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Irreducible representation of surface distributions and Piola transformation of external loads sustainable by third gradient continua

Représentation irréductible des distributions surfaciques et transformation de Piola des charges externes soutenables par un continuum de troisième gradient

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Abstract. In this paper Piola transformations are found that relate the Eulerian and Lagrangian external loads which third gradient continua can sustain. As shown by Gabrio Piola and Paul Germain, the most effective postulation scheme in mechanics is based on the principle of virtual work and therefore continuum mechanics must be mathematically founded based on the theory of distributions and on differential geometry. Using the principle of virtual work, the set of admissible external loads sustainable by third gradient continua is seen to include: i) volume force density, ii) surface density of contact force, iii) surface density of contact double force, iv) surface density of contact triple force, v) line density of edge contact forces, vi) line density of contact edge double forces and vii) contact forces concentrated on wedge points. Following the nomenclature introduced by Paul Germain, forces are dual in virtual work of virtual displacements, surface and line

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double forces are dual of the derivatives of virtual displacements in the normal direction(s) of the surfaces and edges constituting the boundary of the continuum, and surface triple forces are dual of the second normal derivatives of virtual displacements. Volume and surface forces transform as in first gradient Cauchy continua. Moreover we find that: a) the virtual work expended by Eulerian surface triple force, when transformed into the Lagrangian description, must be represented as the work expended by all the kinds of external Lagrangian loads listed in i)-vii); b) Eulerian surface double force transforms into Lagrangian surface double force, surface contact force and edge contact line force; c) Eulerian edge contact line double force transforms into Lagrangian edge contact line double forces, edge line forces and point concentrated wedge forces; d) Eulerian edge and wedge contact line forces transforms into their Lagrangian counterpart only. The Piola transformation formulas deduced in this paper depend on the first, second and third gradients of placement. The presented results allow for the formulation of well-posed boundary condition problems for third gradient continua in the Lagrangian description, and are relevant in computational mechanics. In view of the obtained Piola transformation formulas, the concept of dead loads needs to be modified. We believe to have given an example of how the *Mechanics in the French Style*, as developed on the ideas by D'Alembert and Lagrange, is still a fertile tool of invention.

Résumé. Dans cet article, on trouve les transformations de Piola qui relient les charges externes Eulériennes et Lagrangiennes que les milieux continuous de troisième gradient peuvent soutenir. Comme l'ont montré Gabrio Piola et Paul Germain, le schéma de postulation le plus efficace en mécanique est basé sur le principe des travaux virtuels et, par conséquent, la mécanique des continuums doit être mathématiquement fondée sur la théorie des distributions. En utilisant le principe des travaux virtuels, l'ensemble des charges externes admissibles soutenables par les continuums de troisième gradient comprend : i) la densité de force volumique, ii) la densité de surface de la force de contact, iii) la densité de surface de la double force de contact, iv) la densité de surface de la triple force de contact, v) la densité linéaire des forces de contact de bord, vi) la densité linéaire des doubles forces de contact de bord et vii) les forces de contact concentrées sur les points de coin. Suivant la nomenclature introduite par Paul Germain, les forces sont duales en travail virtuel des déplacements virtuels, les forces doubles de surface et de ligne sont duales des dérivées des déplacements virtuels dans la ou les directions normales des surfaces et des bords constituant la frontière du continuum et les forces triples de surface sont duales des dérivées secondes normales des déplacements virtuels. Les forces de volume et de surface se transforment comme dans les milieux continus de Cauchy à premier gradient. En plus, nous trouvons que a) le travail virtuel dépensé par la force triple de surface Eulérienne, lorsqu'elle est transformée en description Lagrangienne, doit être représentée comme le travail dépensé par tous les types de charges Lagrangiennes externes énumérées aux points i)-vii) ; b) la force double de surface Eulérienne se transforme en force double de surface, en force de contact de surface et en force de ligne de contact Lagrangiennes, c) la force double de ligne de contact Eulérienne se transforme en doubles forces de ligne de contact Lagrangiennes, en forces de ligne et en forces de coin concentrées, d) les forces de ligne de contact de bord et de coin Eulériennes se transforment uniquement en leur contrepartie Lagrangienne. Les formules de transformation de Piola déduites dans cet article dépendent des premier, deuxième et troisième gradients du placement. Les résultats présentés permettent la formulation de problèmes de conditions aux limites bien posés pour les milieux continus de troisième gradient dans la description Lagrangienne et sont pertinents en mécanique computationnelle. Compte tenu des formules de transformation de Piola obtenues, le concept de charges mortes doit être aussi modifié. Nous pensons avoir donné un exemple de la façon dont la « Mécanique à la française », telle qu'elle a été développée à partir des idées de D'Alembert et de Lagrange, est toujours un outil de découverte fertile.

Keywords. Third-gradient materials, Principle of Virtual Work, Piola Transformation, Eulerian description, Lagrangian description, Distribution theory, Differential Geometry.

Mots-clés. Continuum de troisième gradient, Principe des Travaux Virtuels, Transformations de Piola, Description lagrangienne, Description Eulerienne, Théorie des distributions, Géométrie différentielle.

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1. Introduction

The aim of this paper is to find Piola transformations for admissible sustainable loads in third gradient continua. The conceptual framework that we accept is that given by the postulation of continuum mechanics based on the principle of virtual work. This principle has been systematically used since the most ancient known mathematized mechanical theories were formulated in the pseudo-Aristotelian Μεκανικά Προβλήματα . In the following centuries, it revealed itself to be the most reliable guide to invent novel mathematical models for describing physical phenomenology. In mentioned apocryphal collection of exercises of mechanics, included in the *Corpus Aristotelicum,* sometimes attributed to Archytas of Tarentum [1] and dated, depending on the commentators, in the period between the IV century BC and the II century BC, one finds the first "great" unification theory: 35 problems of relevance in Engineering applications, which were studied separately on the basis of different principles, are discussed using always the same and unique principle, the principle of virtual work, with the support of the logical structure given by Euclidean geometry.

The foundational attitude which animated the Μηχανικά Προβλήματα has been revived many centuries later by the French-Italian school of mechanics headed by D'Alembert, Lagrange and Gabrio Piola. Albeit the principle of virtual work has been too often considered "controversial" (see already [2, 3]) it has been recognized many times to be the most effective tools for guiding the invention of novel mathematical models in mechanical sciences (see [4, 5]).

D'Alembert in his *Traité de Dynamique* (1768) explicitly states that the range of application of mechanics must be "*enlarged in reducing the number of the principles on which it is based*". This is exactly was has been done by Paul Germain, when he extended the the possibilities of application for continuum mechanics by renouncing to use many balance laws (balance of forces, of torques of microforces and so on) and by concentrating in the systematic use of the principle of virtual work for formulating generalized continuum models (see [6–8] and the more recent works [9,10]).

In the present paper, we want to give a further example of the application of the principle of virtual work to perfect some parts of continuum mechanics, which cannot be developed by using the postulation based on the balance of forces and torques, and which seems very promising in their applications to their use for the design of novel metamaterials [11–13]. The mathematical tools used here are taken from a more modern version of Euclidean geometry: modern differential geometry, originated from the works of Gauss and Riemann and developed using absolute tensor calculus by Levi–Civita and Ricci in the form presented in the fundamental textbook by Lichnerowicz [14].

In this way we want to contribute to substantiate the following statement:

« "Mechanics in the French Style", as developed on the ideas by D'Alembert and Lagrange, remains the most powerful and fertile tool of invention in mechanical sciences ».

1.1. *Third-gradient continua: summary of main concepts*

The virtual work functionals to be used for third-gradient continua are third-order distributions (see [15] and [6–8]), when virtual displacements are regarded as test functions. Therefore, their deformation energy density depends on the derivatives of the placement field up to the third order, see e.g. [15, 16]. Third gradient internal virtual work functional requires a second-order stress tensor and hyper-stress tensors of order three and four, called double and triple stresses, see e.g. [17, 18]. By iterated integrations by parts, also extended to the faces, edges and wedges of the continuum boundary, all regarded as embedded manifolds with boundary [15, 19, 20], the internal virtual work functional can be represented as the sum of distributions concentrated on volumes, surfaces, curves and points [15, 21].

As proven in [15], also for higher-gradient continua a Cauchy representation theorem for contact loads in terms of hyper-stress tensors can be proven, using an integration by parts argument. Therefore the class of admissible sustainable external loads for n^{th} gradient continua is easily determined once the Principle of Virtual Work is accepted. External loads applicable to third-gradient continua include surface *k-forces* with $k = 1,2,3$, line *h-forces* with $k = 1,2$ and wedge forces: well-posed boundary problems for third gradient continua are thus easily deduced without any *ad hoc* assumption.

1.2. *Meta-materials architectures and higher gradient continuum models*

Higher-gradient continua are models suitable to describe the macroscopic behaviour of mechanical systems exhibiting multi-scale structure. We simply cite here few interesting phenomenological context plural where size effects, determined by characteristic multiple length scales, play a relevant role: materials with nano architecture [22], systems where boundary layers [23–25] or contact lines [26] arise, surface tension in fluids [27].

Rather interestingly, in the present context, it has to be recalled that the homogenized continua corresponding to a large class of truss micro-architectures have been studied (see [28–30]): in particular, it has been proven that, by using three length-scales, it is possible to design beams whose deformation energy depends on the third gradient of transverse displacement. Therefore the problem of synthesis of third gradient beams has found a positive solution, so further motivating the study of third gradient continua. In fact, we conjecture that, by using the design scheme of pantographic micro-structures, and replacing the Euler beams with the third gradient beams synthesized in [29] it will be possible to design third gradient 2D and 3D continua (see e.g. [31]).

The theoretical results about higher gradient continua have been exploited to design experiments involving multi-scale 3D printed micro-architectures: correspondingly suitable multi-scale Digital Image Correlation procedures have been conceived, both using images of a free surface (2D DIC) or within the bulk (Digital Volume Correlation DVC) (see e.g. [32–35]): this circumstance proves the importance in the engineering applications of generalized continuum theories.

Finally, it has to be remarked that, to our knowledge, any result about existence, uniqueness and stability of third-gradient models has not been yet found, albeit it seems to us that the methods exploited in [36–38] should be easily generalized: this means that higher gradient continuum theories can give inspiration to more advanced mathematical investigations.

Although Gabrio Piola, already in [3], introduced higher gradient continua, via an asymptotic homogenization procedure, only recently the need of such continua in modelling exotic mechanical phenomena has been fully recognized. This circumstance is mainly due to the opposition of Truesdellian school, in which it is often believed that balance laws, using the approach *à la Cauchy*, are the only basic principles at the basis of mechanics and that "exotic" *h-forces* do nor "exist", see e.g. [4, 39]. Instead, in the case of n^{th} gradient continua, *h-forces* appear naturally (being $h \le n$), by duality in the postulation of mechanics based on the principle of the virtual work [40] and supply a powerful tool for model formulation.

We expect that the study of continuum mechanics based on the principle of virtual work, and in particular the presented investigations about third-gradient continua, will pave the way towards the synthesis of novel metamaterials, exhibiting a larger variety of mechanical behaviour.

