\mu) \mathbf{1}_{\mu<0}\\
&+ \mathbf{1}_{\mu>0} \int_0^\tau \mathrm{e}^{-\kappa_\nu \frac{(\tau-t)}{\mu}} \frac{\kappa_\nu}{\mu} (a_\nu J_\nu^n(t) + (1-a_\nu) B_\nu(T_t^n))\,\mathrm{d}t\\
&+ \mathbf{1}_{\mu<0} \int_\tau^Z \mathrm{e}^{-\kappa_\nu \frac{(t-\tau)}{\mu}} \frac{\kappa_\nu}{\mu} (a_\nu J_\nu^n(t) + (1-a_\nu) B_\nu(T_t^n))\,\mathrm{d}t,
\end{aligned}
\tag{32}
$$

but numerically these are singular integrals while (30),(31) are not. Indeed $\mathrm{e}^{-x/\mu}/\mu$ tends to infinity when $x$ and $\mu$ tend to 0.

(6) Theorem 2 extends a result given in [21] which had unnecessary restrictions on $\kappa_\nu$.

**Proof.** The complete proof will appear in [22]. Here, for simplicity, we consider the case $a_\nu = 0$. Let

$$
S(\tau) := \int_0^\infty \frac{\kappa_\nu}{2} \int_0^1 \left( \mathrm{e}^{-\kappa_\nu \frac{\tau}{\mu}} Q_\nu^+(\mu) + \mathrm{e}^{-\kappa_\nu \frac{Z-\tau}{\mu}} Q_\nu^-(-\mu) \right) \mathrm{d}\mu\,\mathrm{d}\nu.
$$

By (30)

$$
\begin{aligned}
\int_0^\infty \kappa_\nu B_\nu(T_\tau^{n+1})\,\mathrm{d}\nu \;&=\; \int_0^\infty \kappa_\nu J_\nu^{n+1}(\tau)\,\mathrm{d}\nu = S(\tau) + \frac{1}{2}\int_0^\infty \int_0^Z \kappa_\nu^2 E_1(\kappa_\nu |\tau - t|) B_\nu(T_t^n)\,\mathrm{d}t\,\mathrm{d}\nu\\
&\le\; S(\tau) + \frac{1}{2}\max_\kappa \int_0^Z \kappa E_1(\kappa|\tau - t|)\,\mathrm{d}t \sup_{t\in(0,Z)} \int_0^\infty \kappa_\nu B_\nu(T_t^n)\,\mathrm{d}\nu\\
&\le\; C_2 + C_1(\kappa_M) \sup_{t\in(0,Z)} \int_0^\infty \kappa_\nu B_\nu(T_t^n)\,\mathrm{d}\nu,
\end{aligned}
$$

with $C_2 = \sup_{t\in(0,Z)} S(t)$ and $\kappa_M = \sup_\nu \kappa_\nu$, because $\kappa \mapsto C_1(\kappa)$ is monotone increasing. As $C_1(\kappa_M) < 1$ it implies that $B_\nu^n(\tau) := B_\nu(T^n(\tau))$ is bounded for all $\tau$.

Now assume that $T_\tau^n > T_\tau^{n-1}$ for all $\tau > 0$. Then $T \mapsto B_\nu(T)$ being increasing, $B_\nu(T_\tau^n) > B_\nu(T_\tau^{n-1})$, $\forall \tau, \nu$, and so for all $\tau$:

$$\int_0^\infty \kappa_\nu \left(B_\nu(T_\tau^{n+1}) - B_\nu(T_\tau^n)\right) d\nu = \int_0^\infty \kappa_\nu \left(J_\nu^{n+1}(\tau) - J_\nu^n(\tau)\right) d\nu$$
$$= \int_0^\infty \frac{\kappa_\nu^2}{2} \int_0^Z E_1(\kappa_\nu|\tau - t|)\left(B_\nu(T_t^n) - B_\nu(T_t^{n-1})\right) dt\, d\nu > 0. \quad (33)$$