1.3. *Organization of the paper*

In the present paper the results in [20, 41] are generalized: we prove that Eulerian external *h-forces*, under Piola transformations, originate all the lower order Lagrangian forces on lower dimension submanifolds of the continuum boundary.

The paper is organized as follows. In the continuation of this first section we introduce the used notations and the first kinematical and geometrical concepts. A warning is repeated here: Lagrangian quantities and indices are chosen to be represented by capital letters, while Eulerian with lowercase letters. However, in order to decrease the complexity of notations, many general results from differential geometry are presented with capital letters but are clearly valid in general.

In Section 2, the class of admissible generalized loadings which can be sustained by third gradient continua, as found in the literature, is shortly described. The essential nomenclature, based on that introduced by Germain, is also recalled.

In Section 3, some useful preliminary concepts and results from modern differential geometry of embedded manifolds with boundaries, as originated from the works of Gauss and Riemann, are recalled. Also some relationships useful in the sequel are established. The concept of piecewise regular surfaces is introduced and explained via some figures. In piecewise regular surfaces, the field of normal unit vectors jumps on a finite number of curves (more technical details about this point are given in [15]). These surfaces are composed by faces, each of which can be regarded as a two-dimensional embedded manifold with boundary. The curves of discontinuity of normals are to be regarded as parts of the boundary of each concurring face, and are called edges. Finally a finite number of edges are concurring on wedges, where the tangent vectors of concurring curves are discontinuous.

In Section 4, the problem of finding the irreducible representation of second-order surface distributions is confronted in the case in which the surface, on which the distribution is concentrated is piecewise regular. Also some formulas are found, useful for integration by parts.

In Section 5, the preparatory concepts are concluded by proving the Piola transformations formulas for the shapes of differential boundaries of reference into current configurations.

In Section 6, finally the Piola transformations of external loads sustainable by third gradient continua are obtained. They link externally applied loads in the current configuration with those applied in the reference configuration.

In Section 7, some conclusions and research perspectives are presented. In particular it is discussed the possibility of extending the present results to the more general case of n^{th} gradient continua.

1.4. *Used notations, kinematical and geometrical concepts*

We use tensor indicial Levi–Civita notation, with Einstein convention the sum of repeated indices. We believe that the distinction between contravariant and covariant components is essential in the present context, as displacements and velocities are naturally to be identified as having contravariant components, and therefore forces and stresses need covariant components to saturate, in work or power expressions, virtual displacements or velocities.

In order to make the used formulas more readable, Eulerian and Lagrangian quantities and components will be denoted respectively by lowercase and uppercase letters. For instance the Lagrangian and Eulerian gradient operators components will be denoted respectively by the symbols $\frac{\partial}{\partial X^A}$ and $\frac{\partial}{\partial x^a}$. As it plays a crucial role in our presentation, the derivative along the direction of the unit vector N^R will be denoted as $\frac{\partial}{\partial N} := N^R \frac{\partial}{\partial X^R}$.

Given a domain D with dimension i $(i = 3, 2, 1)$, symbol ∂D will denote its differential border, having dimension $(i - 1)$. As the differential border may not be smooth, we will need to consider the p^{th} order border, which is defined iteratively as follows: $\partial^{(0)}\Omega := \Omega$-and $\partial^{(p)}D \equiv \partial\, \partial^{(p-1)}D$. It has to be remarked that the standard formulation of the so-called Poincaré formula (see [42]) does not hold for the domains having the regularity which we are considering here (see [15] for more details on this subject).

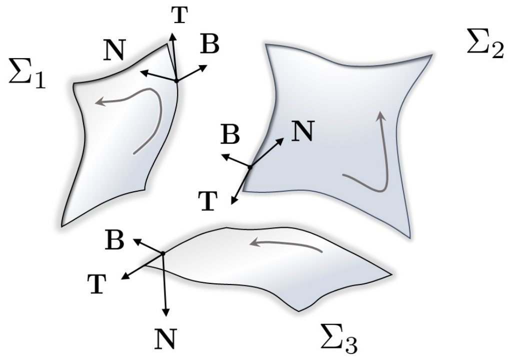

Figure 1. Part of the volume boundary $\partial\Omega$ constituted of oriented regular faces Σ (with the positive normals pointing outwards), with their differential boundaries including in turn piecewise regular edges oriented consistently. The mutually orthogonal, normalized vectors **T**, **B** and **N** specify a Darboux moving basis along such boundary edges.

We will consider here a continuum having the three-dimensional domain Ω, with non-vanishing volume and included in the Euclidean space, as reference configuration. The placement Π of the continuum is assumed to be a bijection sufficiently regular that maps every material point in Ω into its current position, hence invertible being its inverse equally regular.

The domain Ω is called Lagrangian or material or reference configuration, while the domain $\omega := \Pi(\Omega)$ is called Eulerian or spatial or current configuration. The domain Ω (a three-dimensional manifold with piecewise regular differentiable boundary, see [15]) is assumed to be a differential boundary which is composed by faces, edges and wedges. Faces and edges are two-dimensional manifolds and one-dimensional manifolds respectively, whose differential boundaries are the edges and the wedges respectively. Edges are those curves on which faces are concurring and where the faces normals are discontinuous, wedges are those points where edges concur: the concurring edge tangent vectors of different edges may be different, see Fig. 1.

We denote Σ, L, and P the surfaces, curves and points which are the supports of $\partial\Omega$, $\partial\partial\Omega$ and $\partial\partial\partial\Omega$: remark that, a curve, which belongs to the boundary of a face, is also a part of the boundary of another concurring face. Therefore the same curve has to be regarded as part of two different boundaries.

Assuming that the placement is sufficiently regular, the domain $\omega := \Pi(\Omega)$ will have the same differential properties as Ω: in this paper we refrain from the study of the process of edge and wedge formation in the passage from reference to current configuration. All the faces and edges, having co-dimension one and two, are oriented consistently with the orientation of the outward pointing normal field, with respect to Ω, whose components are denoted N^R.

1.5. *Distinction between Eulerian and Lagrangian fields and quantities: important notational warning*

In this paper we will need to discuss some differential geometry properties of the boundaries for both Eulerian and Lagrangian configurations of third-gradient continua. Moreover we will consider both Eulerian and Lagrangian fields to establish several relationships among them. In order to make the presentation less cumbersome we have not formulated in a general formalism all the needed properties, and subsequently reformulated them in either the Lagrangian or Eulerian description, when required. Instead we have discussed them in the Lagrangian description, by

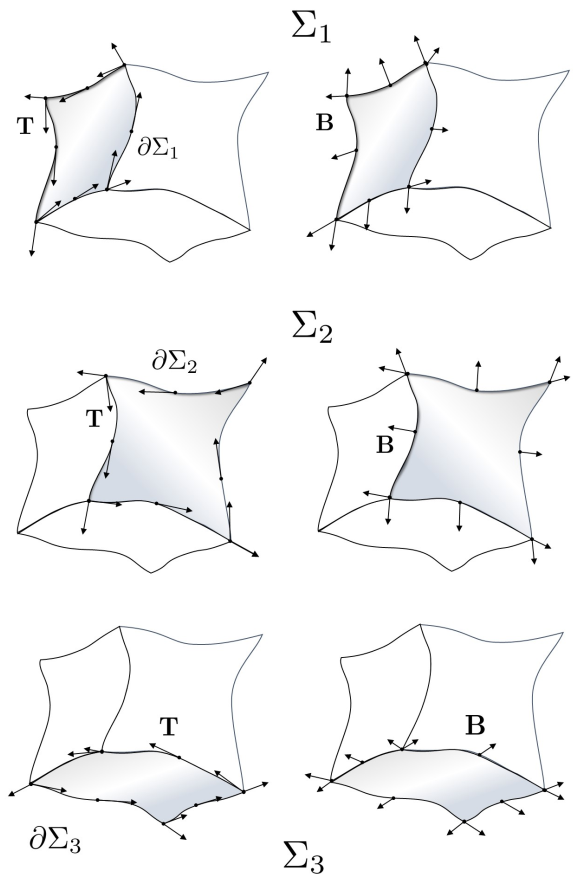

Figure 2. Edge tangent and tangent normal vectors (in the order denoted by symbols **T** and **B**) running along the differential boundary $\partial\Sigma$ of each oriented regular face Σ. Wedges are easily recognized as discontinuity points for the tangent (and for the tangent normal).

using the convention that every relevant Lagrangian quantity is denoted with uppercase letters. The reader will easily reconstruct, when necessary, the required relationships in the Eulerian description simply by denoting the Eulerian counterparts of every Lagrangian quantity or field by using the same letter, but lowercase.

For instance in the Eulerian configuration, we denote σ, $\lambda|$, and p the surfaces, curves and points which are the supports of $\partial\omega$, $\partial\partial\omega$ and $\partial\partial\partial\omega$.

2. Admissible generalized loadings which can be sustained by third gradient continua

We introduce two different vector bases: one for the reference (denoted $\{E_B\}$) and one for the current configuration (denoted $\{e_i\}$). As always needed when distinguishing contravariant from covariant components, we also introduce, in both the reference and actual configurations, Riemannian metric tensors, whose components are denoted in the specified bases by G_{AB} and g_{ij}, respectively.

Along any curved edge, regarded as part of the boundary of a face having normal **N**, one can define the field of bases constituted by the triples **T**, **N** and $\mathbf{B} := \mathbf{T} \wedge \mathbf{N}$, where **T** denoted the edge tangent: both **T** and the normal **B** to the boundary of the face belong to the plane tangent to the face. The vector **B** is also called *the tangent normal* to the edge (see Figs. 1 and 2).

The introduced continuum deformation is specified by the diffeomorphism Π between Ω and ω: therefore many concepts and results of differential geometry [43–45] will be used in the following. This diffeomorphism will play an important role also when developing the theory of generalised continua (see e.g. [18]) where additional kinematic descriptors are needed: in fact Π allows for the establishment of the correspondence between any Lagrangian field $\Psi(X)$ (with $X \in \Omega$) with the Eulerian field $\Psi(\Pi^{-1}(x))$ (with $x \in \omega$).