As $T \mapsto B_\nu(T)$ is continuous, it implies that $T_\tau^{n+1} > T_\tau^n$, $\forall \tau$. Hence for some $T^*(\tau)$, possibly $+\infty$, $T^n \to T^*$. By continuity $B_\nu(T_t^n) \to B_\nu(T_t^*)$, but it has been show above that $B_\nu(T_t^n) = B_\nu^n \to B_\nu^*$, so $B_\nu(T_t^*)$ is finite and so is $T_t^*$. Recall that a bounded increasing sequence converges, so $B_\nu(T_t^n) \to B_\nu(T_t^*)$ for all $t$ and $\nu$ and the convergence of $E_1(\kappa_\nu|\tau - t|)B_\nu(T_t^n) \to E_1(\kappa_\nu|\tau - t|)B_\nu(T_t^*)$ being monotone, the integral converges to the integral of the limit (Beppo Levi's lemma). This shows that $T_\tau^*$ is the solution of the problem. □

**Remark 2.** Note that (33) also shows that if $T^n < T^{n-1}$ then $T^{n+1} < T^n$. Although it is harder to find $T_\tau^0$ with $T_\tau^1 < T_\tau^0$, such a start gives $T_\tau^n > T_\tau^*$.

## 5. Uniqueness, maximum principle

This section follows computations in [23] (in the case $Z = +\infty$ and with $a_\nu = 0$) and in [24].

**Theorem 3.** *Assume* $0 < \kappa_\nu \le \kappa_M$, $0 \le a_\nu < 1$ *for all* $\nu > 0$. *Let* $Q^\pm, R^\pm \in L^1((0,1)\times\mathbb{R}^+)$ *satisfy*

$$0 \le Q_\nu^\pm(\mu) \le R_\nu^\pm(\mu) \quad \textit{for a.e. } (\mu,\nu) \in (0,1)\times(0,\infty).$$

*Then, the solutions* $(I_\nu, T)$ *and* $(I'_\nu, T')$ *of* (29) *with* $Q_\nu^\pm(\mu)$ *and* $R_\nu^\pm(\mu)$ *respectively, satisfy*

$$I_\nu(\tau,\mu) \le I'_\nu(\tau,\mu) \textit{ and } T_\tau \le T'_\tau \quad \textit{for a.e. } (\tau,\mu) \in (-1,1)\times(0,\infty).$$

*In particular,* $Q_\nu^\pm(\mu) = R_\nu^\pm(\mu)$ *for a.e.* $(\mu,\nu) \in (0,1)\times(0,\infty)$ *implies*

$$I_\nu(\tau,\mu) = I'_\nu(\tau,\mu) \textit{ and } T_\tau = T'_\tau \quad \textit{for a.e. } (\tau,\mu) \in (-1,1)\times(0,\infty).$$

One has also the following form of a Maximum Principle.

**Corollary 1.** *Let the hypotheses of Theorem 2 hold. Let* $Q_\nu^\pm(\mu) \le B_\nu(T_M)$ *(resp.* $Q_\nu^\pm(\mu) \ge B_\nu(T_m)$*) for a.e.* $(\mu,\nu) \in (0,1)\times\mathbb{R}^+$. *Then a.e.* $(\tau,\mu) \in (-1,1)\times(0,\infty)$,

$$I_\nu(\tau,\mu) \le B_\nu(T_M) \textit{ and } T_\tau \le T_M \quad \textit{resp. } I_\nu(\tau,\mu) \ge B_\nu(T_m) \textit{ and } T_\tau \ge T_m.$$

The proof relies partially on a difficult argument due to [24]. It will be published in [25].

## 6. An application to the temperature in the earth's atmosphere

A numerical test is reported on Figures 4 and 5. It is an attempt at the simulation of the effect of an increase of $CO_2$ in the atmosphere. Our purpose is only to assess that the numerical method can detect such a small change of $\kappa_\nu$.

Equation (31) is solved by a few steps of dichotomy followed by a few new steps. When $\kappa_\nu$ is larger than 4 some instabilities occur, probably in the exponential integrals. This point will be investigated in the future.

The physical and numerical parameters are

- Atmosphere thickness: 12 km
- Scaled sunlight power hitting the top of the atmosphere: $3.042 \times 10^{-5}$
- Percentage of sunlight reaching the ground unaffected: 0.99
- Percentage reemitted (Earth albedo): 10%.

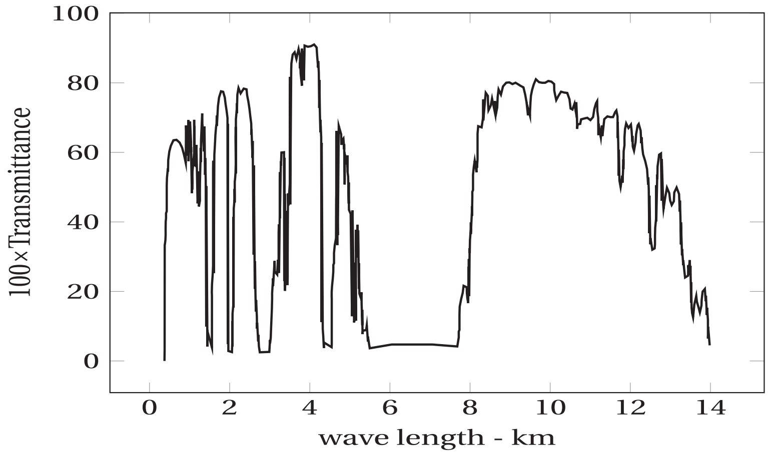


**Figure 4.** Transmittance $t_e$ versus wave-length digitized from https://commons.wikimedia.org/wiki/File:Atmosfaerisk_spredning-ru.svg. The window around 3 is blocked by $CO_2$. The absorption is related to the transmittance $t_e$ by $\kappa_\nu = -\log t_e$.

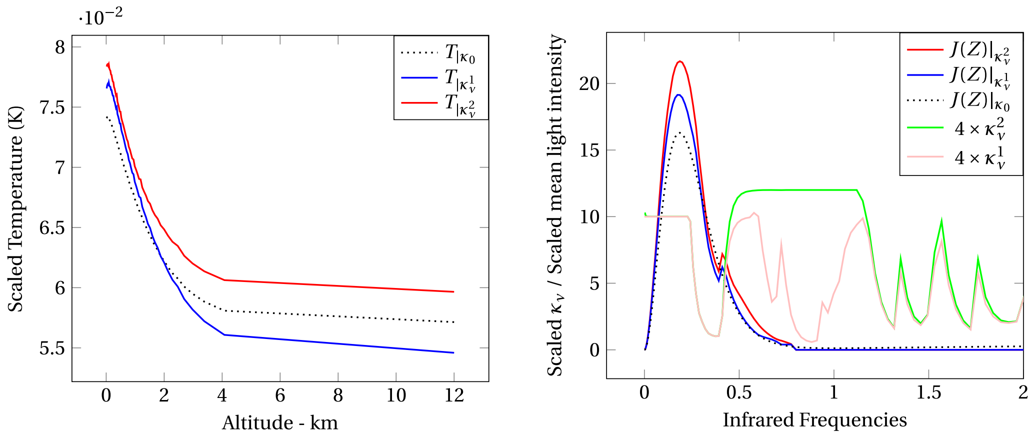


**Figure 5.** Scaled temperatures (left): 3 curves $z \to T(z)$ are plotted. One computed with $\kappa_0 = 1.225$ which corresponds to a grey atmosphere. One with $\kappa_\nu$ shown on the right in pink which corresponds to Figure 4. The third one is with $\kappa_\nu$ shown in green on the right where the transparent window around frequency 1 has been blocked. On the right the mean light intensity at altitude $Z$ is shown (mostly outgoing waves). Filling the transparent window results in an elevation of temperature.

- Percentage of sunlight being a source at high altitude ($Q^-$): 0.1%
- Cloud (isotropic) scattering: 20%. Cloud position: between 6 and 9 km
- Rayleigh scattering: 20% above 9 km
- Average absorption coefficient $\kappa_0 = 1.225$
- Density drop versus altitude : $\rho_0 \exp(-z)$
- Discretization: 60 altitude stations, 300 frequencies (unevenly distributed)
- Number of iterations: 22. Computing time: 30″ per cases.

The results are very sensitive to the value of $Q^-$ and the Earth albedo. The values for $\kappa_\nu$ are taken from Russian measurements posted on wikipedia https://commons.wikimedia.org/wiki/File:Atmosfaerisk_spredning-ru.svg.

## 7. Computer implementation details

The 1D case, uncoupled from the 2D or 3D temperature equation has been implemented in C++. Some care must be applied to the computation of the exponential integrals, as follows:

```
double expint_E1(const double t){
    const int K=8; // precision in the exponential integral function E1
    const double epst=1e-5, gamma =0.577215664901533;
    if(t==0) return -1e12;
    double abst=fabs(t);
    if(abst<epst) return -abst*(gamma + log(abst)-1);
    double ak=abst, somme=-gamma - log(abst)+ak;
    for(int k=2;k<K;k++){
        ak *= -abst*(k-1)/sqr(k); somme += ak; }
    return somme;
}
```

The rest is straigthforward.