Finally we will denote the placement gradient with the symbol

$$\mathbf{F} = \partial\chi/\partial\mathbf{X},$$

and assume that

$$J := \det(\mathbf{F}) > 0.$$

For third-gradient continua (see e.g. [15, 21] and in the linear case [46]), the sustainable admissible external virtual work functional, in Lagrangian form, has the form:

$$\begin{aligned}\delta\mathscr{W}^{\text{ext}} = \int_{\Omega} \mathscr{F}^{\text{ext}}_{\Omega i}\,\delta\Pi^i\, d\Omega\ {\scriptstyle (V)} + \int_{\Sigma} \mathscr{F}^{\text{ext}}_{\Sigma i}\,\delta\Pi^i\, d\Sigma\ {\scriptstyle (S)} \\ + \int_{\Sigma} \mathscr{F}^{\text{ext}}_{\Sigma N i}\,\frac{\partial\delta\Pi^i}{\partial N}\, d\Sigma\ {\scriptstyle (SN)} + \int_{\Sigma} \mathscr{F}^{\text{ext}}_{\Sigma NN i}\,\frac{\partial^2\delta\Pi^i}{\partial N^2}\, d\Sigma\ {\scriptstyle (SNN)} + \\ + \int_{L} \mathscr{F}^{\text{ext}}_{L i}\,\delta\Pi^i\, dL\ {\scriptstyle (L)} + \int_{L} \mathscr{F}^{\text{ext}}_{LN i}\,\frac{\partial\delta\Pi^i}{\partial N}\, dL\ {\scriptstyle (LN)} + \int_{L} \mathscr{F}^{\text{ext}}_{LB i}\,\frac{\partial\delta\Pi^i}{\partial B}\, dL\ {\scriptstyle (LB)} + \\ + \sum_{\mathrm{w}=1}^{\sharp\text{wedge}} \mathscr{F}^{\text{ext}}_{P_{\mathrm{w}} i}\,\delta\Pi^i(\mathbf{P}_{\mathrm{w}})\ {\scriptstyle (W)}.\end{aligned} \tag{1}$$

Each addend has been labeled with a letter having an obvious meaning: volume lowercase (V) surface (S), edge (L) and wedge (W) addend. When the considered load is dual of a first or second order normal derivative of virtual displacement the corresponding letters N, B, NN have been added.

The fields $\mathscr{F}^{\text{ext}}_{\Omega i}(\mathbf{X})$, $\mathscr{F}^{\text{ext}}_{\Sigma i}(\mathbf{X})$, $\mathscr{F}^{\text{ext}}_{L i}(\mathbf{X})$ and $\mathscr{F}^{\text{ext}}_{P_{\mathrm{w}} i}$ map the material particle X or P_w belonging to the reference configuration into Eulerian vectors. These fields are defined, respectively in the domain Ω, on its faces, on its edges and on its wedges: they are force densities per unit volume, per unit surface, per unit length and point forces.

Following (and extending) the nomenclature by Paul Germain (see [6–8])

(1) the vector fields $\mathscr{F}^{\text{ext}}_{\Sigma N i}(\mathbf{X})$ and $\mathscr{F}^{\text{ext}}_{\Sigma NN i}(\mathbf{X})$ can be called *external surface double* and *triple forces*, respectively: they are, respectively, a work (force times length) per unit surface, and a force times length squared per unit surface;
(2) the vector fields $\mathscr{F}^{\text{ext}}_{LN i}(\mathbf{X})$ and $\mathscr{F}^{\text{ext}}_{LB i}(\mathbf{X})$ can be called *external edge double forces*, that are work (force times length) per unit line.

As the placement is assumed to be a diffeomorphism, the structure of external virtual work functional, as established by the theorem of Laurent Schwartz (see [47] and also [15]), is the same in both the Lagrangian and the Eulerian descriptions.

Therefore one can obtain the expression for the Eulerian external work functional from the equation (1) *simply replacing with lowercase letters the uppercase letters, and redefining the involved quantities in the Eulerian description.*

The reader should remark that the equation (1) is a particularization of the general representation formula for the external virtual work for a n^{th} gradient material, provided in [15].

3. Some useful preliminary concepts and results from differential geometry

The most important ancillary mathematical theory for continuum mechanics is differential geometry, as it is obviously understood once that the placement is identified as a diffeomorphism between reference and current configuration, and that the deformation can be interpreted as a change of Riemannian metrics in the reference configuration.

In this section, for seek of self consistency, we recall some results which we need in the development of our presentation. The reader needs to familiarize with them, and with the used Ricci–Levi–Civita notation, in order to follow the subsequent deductions.

3.1. *Definitory properties of faces and edges projectors*

The content of this section is illustrated by the Figs. 1 and 2. Because of the accepted regularity assumptions, at each point of a regular face of $\partial\Omega$, its linear tangential and normal projection operators are well defined. Denoting them by $[M_{\|}]^A_B$ and $[M_{\perp}]^A_B$, respectively, they enjoy the following properties:

$$\left[M_{\|}\right]^A_B + [M_{\perp}]^A_B = G^A_B, \qquad \mathbf{M}_{\|} + \mathbf{M}_{\perp} = \mathbf{1}, \tag{2}$$

$$[M_{\perp}]^C_A = N^C N_A, \qquad [\mathbf{M}_{\perp}] = \mathbf{N}\otimes\mathbf{N}, \tag{3}$$

$$\left[M_{\|}\right]^C_A = G^C_A - N^C N_A, \qquad \left[\mathbf{M}_{\|}\right] = \mathbf{1} - \mathbf{N}\otimes\mathbf{N}, \tag{4}$$

$$\left[M_{\|}\right]^A_B \left[M_{\|}\right]^B_C = \left[M_{\|}\right]^A_C, \qquad \mathbf{M}^2_{\|} = \mathbf{M}_{\|}, \tag{5}$$

$$[M_{\perp}]^A_B [M_{\perp}]^B_C = [M_{\perp}]^A_C, \qquad \mathbf{M}^2_{\perp} = \mathbf{M}_{\perp}, \tag{6}$$

where G^A_B are the components of the unit operator, coincident with the mixed form of the metric tensor.

Similarly, at each point of an edge in $\partial\partial\Omega$, its linear tangential and normal projection operators are well defined. These projectors are denoted in the order by the symbols $[M_{L\|}]^A_B$ and $[M_{L\perp}]^A_B$ and enjoy the following properties:

$$\left[M_{L\|}\right]^E_A = T^E T_A, \qquad \mathbf{M}_{L\|} = \mathbf{T}\otimes\mathbf{T}, \tag{7}$$

$$[M_{L\perp}]^E_A = B^E B_A + N^E N_A, \quad \mathbf{M}_{L\perp} = \mathbf{B}\otimes\mathbf{B} + \mathbf{N}\otimes\mathbf{N}, \tag{8}$$

$$G^E_A = \left[M_{L\|}\right]^E_A + [M_{L\perp}]^E_A = T^E T_A + B^E B_A + N^E N_A, \tag{9}$$

which can be rewritten as

$$\mathbf{1} = \mathbf{M}_{L\|} + \mathbf{M}_{L\perp} = \mathbf{T}\otimes\mathbf{T} + \mathbf{B}\otimes\mathbf{B} + \mathbf{N}\otimes\mathbf{N}.$$

3.2. *Deformative Riemannian metrics*

As already understood by Piola (see [16]), the deformation of a continuum can be mathematically described by changing the inner product in the reference configuration. The placement-induced Riemannian metrics defined as

$$G^{\star RS} := g^{rs}\left(\mathbf{F}^{-1}\right)^R_r \left(\mathbf{F}^{-1}\right)^S_s, \tag{10}$$

$$G^{\star}_{AB} := \left(g^{rs}\left(\mathbf{F}^{-1}\right)^A_r \left(\mathbf{F}^{-1}\right)^B_s\right)^{-1} = g_{lm} F^l_A F^m_B, \tag{11}$$

is sometimes called the *pull-back* of the Eulerian metric tensor: it is naturally defined in its doubly contravariant form.

One obviously has that[1] for every co-vector V

$$G^{\star RQ} V_R V_Q = \left\| \mathbf{F}^{-T} \mathbf{V} \right\|_g^2, \tag{12}$$

(see [16, 19, 43]), where $\| v \|_g = (g_{ab} v^a v^b)^{\frac{1}{2}}$ denotes the Eulerian norm.

It is easy to check that

$$\langle \mathbf{FU}, \mathbf{FV} \rangle_g = g_{lm} F^l_A F^m_B U^A V^B = G^{\star}_{RQ} U^R V^Q. \tag{13}$$

In what follows we will need to introduce the following vector associated to V

$$V^{\star R} := G^{\star RS} V_S. \tag{14}$$

It is the contravariant vector associated to V by using the pull-back metrics: obviously, in general, it differs from the contravariant form of the same vector (generated by the usual metrics): $V^{\star R} \neq V^R$. In particular we will need the previous formula for the normal

$$N^{\star R} := G^{\star RS} N_S. \tag{15}$$

3.3. *The divergence theorem for submanifolds with boundary and a notational warning*

This theorem is probably the one which initiated differential geometry, and is essentially due to Gauss and Stokes. It applies to every Riemannian submanifolds, may it be one-dimensional (as a compact curve) or two-dimensional (as a surface): the interested reader may consult e.g. [44, 45] or [15] for further details.

Let us consider a submanifold S embedded in the Euclidean space and let ∂S denote its differential boundary. Let Q be the parallel projector on the tangent space to S.

Gauss divergence theorem on S states that for every vector field W defined on S the following equality holds

$$\int_S Q^C_A \frac{\partial}{\partial X^C} \left(Q^A_B W^B \right) dS = \int_{\partial S} Q^B_A W^A D_B \, d\partial S, \tag{16}$$

where the vector D is the vector tangent to S which is orthogonal to ∂S.

The integrand appearing at LHS of the equality defines the surface divergence of the vector field W sometimes also denoted with the symbol $\mathrm{DIV}_S(\mathbf{W})$.

In the present paper this theorem is applied in four different instances: for Eulerian or Lagrangian faces or edges, constituting the differential boundary of the reference or for the current configurations.

In the present paper we will need to consider domains (in both the reference and current configuration) whose differential boundary is a piecewise regular surface, which can be regarded as the union of faces, edges and wedges (see Figs. 1-2). Each face can be regarded as a 2D submanifold, whose boundary is constituted by all the edges on which the face is concurring. Each edge can be regarded as part of the differential boundary of all the faces concurring on it: the limits on the edge of the face normals from every face are, in general, different, as different are the relative tangent normal vectors (see Figs. 2-3). Therefore, when applying the divergence theorem to every face of the considered piecewise regular surface, different edge fluxes coming from every face arise.

---

[1] Recalling that the normal to a manifold is naturally recognized as a co-vector, in the literature the Lagrangian co-vector having components $(\mathbf{F}^{-1})^R_r N_R$ is denoted $\mathbf{F}^{-T}\mathbf{N}$.

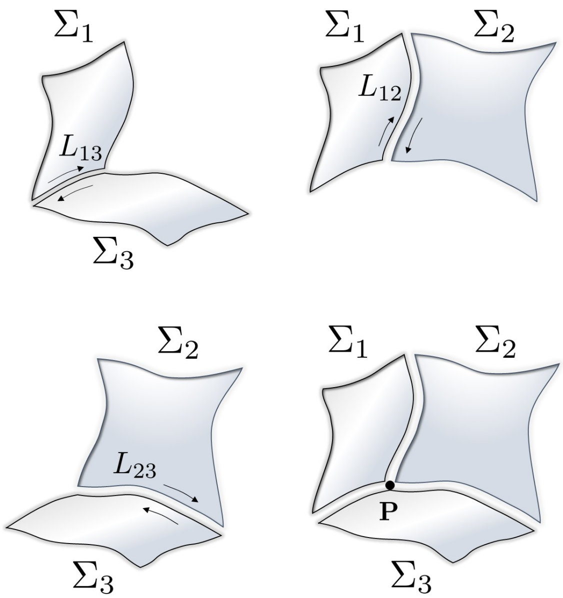

Figure 3. Pairs of boundary faces Σ which share parts of the support of their differential border $\partial\Sigma$. Opposite orientations are induced along each border edge, when regarded as belonging to the boundary of one face or of the contiguous one. Herein the wedge point **P** is shared among three faces, and belongs to three regular edges of their differential boundaries.