In fifty years of CFD studies the research problems have become increasingly complex and without the joint development of computers and programming tools it would not be possible for a single individual to contribute or even test his or her ideas. The second author is part of the team which developed the PDE solver FreeFem++ [26] (see fr.wikipedia.org/wiki/FreeFem%2B%2B). It is a high level language which accepts instructions like

```
solve a(u,uh)=int2d(Th)(grad(u)'*grad(uh) + grad(u)*q + grad(uh)*p -eps*p*q)
  + on(1,u=0,v=1) + on(2,3,4,u=0,v=0);
```

where Th is the mesh of a square with borders labelled 1,2,3,4. This solves the driven cavity problem for the Stokes equations.

The algorithms discussed here have been implemented with this tool in a very short time. The discretization of Lake Leman is part of the examples in [26], written by F. Hecht.

Integrals on a line are computed by interpolating the integrand as a $P^2$ function on a 1D mesh which is the intersection of the 2D or 3D mesh of $\Omega$ with the line. It is an imbedded functionality in FreeFem++. The complexity of the algorithm is $N\sqrt{N}$ plus the complexity of a Navier–Stokes solver. All computing times are less than a few minutes in all cases.

The precision is $O(h)$ in $H^1$-norm for the temperature.

The method will be analyzed further in a forthcoming paper which will appear in the SIAM Journal Numerical Analysis (SINUM).

## 8. Conclusion

Results obtained here are in continuation of [23, 24, 27], recently reviewed for possible applications to climatology in [22] and [21]. Existence and uniqueness for the radiative transfer equations had remained open in the context of nuclear engineering. For incompressible fluids it is not unrealistic to assume that the dependence of the absorption coefficient $\kappa_\nu$ upon the temperature can be replaced by an explicit dependence on altitude. This is the key simplification by which existence, uniqueness and monotone fast and accurate numerical schemes could be found. Hence, adding RT to a Navier–Stokes solver is easy and fast when radiations come from one direction only.

As a final remark note that it seems doable to extend the method to the general case where $\kappa_\nu$ depends on $\tau$ and $T$. Indeed if the dependency $\tau \mapsto \kappa_\nu(\tau)$ is guessed only approximately, then knowing $\kappa_\nu^M > \kappa_\nu(\tau)$ independent of $\tau$ is enough to apply the method with $\kappa_M$ on the left of the equation for $I_\nu$ with a correction on the right equal to $(\kappa_\nu^M - \kappa_\nu(\tau))I_\nu$; this correction seems compatible with the monotone convergence of the temperature. Then the method could also be extended to the case $\kappa$ function of $T$ by an additional algorithmic m-loop using $\kappa(T^m)$ instead of $\kappa(T)$ and then updating $T^m$ to the $T$ just computed.

In this article the numerical computations are only given for showing the potential of the method. Real life applications, coupling RT to the full Navier–Stokes equations requires supercomputing power and will be done later.

## Conflicts of interest

Authors have no conflict of interest to declare.

## References


[1] E. Cancès, M. Defranceschi, W. Kutzelnigg, C. Le Bris, Y. Maday, “Computational quantum chemistry: A primer”, in *Special Volume, Computational Chemistry*, Handbook of Numerical Analysis, vol. 10, Elsevier, Amsterdam, 2003, p. 3-270, https://www.sciencedirect.com/science/article/pii/S1570865903100038.

[2] J.-L. Lions, R. Temam, S. Wang, “Geostrophic asymptotics of the primitive equations of the atmosphere”, *Topol. Methods Nonlinear Anal.* **4** (1994), no. 2, p. 253-287.

[3] P. Azérad, F. Guillén, “Mathematical justification of the hydrostatic approximation in the primitive equations of geophysical fluid dynamics”, *SIAM J. Math. Anal.* **33** (2001), p. 847-859.

[4] Q. L. C. Cao, E. S. Titi, “On the well-posedness of reduced 3D primitive geostrophic adjustment model with weak dissipation”, *J. Math. Fluid Mech.* **22** (2020), article no. 32.

[5] M. Castro Díaz, J. López-García, C. Parés, “High order exactly well-balanced numerical methods for shallow water systems”, *J. Comput. Phys.* **246** (2013), p. 242-264, https://www.sciencedirect.com/science/article/pii/S0021999113002118.