3.4. *Some useful identities involving the face parallel projectors*

It is easy to prove the following identities, which we will systematically use in the sequel. We have used a notation involving uppercase letters only: however, the involved properties in the demonstrations are purely geometrical. The reader will easily transform all formulas in lowercase letters so obtaining the needed identities in the Eulerian configuration.

3.4.1. *Representation for face parallel projector restricted to edges*

For the edge tangent vector and the tangent normal to the edge, the following identities hold:

$$[M_{\|}]^E_A T^A = T^E, \quad [M_{\|}]^E_A B^A = B^E. \tag{17}$$

Therefore, the following representation for face projectors holds:

$$[M_{\|}]^E_{A'} = T^E T_{A'} + B^E B_{A'}. \tag{18}$$

3.4.2. *Surface divergence of surface parallel projector*

The surface divergence of the parallel projector can be calculated as follows

$$[M_{\|}]^S_Q \frac{\partial [M_{\|}]^K_A}{\partial X^S} = -[M_{\|}]^S_Q \frac{\partial \left(N^K N_A\right)}{\partial X^S} = -[M_{\|}]^S_Q \left(\frac{\partial N_A}{\partial X^S} N^K + \frac{\partial N^K}{\partial X^S} N_A\right), \tag{19}$$

and the following definitory identity is very well-known in Riemannian differential geometry:

$$[M_{\|}]^R_S \frac{\partial}{\partial X^R} [M_{\|}]^S_D = -N_D \frac{\partial N^S}{\partial X^S} =: \frac{2}{R_m} N_D, \tag{20}$$

where symbol R_m is called the local mean curvature over the boundary face.

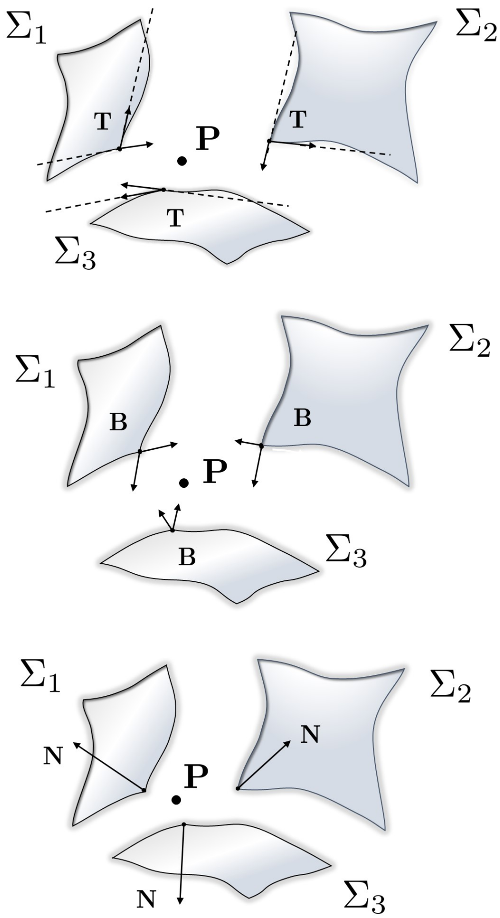

Figure 4. Darboux basis vectors belonging to the differential boundaries of each face concurring to the wedge point **P**. When considering separately each face with a continuous normal field, the edge tangent and edge tangent normal vectors, i.e. **T** and **B**, are discontinuous at the wedge points: instead, when passing from one face to the contiguous one sharing the support of an edge, **T**, **B** and also **N** are discontinuous (in particular, the tangents equal opposite).

3.4.3. *Surface divergence for the fully parallel projection of a two times contravariant tensor* $\mathbb{A}_i^{AB}$

It is calculated with the following equality chain:

$$\begin{aligned}
&[M_\parallel]_Q^S \frac{\partial\left(\mathbb{A}_i^{AB}\,[M_\parallel]_A^K\,[M_\parallel]_B^Q\right)}{\partial X^S} \\
&\quad = [M_\parallel]_Q^S \frac{\partial\left(\mathbb{A}_i^{AB}\,\delta_A^K\,[M_\parallel]_B^Q\right)}{\partial X^S} - [M_\parallel]_Q^S \frac{\partial\left(\mathbb{A}_i^{AB}\,N^K N_A\,[M_\parallel]_B^Q\right)}{\partial X^S} \\
&\quad = [M_\parallel]_Q^S \frac{\partial\left(\mathbb{A}_i^{KB}\,[M_\parallel]_B^Q\right)}{\partial X^S} - N^K\,[M_\parallel]_Q^S \frac{\partial\left(\mathbb{A}_i^{AB}\,N_A\,[M_\parallel]_B^Q\right)}{\partial X^S} - \mathbb{A}_i^{AB}\,N_A\,[M_\parallel]_B^S \frac{\partial N^K}{\partial X^S}\,.
\end{aligned} \tag{21}$$

By projecting the equation (21) along the direction normal to the surface, we get

$$N_K \left[M_\parallel\right]_Q^S \frac{\partial \left(\mathbb{A}_i^{AB} \left[M_\parallel\right]_A^K \left[M_\parallel\right]_B^Q\right)}{\partial X^S} = N_K \left[M_\parallel\right]_Q^S \frac{\partial \left(\mathbb{A}_i^{KB} \left[M_\parallel\right]_B^Q\right)}{\partial X^S} - \left[M_\parallel\right]_Q^S \frac{\partial \left(\mathbb{A}_i^{AB} N_A \left[M_\parallel\right]_B^Q\right)}{\partial X^S}. \tag{22}$$

By projecting the equation (21) along the direction B, we get

$$B_K \left[M_\parallel\right]_Q^S \frac{\partial \left(\mathbb{A}_i^{AB} \left[M_\parallel\right]_A^K \left[M_\parallel\right]_B^Q\right)}{\partial X^S} = B_K \left[M_\parallel\right]_Q^S \frac{\partial \left(\mathbb{A}_i^{KB} \left[M_\parallel\right]_B^Q\right)}{\partial X^S} - \mathbb{A}_i^{AB} N_A \left[M_\parallel\right]_B^S \frac{\partial N^K}{\partial X^S} B_K. \tag{23}$$

By projecting the equation (21) along the tangent plane to the surface, we get

$$\begin{aligned} &\left[M_\parallel\right]_K^{K'} \left[M_\parallel\right]_Q^S \frac{\partial \left(\mathbb{A}_i^{AB} \left[M_\parallel\right]_A^K \left[M_\parallel\right]_B^Q\right)}{\partial X^S} \\ &\qquad = \left(\left[M_\parallel\right]_K^{K'} \left[M_\parallel\right]_Q^S \frac{\partial \left(\mathbb{A}_i^{KB} \left[M_\parallel\right]_B^Q\right)}{\partial X^S} - \mathbb{A}_i^{AB} N_A \left[M_\parallel\right]_B^S \left[M_\parallel\right]_K^{K'} \frac{\partial N^K}{\partial X^S}\right). \end{aligned} \tag{24}$$

3.4.4. *A first useful reduction*

The term

$$\left[M_\parallel\right]_K^{K'} \left[M_\parallel\right]_Q^S \frac{\partial \left(\mathbb{A}_i^{KB} \left[M_\parallel\right]_B^Q\right)}{\partial X^S} + \left[M_\parallel\right]_K^{K'} \mathbb{A}_i^{AB} N_A \left[M_\parallel\right]_B^R \frac{\partial N^K}{\partial X^R}$$

appears in the irreducible representation for a surface second-order distribution, which we will get in the following as surface dual in work of virtual displacement.

We get the following chain of equalities:

$$\begin{aligned} &\left[M_\parallel\right]_K^{K'} \left(\left[M_\parallel\right]_Q^S \frac{\partial \left(\mathbb{A}_i^{KB} \left[M_\parallel\right]_B^Q\right)}{\partial X^S} + \mathbb{A}_i^{AB} N_A \left[M_\parallel\right]_B^R \frac{\partial N^K}{\partial X^R}\right) \\ &= \left[M_\parallel\right]_K^{K'} \left(\left[M_\parallel\right]_Q^S \frac{\partial \left(\left[M_\parallel\right]_B^Q\right)}{\partial X^S} \mathbb{A}_i^{KB} + \left[M_\parallel\right]_Q^S \left[M_\parallel\right]_B^Q \frac{\partial \mathbb{A}_i^{KB}}{\partial X^S} + \mathbb{A}_i^{AB} N_A \left[M_\parallel\right]_B^R \frac{\partial N^K}{\partial X^R}\right) \\ &= \left[M_\parallel\right]_K^{K'} \left(\left[M_\parallel\right]_Q^S \frac{\partial \left(-N^Q N_B\right)}{\partial X^S} \mathbb{A}_i^{KB} + \left[M_\parallel\right]_B^S \frac{\partial \mathbb{A}_i^{KB}}{\partial X^S} + \mathbb{A}_i^{AB} N_A \frac{\partial N^K}{\partial X^B}\right) \\ &= \left[M_\parallel\right]_K^{K'} \left(\left[M_\parallel\right]_Q^S N_B \frac{\partial \left(-N^Q\right)}{\partial X^S} \mathbb{A}_i^{KB} + \left[M_\parallel\right]_Q^S N^Q \frac{\partial \left(-N_B\right)}{\partial X^S} \mathbb{A}_i^{KB} + \left[M_\parallel\right]_B^S \frac{\partial \mathbb{A}_i^{KB}}{\partial X^S} + \mathbb{A}_i^{AB} N_A \frac{\partial N^K}{\partial X^B}\right) \\ &= \left[M_\parallel\right]_K^{K'} \left(-\frac{\partial N^Q}{\partial X^Q} \mathbb{A}_i^{KB} N_B + \left[M_\parallel\right]_B^S \frac{\partial \mathbb{A}_i^{KB}}{\partial X^S} + \mathbb{A}_i^{AB} N_A \frac{\partial N^K}{\partial X^B}\right) \\ &= \left[M_\parallel\right]_K^{K'} \left(\left[M_\parallel\right]_B^S \frac{\partial \mathbb{A}_i^{KB}}{\partial X^S} + \mathbb{A}_i^{AB} N_A \frac{\partial N^K}{\partial X^B} - \frac{\partial N^Q}{\partial X^Q} \mathbb{A}_i^{KB} N_B\right). \end{aligned} \tag{25}$$

3.4.5. *Another useful reduction*

The term

$$\left[M_\parallel\right]_Q^S \frac{\partial}{\partial X^S} \left(\mathbb{A}_i^{AB} N_A \left[M_\parallel\right]_B^Q\right) + N_K \left[M_\parallel\right]_Q^S \frac{\partial \left(\mathbb{A}_i^{KB} \left[M_\parallel\right]_B^Q\right)}{\partial X^S}$$

appears in the irreducible representation for a surface second-order distribution, which we will get in the following as surface dual in work of normal derivative of virtual displacement.

We get the following chain of equalities:

$$
\begin{aligned}
&[M_\parallel]_Q^S \frac{\partial}{\partial X^S}\left(\mathbb{A}_i^{AB} N_A [M_\parallel]_B^Q\right) + N_K [M_\parallel]_Q^S \frac{\partial\left(\mathbb{A}_i^{KB}[M_\parallel]_B^Q\right)}{\partial X^S} \\
&\qquad = [M_\parallel]_Q^S \frac{\partial}{\partial X^S}\left(\mathbb{A}_i^{AB} N_A [M_\parallel]_B^Q N^K\right) N_K + N_K [M_\parallel]_Q^S \frac{\partial\left(\mathbb{A}_i^{KB}[M_\parallel]_B^Q\right)}{\partial X^S} \\
&\qquad = N_K [M_\parallel]_Q^S \left(\frac{\partial\left(\mathbb{A}_i^{AB} N_A [M_\parallel]_B^Q N^K + \mathbb{A}_i^{KB}[M_\parallel]_B^Q\right)}{\partial X^S}\right).
\end{aligned}
\tag{26}
$$

3.4.6. *Derivatives of the vectors* **T**, **B** *and* **N**

As the vectors **T** and **B** have norm equal to one, we have

$$\frac{d\mathbf{T}}{dS}\cdot\mathbf{T}=0, \quad \frac{d\mathbf{B}}{dS}\cdot\mathbf{B}=0, \tag{27}$$

where S denotes the curvilinear abscissa along the boundary edge. Therefore we can deduce that:

$$T^A \frac{\partial T^E}{\partial X^A} T_E = [M_{L\parallel}]_E^A \frac{\partial T^E}{\partial X^A} = 0. \tag{28}$$

The reader will remark that, to introduce the gradient of **T**, it is necessary to smoothly prolong the field **T** in an open neighborhood of the curved edge, with constant value in any direction orthogonal to the edge: this requirement is met in the vicinity of any of its regular points. Using this constant orthogonal extension (i.e. wrt the edge tangent) and its analogue for the field **B**, it is also easy to verify that

$$B_{A'} \frac{\partial T^{A'}}{\partial X^E} = 0, \quad N_{A'} \frac{\partial T^{A'}}{\partial X^E} = 0, \tag{29}$$

$$B^E \frac{\partial B^{A'}}{\partial X^E} = 0, \quad N^E \frac{\partial B^{A'}}{\partial X^E} = 0. \tag{30}$$

A similar extension of the normal field to every face is also needed for defining its gradient. This strategy is used in Riemannian geometry to define the surface curvature tensor. It is simple to verify that:

$$\frac{\partial N_A}{\partial X_E} N^A = 0, \quad \frac{\partial N_A}{\partial X_E} N^E = 0. \tag{31}$$

4. Irreducible representation of second order surface distributions and some integration by parts formulas

When using the principle of virtual work it is essential to get the irreducible representation of virtual work functionals: i.e., their representation in which the lowest possible order of derivatives of test functions appears (see [15] and [47]). To this aim it is necessary to apply systematically the divergence theorem and the integration by parts techniques.

We explicitly warn the reader that the surface on which the distribution is concentrated is a piecewise regular surface Σ (as defined in the previous sections): therefore when we write $\partial\Sigma$ we mean the union of the boundaries of all the m faces Σ_r,with $r = 1, \cdots$, constituting Σ (see Figs. 1-3).

Therefore we will use systematically the following notation

$$\int_{\partial\Sigma} (\bullet)\, d\partial\Sigma := \sum_{r=1}^{m} \int_{\partial\Sigma_r} (\bullet)\, d\partial\Sigma_r.$$

4.1. *Projector-based decomposition of second order surface distributions*

In order to get its irreducible representation, we will need repeatedly to integrate by parts two times the following work functional, which is a second order distribution concentrated on a surface:

$$\int_{\Sigma} \mathbb{A}_i^{RS} \frac{\partial^2 \delta\Pi^i}{\partial X^R \partial X^S} d\Sigma, \tag{32}$$

where $\mathbb{A}_i^{RS}$ is a generic (symmetric!) tensor field defined in the neighbourhood of the surface Σ.

To do so, we start by decomposing the identity in the space of two times covariant tensors and considering the tensor product:

$$\begin{aligned} \delta_A^R \delta_B^S &= \left([M_\parallel]_A^R + N^R N_A \right)\left([M_\parallel]_B^S + N^S N_B \right) \\ &= [M_\parallel]_A^R [M_\parallel]_B^S + N^R N_A [M_\parallel]_B^S + N_B N^S [M_\parallel]_A^R + N_B N^S N^R N_A . \end{aligned} \tag{33}$$

The work functional (32) can be split as the sum of the three addends:

- completely parallel projection:

$$\int_{\Sigma} \mathbb{A}_i^{AB} [M_\parallel]_A^R [M_\parallel]_B^S \frac{\partial^2 \delta\Pi^i}{\partial X^R \partial X^S} d\Sigma; \tag{34}$$

- one time normal and one time parallel projection. This addend, because of the symmetry of $\mathbb{A}_i^{AB}$ and of the Hessian matrix of the field $\delta\Pi^i$, is equal to

$$\int_{\Sigma} \left(2\mathbb{A}_i^{AB} N_A [M_\parallel]_B^R \right) N^S \frac{\partial^2 \delta\Pi^i}{\partial X^R \partial X^S} d\Sigma, \tag{35}$$

which, by using the equality

$$N^S \frac{\partial^2 \delta\Pi^i}{\partial X^R \partial X^S} = \frac{\partial}{\partial X^R}\left(N^S \frac{\partial \delta\Pi^i}{\partial X^S} \right) - \frac{\partial N^S}{\partial X^R}\left(\frac{\partial \delta\Pi^i}{\partial X^S} \right) = \frac{\partial}{\partial X^R}\left(\frac{\partial \delta\Pi^i}{\partial N} \right) - \frac{\partial N^S}{\partial X^R}\left(\frac{\partial \delta\Pi^i}{\partial X^S} \right)$$

becomes

$$\int_{\Sigma} \left(2\mathbb{A}_i^{AB} N_A [M_\parallel]_B^R \right) \left(\frac{\partial}{\partial X^R}\left(\frac{\partial \delta\Pi^i}{\partial N} \right) - \frac{\partial N^S}{\partial X^R}\left(\frac{\partial \delta\Pi^i}{\partial X^S} \right) \right) d\Sigma; \tag{36}$$

- completely normal projection:

$$\int_{\Sigma} \left(\mathbb{A}_i^{AB} N_B N_A \right) N^S N^R \frac{\partial^2 \delta\Pi^i}{\partial X^R \partial X^S} d\Sigma. \tag{37}$$

4.1.1. *Irreducible representation of the first addend* (34)

To reduce the first addend (34) we need to integrate by parts and to apply the surface divergence theorem two times. We start with the needed chain of equalities, by applying the Leibniz rule and the surface divergence theorem and recalling idempotence of projectors:

$$\begin{aligned} &\int_{\Sigma} \left(\mathbb{A}_i^{AB} [M_\parallel]_A^R [M_\parallel]_B^Q \right) [M_\parallel]_Q^S \frac{\partial}{\partial X^S}\left(\frac{\partial \delta\Pi^i}{\partial X^R} \right) d\Sigma \\ &\quad = \int_{\Sigma} [M_\parallel]_Q^S \frac{\partial}{\partial X^S}\left(\mathbb{A}_i^{AB} [M_\parallel]_A^R [M_\parallel]_B^Q \frac{\partial \delta\Pi^i}{\partial X^R} \right) d\Sigma \\ &\quad\quad - \int_{\Sigma} [M_\parallel]_Q^S \frac{\partial}{\partial X^S}\left(\mathbb{A}_i^{AB} [M_\parallel]_A^R [M_\parallel]_B^Q \right)\left(\frac{\partial \delta\Pi^i}{\partial X^R} \right) d\Sigma \\ &\quad = \int_{\partial\Sigma} \left(B_B \mathbb{A}_i^{AB} [M_\parallel]_A^R \right) \frac{\partial \delta\Pi^i}{\partial X^R} d\partial\Sigma - \int_{\Sigma} [M_\parallel]_Q^S \frac{\partial}{\partial X^S}\left(\mathbb{A}_i^{AB} [M_\parallel]_A^R [M_\parallel]_B^Q \right)\left(\frac{\partial \delta\Pi^i}{\partial X^R} \right) d\Sigma = \end{aligned} \tag{38}$$

The chain of equalities continues by using the decompositions of identities Eqs. (2) and (9), namely

$$\begin{aligned}
&= \int_{\partial\Sigma} \left(B_B \mathbb{A}_i^{AB} \left[M_\parallel\right]_A^K\right) \left(T^R T_K + B^R B_K + N^R N_K\right) \frac{\partial \delta\Pi^i}{\partial X^R} d\partial\Sigma \\
&\quad - \int_\Sigma \left[M_\parallel\right]_Q^S \frac{\partial \left(\mathbb{A}_i^{AB} \left[M_\parallel\right]_A^K \left[M_\parallel\right]_B^Q\right)}{\partial X^S} \left(\left[M_\parallel\right]_K^R + N^R N_K\right) \left(\frac{\partial \delta\Pi^i}{\partial X^R}\right) d\Sigma \\
&= \int_{\partial\Sigma} \left(B_B \mathbb{A}_i^{AB} T_A\right) \left(\frac{\partial \delta\Pi^i}{\partial X^R} T^R\right) d\partial\Sigma \\
&\quad + \int_{\partial\Sigma} \left(B_B \mathbb{A}_i^{AB} B_K\right) \left(\frac{\partial \delta\Pi^i}{\partial B}\right) d\partial\Sigma \\
&\quad - \int_\Sigma N_K \left[M_\parallel\right]_Q^S \frac{\partial \left(\mathbb{A}_i^{AB} \left[M_\parallel\right]_A^K \left[M_\parallel\right]_B^Q\right)}{\partial X^S} \left(\frac{\partial \delta\Pi^i}{\partial N}\right) d\Sigma \\
&\quad - \int_\Sigma \left(\left[M_\parallel\right]_Q^S \frac{\partial \left(\mathbb{A}_i^{AB} \left[M_\parallel\right]_A^K \left[M_\parallel\right]_B^Q\right)}{\partial X^S}\right) \left[M_\parallel\right]_K^R \frac{\partial}{\partial X^R} \left(\delta\Pi^i\right) d\Sigma
\end{aligned} \tag{39}$$

where we have used the notation $\frac{\partial \delta\Pi^i}{\partial B} := \frac{\partial \delta\Pi^i}{\partial X^R} B^R$.