[6] T. Dubos, S. Dubey, M. Tort, R. Mittal, Y. Meurdesoif, F. Hourdin, “DYNAMICO-1.0, an icosahedral hydrostatic dynamical core designed for consistency and versatility”, *Geosci. Model Dev.* **8** (2015), p. 3131-3150.

[7] C. Eldred, T. Dubos, E. Kritsikis, “A quasi-Hamiltonian discretization of the thermal shallow water equations”, *J. Comput. Phys.* **379** (2019), p. 1-31.

[8] S. Chandrasekhar, *Radiative Transfer*, Clrarendon Press, Oxford, 1950.

[9] R. Dautray, J. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology*, vol. 3, Springer Verlag, New York, 2000.

[10] J. Amaya, O. Cabrit, D. Poitou, B. Cuenot, M. ElHafi, “Unsteady coupling of Navier–Stokes and Radiative Heat Transfer solvers applied to an anisothermal multicomponent turbulent channel flow”, *J. Quant. Spectrosc. Radiat. Transf.* **111** (2010), no. 2, p. 295-301.

[11] S. Zeng, S. Pian, M. Su, Z. Wang, M. Wu, X. Liu, M. Chen, Y. Xiang, J. Wu, M. Zhang, Q. Cen, Y. Tang, X. Zhou, Z. Huang, R. Wang, A. Tunuhe, X. Sun, Z. Xia, M. Tian, M. Chen, X. Ma, L. Yang, J. Zhou, H. Zhou, Q. Yang, X. Li, Y. Ma, G. Tao, “Hierarchical-morphology metafabric for scalable passive daytime radiative cooling”, *Science* **373** (2021), p. 692-696.

[12] R. Goody, Y. Yung, *Atmospheric Radiation*, Oxford University Press, Oxford, UK, 1961.

[13] C. F. Bohren, *Fundamentals of Atmospheric Radiation*, Cambridge University Press, Cambridge, UK, 2006.

[14] W. Zdunkowski, T. Trautmann, *Radiation in the Atmosphere*, Cambridge University Press, Cambridge, UK, 2003.

[15] Y. Fouquart, “Radiative transfer in climate models”, in *NATO ASI Series* (M. E. Schlesinger, ed.), NATO, New York, 1988.

[16] J.-J. Morette, “Radiation and cloud radiative properties in the european centre for medium range weather forecasts forecasting system”, *J. Geophys. Res.* **91** (1991), no. D5, p. 9121-9132.

[17] A. Fowler, *Mathematical Geoscience*, Springer Verlag, New York, 2011.

[18] G. Pomraning, *The Equations of Radiation Hydrodynamics*, Pergamon Press, New York, 1973.

[19] P.-L. Lions, *Mathematical Topics in Fluid Dynamics, Vol. 1, Incompressible Models*, Oxford University Press, Oxford, UK, 1996.

[20] A. Attaoui, “Existence of solutions for a nonlinear Boussinesq–Stefan system”, *Adv. Differ. Equ.* **14** (2009), no. 9, p. 985-1018.
[21] O. Pironneau, “A fast and accurate numerical method for radiative transfer in the atmosphere”, *C. R. Math.* **359** (2021), no. 9, p. 1179-1189.
[22] C. Bardos, O. Pironneau, “The radiative transfer model for the greenhouse effect”, *SeMA J.* **79** (2022), no. 3, p. 489-525, Special Issue ICIAM 2019.
[23] F. Golse, “The milne problem for the radiative transfer equations (with frequency dependence)”, *Trans. Amer. Math. Soc.* **303** (1987), p. 125-143.
[24] B. Mercier, “Application of accretive operators theory to the radiative transfer equations”, *SIAM J. Math. Anal.* **18** (1987), no. 2, p. 393-408.
[25] F. Golse, O. Pironneau, *Radiative Transfer in a Fluid*, RACSAM, Springer, 2022, Volume dedicated to I. Diaz.
[26] F. Hecht, “New developments in FreeFem++ (www.freefem.org)”, *J. Numer. Math.* **20** (2012), p. 251-265.
[27] C. Bardos, F. Golse, B. Perthame, R. Sentis, “The nonaccretive radiative transfer equations: existence of solutions and Rosseland approximation”, *J. Funct. Anal.* **77** (1988), p. 434-460.