The obtained equality can be further simplified by using the equality (21)

$$\begin{aligned}
&= \int_{\partial\Sigma} \left(B_B \mathbb{A}_i^{AB} T_A\right) \left(\frac{\partial \delta\Pi^i}{\partial X^R} T^R\right) d\partial\Sigma + \int_{\partial\Sigma} \left(B_B \mathbb{A}_i^{AB} B_K\right) \left(\frac{\partial \delta\Pi^i}{\partial B}\right) d\partial\Sigma \\
&\quad - \int_\Sigma \left(N_K \left[M_\parallel\right]_Q^S \frac{\partial \left(\mathbb{A}_i^{KB} \left[M_\parallel\right]_B^Q\right)}{\partial X^S} - \left[M_\parallel\right]_Q^S \frac{\partial \left(\mathbb{A}_i^{AB} N_A \left[M_\parallel\right]_B^Q\right)}{\partial X^S}\right) \left(\frac{\partial \delta\Pi^i}{\partial N}\right) d\Sigma \\
&\quad - \int_\Sigma \left[M_\parallel\right]_{K'}^R \frac{\partial}{\partial X^R} \left(\left[M_\parallel\right]_K^{K'} \left[M_\parallel\right]_Q^S \frac{\partial \left(\mathbb{A}_i^{AB} \left[M_\parallel\right]_A^K \left[M_\parallel\right]_B^Q\right)}{\partial X^S} \delta\Pi^i\right) d\Sigma \\
&\quad + \int_\Sigma \left[M_\parallel\right]_{K'}^R \frac{\partial}{\partial X^R} \left(\left[M_\parallel\right]_K^{K'} \left[M_\parallel\right]_Q^S \frac{\partial \left(\mathbb{A}_i^{AB} \left[M_\parallel\right]_A^K \left[M_\parallel\right]_B^Q\right)}{\partial X^S}\right) \delta\Pi^i d\Sigma =
\end{aligned} \tag{40}$$

which becomes through the divergence theorem

$$\begin{aligned}
&= \int_{\partial\Sigma} \left(B_B \mathbb{A}_i^{AB} T_A\right) \left(\frac{\partial \delta\Pi^i}{\partial X^R} T^R\right) d\partial\Sigma + \int_{\partial\Sigma} \left(B_B \mathbb{A}_i^{AB} B_K\right) \left(\frac{\partial \delta\Pi^i}{\partial B}\right) d\partial\Sigma \\
&\quad - \int_\Sigma \left(N_K \left[M_\parallel\right]_Q^S \frac{\partial \left(\mathbb{A}_i^{KB} \left[M_\parallel\right]_B^Q\right)}{\partial X^S}\right) \left(\frac{\partial \delta\Pi^i}{\partial N}\right) d\Sigma \\
&\quad + \int_\Sigma \left(\left[M_\parallel\right]_Q^S \frac{\partial \left(\mathbb{A}_i^{AB} N_A \left[M_\parallel\right]_B^Q\right)}{\partial X^S}\right) \left(\frac{\partial \delta\Pi^i}{\partial N}\right) d\Sigma \\
&\quad - \int_{\partial\Sigma} \left(B_K \left[M_\parallel\right]_Q^S \frac{\partial \left(\mathbb{A}_i^{AB} \left[M_\parallel\right]_A^K \left[M_\parallel\right]_B^Q\right)}{\partial X^S}\right) \delta\Pi^i d\partial\Sigma \\
&\quad + \int_\Sigma \left[M_\parallel\right]_{K'}^R \frac{\partial}{\partial X^R} \left(\left[M_\parallel\right]_K^{K'} \left[M_\parallel\right]_Q^S \frac{\partial \left(\mathbb{A}_i^{AB} \left[M_\parallel\right]_A^K \left[M_\parallel\right]_B^Q\right)}{\partial X^S}\right) \delta\Pi^i d\Sigma.
\end{aligned} \tag{41}$$

The reduction of the first addend (34) is then continued by applying the Leibniz rule to the derivation along the edge and on the surface to get:

$$
\begin{aligned}
&= \int_{\partial\Sigma} T^R \frac{\partial}{\partial X^R}\left(B_B \mathbb{A}_i^{AB} T_A \delta\Pi^i\right) d\partial\Sigma - \int_{\partial\Sigma} T^R \frac{\partial}{\partial X^R}\left(B_B \mathbb{A}_i^{AB} T_A\right)\delta\Pi^i\, d\partial\Sigma \\
&+ \int_{\partial\Sigma}\left(B_B \mathbb{A}_i^{AB} B_K\right)\left(\frac{\partial\delta\Pi^i}{\partial B}\right) d\partial\Sigma - \int_{\Sigma}\left(N_K \left[M_\parallel\right]_Q^S \frac{\partial\left(\mathbb{A}_i^{KB}\left[M_\parallel\right]_B^Q\right)}{\partial X^S}\right)\left(\frac{\partial\delta\Pi^i}{\partial N}\right) d\Sigma \\
&+ \int_{\Sigma}\left(\left[M_\parallel\right]_Q^S \frac{\partial\left(\mathbb{A}_i^{AB} N_A \left[M_\parallel\right]_B^Q\right)}{\partial X^S}\right)\left(\frac{\partial\delta\Pi^i}{\partial N}\right) d\Sigma \\
&- \int_{\partial\Sigma}\left(B_K \left[M_\parallel\right]_Q^S \frac{\partial\left(\mathbb{A}_i^{KB}\left[M_\parallel\right]_B^Q\right)}{\partial X^S} - \mathbb{A}_i^{AB} N_A \left[M_\parallel\right]_B^S \frac{\partial N^K}{\partial X^S} B_K\right)\delta\Pi^i\, d\partial\Sigma \\
&+ \int_{\Sigma}\left[M_\parallel\right]_{K'}^R \frac{\partial}{\partial X^R}\left(\left[M_\parallel\right]_K^{K'}\left[M_\parallel\right]_Q^S \frac{\partial\left(\mathbb{A}_i^{KB}\left[M_\parallel\right]_B^Q\right)}{\partial X^S} - \mathbb{A}_i^{AB} N_A \left[M_\parallel\right]_B^S \left[M_\parallel\right]_K^{K'}\frac{\partial N^K}{\partial X^S}\right)\delta\Pi^i\, d\Sigma =
\end{aligned}
\tag{42}
$$

which, by applying again divergence theorem on edges and faces, becomes the searched irreducible representation

$$
\begin{aligned}
&= \sum_{\partial\partial\Sigma}\left(B_B \mathbb{A}_i^{AB} T_A \delta\Pi^i\right) \\
&- \int_{\partial\Sigma} T^R \frac{\partial}{\partial X^R}\left(B_B \mathbb{A}_i^{AB} T_A\right)\delta\Pi^i\, d\partial\Sigma + \int_{\partial\Sigma}\left(B_B \mathbb{A}_i^{AB} B_K\right)\left(\frac{\partial\delta\Pi^i}{\partial B}\right) d\partial\Sigma \\
&- \int_{\Sigma}\left(N_K \left[M_\parallel\right]_Q^S \frac{\partial\left(\mathbb{A}_i^{KB}\left[M_\parallel\right]_B^Q\right)}{\partial X^S}\right)\left(\frac{\partial\delta\Pi^i}{\partial N}\right) d\Sigma \\
&+ \int_{\Sigma}\left(\left[M_\parallel\right]_Q^S \frac{\partial\left(\mathbb{A}_i^{AB} N_A \left[M_\parallel\right]_B^Q\right)}{\partial X^S}\right)\left(\frac{\partial\delta\Pi^i}{\partial N}\right) d\Sigma \\
&- \int_{\partial\Sigma}\left(B_K \left[M_\parallel\right]_Q^S \frac{\partial\left(\mathbb{A}_i^{KB}\left[M_\parallel\right]_B^Q\right)}{\partial X^S}\right)\delta\Pi^i\, d\partial\Sigma + \int_{\partial\Sigma}\left(\mathbb{A}_i^{AB} N_A \left[M_\parallel\right]_B^S \frac{\partial N^K}{\partial X^S} B_K\right)\delta\Pi^i\, d\partial\Sigma \\
&+ \int_{\Sigma}\left[M_\parallel\right]_{K'}^R \frac{\partial}{\partial X^R}\left(\left[M_\parallel\right]_K^{K'}\left[M_\parallel\right]_Q^S \frac{\partial\left(\mathbb{A}_i^{KB}\left[M_\parallel\right]_B^Q\right)}{\partial X^S}\right)\delta\Pi^i\, d\Sigma \\
&- \int_{\Sigma}\left[M_\parallel\right]_{K'}^R \frac{\partial}{\partial X^R}\left(\mathbb{A}_i^{AB} N_A \left[M_\parallel\right]_B^S \left[M_\parallel\right]_K^{K'}\frac{\partial N^K}{\partial X^S}\right)\delta\Pi^i\, d\Sigma.
\end{aligned}
\tag{43}
$$

In the first line of the above equation we have used the following notation

$$
\sum_{\partial\partial\Sigma}(\varphi) = \sum_{\alpha=q}^{l}\left(\sum_{\beta=1}^{k_\alpha}\left(\lim \varphi\left(e_{\beta,\alpha}, P_\alpha\right)\right)\right), \tag{44}
$$

where: the symbol P_α denotes the α^{th} wedge among the q wedges in the piecewise regular surface Σ; the symbol $e_{\beta,\alpha}$ denotes the β^{th} edge among the k_α concurring in the wedge P_α; symbol $\lim\varphi(e_{\beta,\alpha}, P_\alpha)$ denotes the sum of all the limits of the quantity φ calculated along the edge $e_{\beta,\alpha}$ towards the point P_α when $e_{\beta,\alpha}$ is regarded as part of the boundary of all faces concurring on it (see Figs. 3 and 4).

4.1.2. *Irreducible representation of the second addend (Eq. 36)*

To complete the irreducible representation of Eq. (32), we must reduce the second addend in Eq. (36). This is done with the further splitting it into two addends, for which two chains of equalities can be developed.

The first of these chains of equalities is obtained by applying again and again the integration by parts and the surface divergence theorem, namely:

$$
\begin{aligned}
&\int_{\Sigma}\left(2\mathbb{A}_i^{AB}N_A\left[M_{\|}\right]_B^S\right)\frac{\partial}{\partial X^S}\left(\frac{\partial\delta\Pi^i}{\partial N}\right)d\Sigma=\int_{\Sigma}\left(2\mathbb{A}_i^{AB}N_A\left[M_{\|}\right]_B^Q\left[M_{\|}\right]_Q^S\frac{\partial}{\partial X^S}\left(\frac{\partial\delta\Pi^i}{\partial N}\right)\right)d\Sigma\\
&=\int_{\Sigma}\left[M_{\|}\right]_Q^S\frac{\partial}{\partial X^S}\left\{\left(2\mathbb{A}_i^{AB}N_A\left[M_{\|}\right]_B^Q\right)\left(\frac{\partial\delta\Pi^i}{\partial N}\right)\right\}d\Sigma+-\int_{\Sigma}\left[M_{\|}\right]_Q^S\frac{\partial}{\partial X^S}\left(2\mathbb{A}_i^{AB}N_A\left[M_{\|}\right]_B^Q\right)\left(\frac{\partial\delta\Pi^i}{\partial N}\right)d\Sigma\\
&=\int_{\partial\Sigma}\left(2\mathbb{A}_i^{AB}N_AB_B\right)\left(\frac{\partial\delta\Pi^i}{\partial N}\right)d\partial\Sigma+-\int_{\Sigma}\left[M_{\|}\right]_Q^S\frac{\partial}{\partial X^S}\left(2\mathbb{A}_i^{AB}N_A\left[M_{\|}\right]_B^Q\right)\left(\frac{\partial\delta\Pi^i}{\partial N}\right)d\Sigma.
\end{aligned}
\tag{45}
$$

The second chain of equality is, by recalling that $\frac{\partial N^S}{\partial X^R}N_S=0$,

$$
\begin{aligned}
&-\int_{\Sigma}\left\{\left(2\mathbb{A}_i^{AB}N_A\left[M_{\|}\right]_B^R\right)\frac{\partial N^S}{\partial X^R}\right\}\frac{\partial\delta\Pi^i}{\partial X^S}d\Sigma=-\int_{\Sigma}\left\{\left(2\mathbb{A}_i^{AB}N_A\left[M_{\|}\right]_B^R\right)\frac{\partial N^S}{\partial X^R}\right\}N_S\frac{\partial\delta\Pi^i}{\partial N}d\Sigma+\\
&-\int_{\Sigma}\left\{\left(2\mathbb{A}_i^{AB}N_A\left[M_{\|}\right]_B^R\right)\frac{\partial N^S}{\partial X^R}\right\}\left[M_{\|}\right]_S^Q\frac{\partial}{\partial X^Q}\left(\delta\Pi^i\right)d\Sigma\\
&=-\int_{\Sigma}\left[M_{\|}\right]_{S'}^Q\frac{\partial}{\partial X^Q}\left\{\left[M_{\|}\right]_S^{S'}\left(2\mathbb{A}_i^{AB}N_A\left[M_{\|}\right]_B^R\right)\frac{\partial N^S}{\partial X^R}\delta\Pi^i\right\}d\Sigma\\
&\quad+\int_{\Sigma}\left[M_{\|}\right]_{S'}^Q\frac{\partial}{\partial X^Q}\left\{\left[M_{\|}\right]_S^{S'}\left(2\mathbb{A}_i^{AB}N_A\left[M_{\|}\right]_B^R\right)\frac{\partial N^S}{\partial X^R}\right\}\delta\Pi^i d\Sigma\\
&=-\int_{\partial\Sigma}B_S\left(2\mathbb{A}_i^{AB}N_A\left[M_{\|}\right]_B^R\right)\frac{\partial N^S}{\partial X^R}\delta\Pi^i d\partial\Sigma\\
&\quad+\int_{\Sigma}\left[M_{\|}\right]_{S'}^Q\frac{\partial}{\partial X^Q}\left\{\left[M_{\|}\right]_S^{S'}\left(2\mathbb{A}_i^{AB}N_A\left[M_{\|}\right]_B^R\right)\frac{\partial N^S}{\partial X^R}\right\}\delta\Pi^i d\Sigma.
\end{aligned}
\tag{46}
$$

By adding the final expression for the two previous chains of equalities, we get for the second addend (36) the following irreducible representation:

$$
\begin{aligned}
&\int_{\Sigma}\left[M_{\|}\right]_{S'}^Q\frac{\partial}{\partial X^Q}\left\{\left[M_{\|}\right]_S^{S'}\left(2\mathbb{A}_i^{AB}N_A\left[M_{\|}\right]_B^R\right)\frac{\partial N^S}{\partial X^R}\right\}\delta\Pi^i d\Sigma\\
&\qquad+-\int_{\Sigma}\left[M_{\|}\right]_Q^S\frac{\partial}{\partial X^S}\left(2\mathbb{A}_i^{AB}N_A\left[M_{\|}\right]_B^Q\right)\left(\frac{\partial\delta\Pi^i}{\partial N}\right)d\Sigma+\\
&\qquad-\int_{\partial\Sigma}B_S\left(2\mathbb{A}_i^{AB}N_A\left[M_{\|}\right]_B^R\right)\frac{\partial N^S}{\partial X^R}\delta\Pi^i d\partial\Sigma+\int_{\partial\Sigma}\left(2\mathbb{A}_i^{AB}N_AB_B\right)\left(\frac{\partial\delta\Pi^i}{\partial N}\right)d\partial\Sigma
\end{aligned}
$$

4.1.3. *Irreducible representation of second-order surface distributions*

Gathering the results obtained in the previous subsections, we get the following irreducible representation of a second-order distribution concentrated on a surface:

$$
\begin{aligned}
&\int_{\Sigma} \mathbb{A}_i^{RS} \frac{\partial^2 \delta \Pi^i}{\partial X^R \partial X^S} d\Sigma = \\
&= \int_{\Sigma} \left(\mathbb{A}_i^{AB} N_B N_A\right) \frac{\partial^2 \delta \Pi^i}{\partial N^2} d\Sigma + \int_{\Sigma} \left[M_{\|}\right]_{K'}^{R} \frac{\partial}{\partial X^R} \left(\left[M_{\|}\right]_{K}^{K'} \left[M_{\|}\right]_{Q}^{S} \frac{\partial \left(\mathbb{A}_i^{KB} \left[M_{\|}\right]_{B}^{Q} \right)}{\partial X^S} \right) \delta \Pi^i d\Sigma \\
&- \int_{\Sigma} \left[M_{\|}\right]_{K'}^{R} \frac{\partial}{\partial X^R} \left(\mathbb{A}_i^{AB} N_A \left[M_{\|}\right]_{B}^{S} \left[M_{\|}\right]_{K}^{K'} \frac{\partial N^K}{\partial X^S} \right) \delta \Pi^i d\Sigma \\
&+ \int_{\Sigma} \left[M_{\|}\right]_{S'}^{Q} \frac{\partial}{\partial X^Q} \left\{ \left[M_{\|}\right]_{S}^{S'} \left(2\mathbb{A}_i^{AB} N_A \left[M_{\|}\right]_{B}^{R} \right) \frac{\partial N^S}{\partial X^R} \right\} \delta \Pi^i d\Sigma \\
&+ \int_{\Sigma} \left(\left[M_{\|}\right]_{Q}^{S} \frac{\partial \left(\mathbb{A}_i^{AB} N_A \left[M_{\|}\right]_{B}^{Q} \right)}{\partial X^S} \right) \left(\frac{\partial \delta \Pi^i}{\partial N} \right) d\Sigma - \int_{\Sigma} \left[M_{\|}\right]_{Q}^{S} \frac{\partial}{\partial X^S} \left(2\mathbb{A}_i^{AB} N_A \left[M_{\|}\right]_{B}^{Q} \right) \left(\frac{\partial \delta \Pi^i}{\partial N} \right) d\Sigma \\
&- \int_{\Sigma} \left(N_K \left[M_{\|}\right]_{Q}^{S} \frac{\partial \left(\mathbb{A}_i^{KB} \left[M_{\|}\right]_{B}^{Q} \right)}{\partial X^S} \right) \left(\frac{\partial \delta \Pi^i}{\partial N} \right) d\Sigma + \int_{\partial\Sigma} \left(\mathbb{A}_i^{AB} N_A \left[M_{\|}\right]_{B}^{S} \frac{\partial N^K}{\partial X^S} B_K \right) \delta \Pi^i d\partial\Sigma \\
&- \int_{\partial\Sigma} B_S \left(2\mathbb{A}_i^{AB} N_A \left[M_{\|}\right]_{B}^{R} \right) \frac{\partial N^S}{\partial X^R} \delta \Pi^i d\partial\Sigma - \int_{\partial\Sigma} \left(B_K \left[M_{\|}\right]_{Q}^{S} \frac{\partial \left(\mathbb{A}_i^{KB} \left[M_{\|}\right]_{B}^{Q} \right)}{\partial X^S} \right) \delta \Pi^i d\partial\Sigma \\
&- \int_{\partial\Sigma} T^R \frac{\partial}{\partial X^R} \left(B_B \mathbb{A}_i^{AB} T_A \right) \delta \Pi^i d\partial\Sigma + \int_{\partial\Sigma} \left(B_B \mathbb{A}_i^{AB} B_K \right) \left(\frac{\partial \delta \Pi^i}{\partial B} \right) d\partial\Sigma \\
&+ \int_{\partial\Sigma} \left(2\mathbb{A}_i^{AB} N_A B_B \right) \left(\frac{\partial \delta \Pi^i}{\partial N} \right) d\partial\Sigma + \sum_{\partial\partial\Sigma} \left(B_B \mathbb{A}_i^{AB} T_A \delta \Pi^i \right).
\end{aligned}
\tag{47}
$$

By gathering and simplifying one gets:

$$
\begin{aligned}
&\int_{\Sigma} \mathbb{A}_i^{RS} \frac{\partial^2 \delta \Pi^i}{\partial X^R \partial X^S} d\Sigma \\
&\quad = \int_{\Sigma} \left(\mathbb{A}_i^{AB} N_B N_A\right) \frac{\partial^2 \delta \Pi^i}{\partial N^2} d\Sigma + \int_{\Sigma} \left[M_{\|}\right]_{K'}^{R} \frac{\partial}{\partial X^R} \left(\left[M_{\|}\right]_{K}^{K'} \left[M_{\|}\right]_{Q}^{S} \frac{\partial \left(\mathbb{A}_i^{KB} \left[M_{\|}\right]_{B}^{Q} \right)}{\partial X^S} \right) \delta \Pi^i d\Sigma \\
&\quad + \int_{\Sigma} \left[M_{\|}\right]_{S'}^{Q} \frac{\partial}{\partial X^Q} \left\{ \left[M_{\|}\right]_{S}^{S'} \mathbb{A}_i^{AB} N_A \left[M_{\|}\right]_{B}^{R} \frac{\partial N^S}{\partial X^R} \right\} \delta \Pi^i d\Sigma \\
&\quad - \int_{\Sigma} \left[M_{\|}\right]_{Q}^{S} \frac{\partial}{\partial X^S} \left(\mathbb{A}_i^{AB} N_A \left[M_{\|}\right]_{B}^{Q} \right) \left(\frac{\partial \delta \Pi^i}{\partial N} \right) d\Sigma \\
&\quad - \int_{\Sigma} \left(N_K \left[M_{\|}\right]_{Q}^{S} \frac{\partial \left(\mathbb{A}_i^{KB} \left[M_{\|}\right]_{B}^{Q} \right)}{\partial X^S} \right) \left(\frac{\partial \delta \Pi^i}{\partial N} \right) d\Sigma \\
&\quad - \int_{\partial\Sigma} \left(\mathbb{A}_i^{AB} N_A \left[M_{\|}\right]_{B}^{R} \right) \frac{\partial N^S}{\partial X^R} B_S \delta \Pi^i d\partial\Sigma - \int_{\partial\Sigma} \left(B_K \left[M_{\|}\right]_{Q}^{S} \frac{\partial \left(\mathbb{A}_i^{KB} \left[M_{\|}\right]_{B}^{Q} \right)}{\partial X^S} \right) \delta \Pi^i d\partial\Sigma \\
&\quad - \int_{\partial\Sigma} T^R \frac{\partial}{\partial X^R} \left(B_B \mathbb{A}_i^{AB} T_A \right) \delta \Pi^i d\partial\Sigma + \int_{\partial\Sigma} \left(B_B \mathbb{A}_i^{AB} B_K \right) \left(\frac{\partial \delta \Pi^i}{\partial B} \right) d\partial\Sigma \\
&\quad + \int_{\partial\Sigma} \left(2\mathbb{A}_i^{AB} N_A B_B \right) \left(\frac{\partial \delta \Pi^i}{\partial N} \right) d\partial\Sigma + \sum_{\partial\partial\Sigma} \left(B_B \mathbb{A}_i^{AB} T_A \delta \Pi^i \right).
\end{aligned}
\tag{48}
$$

A final version, obtained using the equation (25), is

$$
\begin{aligned}
&\int_{\Sigma} \mathbb{A}_i^{RS} \frac{\partial^2 \delta\Pi^i}{\partial X^R \partial X^S} d\Sigma \\
&= \int_{\Sigma} \left(\mathbb{A}_i^{AB} N_B N_A\right) \frac{\partial^2 \delta\Pi^i}{\partial N^2} d\Sigma \\
&+ \int_{\Sigma} [M_\|]^R_{K'} \frac{\partial}{\partial X^R} \left\{ [M_\|]^{K'}_K \left([M_\|]^S_B \frac{\partial \mathbb{A}_i^{KB}}{\partial X^S} + \mathbb{A}_i^{AB} N_A \frac{\partial N^K}{\partial X^B} - \frac{\partial N^Q}{\partial X^Q} \mathbb{A}_i^{KB} N_B \right) \right\} \delta\Pi^i d\Sigma \\
&- \int_{\Sigma} \left\{ N_K [M_\|]^S_Q \left(\frac{\partial \left(\mathbb{A}_i^{AB} N_A [M_\|]^Q_B N^K + \mathbb{A}_i^{KB} [M_\|]^Q_B \right)}{\partial X^S} \right) \right\} \left(\frac{\partial \delta\Pi^i}{\partial N} \right) d\Sigma \\
&- \int_{\partial\Sigma} \left\{ \mathbb{A}_i^{AB} N_A \frac{\partial N^S}{\partial X^B} B_S + B_K [M_\|]^S_Q \frac{\partial \left(\mathbb{A}_i^{KB} [M_\|]^Q_B \right)}{\partial X^S} + T^R \frac{\partial}{\partial X^R} \left(B_B \mathbb{A}_i^{AB} T_A \right) \right\} \delta\Pi^i d\partial\Sigma \\
&+ \int_{\partial\Sigma} \left(B_B \mathbb{A}_i^{AB} B_K \right) \left(\frac{\partial \delta\Pi^i}{\partial B} \right) d\partial\Sigma + \int_{\partial\Sigma} \left(2\mathbb{A}_i^{AB} N_A B_B \right) \left(\frac{\partial \delta\Pi^i}{\partial N} \right) d\partial\Sigma + \sum_{\partial\partial\Sigma} \left(B_B \mathbb{A}_i^{AB} T_A \delta\Pi^i \right).
\end{aligned}
\tag{49}
$$

5. Piola Transformations of the shapes of differential boundaries of reference into current configurations

In this section we recall the Piola transformation formulas for the vectors characterizing the shapes of the configurations boundaries as needed for third gradient continua: they were already used in [19, 41].

5.1. *The Eulerian-Lagrangian transformation formulas for the edge tangent vector*

The regularity assumptions about placement imply that:

$$
t^r = \frac{F^r_R T^R}{\|\mathbf{F}\mathbf{T}\|} = F^r_R T^R \left\|\mathbf{F}^{-1}\mathbf{t}\right\|, \tag{50}
$$

$$
t_r = g_{rs} t^s = g_{rs} \frac{F^s_R T^R}{\|\mathbf{F}\mathbf{T}\|} = g_{rs} \frac{F^s_R G^{RS} T_S}{\|\mathbf{F}\mathbf{T}\|}. \tag{51}
$$

5.2. *The transformation of the normal vector to a boundary face from reference to current configurations*

It is very well-known in the literature (see e.g. [16, 19] also for a detailed reference to the literature)[2] that

$$
n_r = \frac{\left(\mathbf{F}^{-1}\right)^R_r N_R}{\left\|\mathbf{F}^{-T}\mathbf{N}\right\|} = \left(\mathbf{F}^{-1}\right)^R_r \left\|\mathbf{F}^T \mathbf{n}\right\| N_R, \tag{52}
$$

$$
n^r = g^{rs} n_s = \frac{g^{rs} \left(\mathbf{F}^{-1}\right)^R_r g_{RS}}{\|\mathbf{F}^{-T}\mathbf{N}\|} N^S. \tag{53}
$$

[2]For some " sociological reasons" this formula is attributed to Nanson: in fact, it appears already in the works by Piola [3]. How it could be possible to attribute to Piola the transformation of Piola-Lagrange stress into Cauchy-Euler stress and imagine that the transformation formula for the normals to Cauchy cuts was found some decades later is rather difficult to justify.

5.3. *The transformation of contravariant and covariant edge tangent normal*

The Piola transformations for edge tangent normal were found in [19, 41]. They are given by the formulas:

$$b_r = \left\{ \left(\mathbf{F}^{-1}\right)_r^R B_R - \frac{G^{\star LM} B_L N_M}{G^{\star RS} N_S N_R} \left(\mathbf{F}^{-1}\right)_r^R N_R \right\} \frac{\left\|\mathbf{F}^{-T}\mathbf{N}\right\|_g}{\left\|J^{-1}\mathbf{F}\mathbf{T}\right\|_g}; \tag{54}$$

$$b^r = \left\{ F_R^r B^R - \frac{G^{\star}_{LM} B^L T^M}{G^{\star}_{RS} T^R T^S} F_R^r T^R \right\} \frac{\|\mathbf{F}\mathbf{T}\|_g}{\left\|J\mathbf{F}^{-T}\mathbf{N}\right\|_g}. \tag{55}$$

One can easily verify the second formula by imposing that $b_r b^r = 1$, $b_r n^r = 0$, $b_r t^r = 1$.

6. Piola Transformations of external loads sustainable by third gradient continua

Gabrio Piola was the first to introduce n^{th} gradient continua (see [3, 4, 39]), considering internal work functionals that later will be recognized as n^{th} order distributions. However he did not manage to characterize the external loads sustainable by these generalized continua: his last work [2] was published posthumous.[3]

In fact, Piola did manage to study the transformations named after him in the case of first gradient continua: this was necessary to compare his variational deduction with the one, based on force and torque balance laws, put forward by Cauchy.

Here we present, based on some results from Riemannian differential geometry, the transformations that gives Lagrangian sustainable external loads in terms of their Eulerian counterparts, in the case of third gradient continua. The novelty peculiarity of these continua consists in the fact that they can sustain contact forces concentrated on wedges of their differential boundaries.

6.1. *Piola Transformation of external volume and surface-contact forces*

As appearing in the representation formula Eq. (1), third gradient continua can sustain external volume and surface-contact forces. No further difficulties, when comparing the third gradient Piola transformation with the first gradient one, appear in this cases.

The expression for the Eulerian volume work functional is easily transformed into a corresponding Lagrangian work functional. This can be done without too much difficulties. In fact:

$$\int_\omega f^{\text{ext}}_{\omega i}\, \delta\Pi^i \, d\omega = \int_\Omega J\, f^{\text{ext}}_{\omega i}(\Pi(X))\, \delta\Pi^i \, d\Omega, \tag{56}$$

The field $f^{\text{ext}}_{\omega i}(\Pi(X))$ results to be defined in the Lagrangian configuration: this change of variables for the Eulerian fields will not be explicitly indicated in what follows.

Hence, Eulerian external volume forces are transformed into Lagrangian volume forces only

$$\mathscr{F}^{\text{ext}}_{\Omega i} = J\, f^{\text{ext}}_{\omega i}. \tag{57}$$

This transformation is exactly the same that has been originally calculated by Piola for first gradient (or Cauchy) continua.

Also the Eulerian surface external forces are easily transformed into Lagrangian external forces by using the transformation formula for the area elements (see e.g. [16, 19, 43]), namely

$$d\sigma = \left\|J\mathbf{F}^{-T}\mathbf{N}\right\| \, d\Sigma, \tag{58}$$

[3] The mathematical genealogy starting from Piola has, indeed, produced Levi-Civita absolute tensor calculus and some important parts of modern differential geometry, see [48].

in the expression for external work functional as follows:

$$\int_\sigma f^{\text{ext}}_{\sigma i}\,\delta\Pi^i\,d\sigma = \int_\Sigma \left\|J\mathbf{F}^{-T}\mathbf{N}\right\| f^{\text{ext}}_{\Sigma i}\,\delta\Pi^i\,d\Sigma. \tag{59}$$

External Eulerian surface contact forces are transformed into Lagrangian surface contact forces only

$$\mathscr{F}^{\text{ext}}_{\Sigma i} = \left\|J\mathbf{F}^{-T}\mathbf{N}\right\| f^{\text{ext}}_{\sigma i}. \tag{60}$$

6.2. *Piola Transformation of external surface double force*

Among the loads, which can be sustained by third and second gradient continua but not by first gradient continua, one finds surface external double forces: surface double forces expend work on the normal derivative of the virtual displacement vector.

The corresponding Eulerian work functional is expressed as:

$$\int_\sigma f^{\text{ext}}_{\sigma n i}\,\frac{\partial\delta\Pi^i}{\partial n}\,d\sigma. \tag{61}$$

Using the equation (58), the rule for the derivation of composed functions and also Eq. (52), we obtain that the previous work functional can be represented as follows:

$$= \int_\Sigma \left\|J\mathbf{F}^{-T}\mathbf{N}\right\| f^{\text{ext}}_{\sigma n i}\, g^{rs}\,\frac{\left(\mathbf{F}^{-1}\right)^Q_s N_Q}{\left\|\mathbf{F}^{-T}\mathbf{N}\right\|}\left(\mathbf{F}^{-1}\right)^R_r\frac{\partial\delta\Pi^i}{\partial X^R}\,d\Sigma.$$

By simplifying and recalling the definition (15) we get that Eq. (61) can be transformed into:

$$\int_\Sigma J f^{\text{ext}}_{\sigma n i}\,N^{\star R}\,\frac{\partial\delta\Pi^i}{\partial X^R}\,d\Sigma. \tag{62}$$

The definition (15) clearly indicates that the distribution representing Eulerian surface external double forces does NOT transforms, according to the Lagrangian description, into the work functional corresponding to a Lagrangian surface external double forces: in fact, in general, $N^{\star R} \neq N^R$.

To decompose the expression (62), into its "elementary" Lagrangian "lower-order" functionals, we start recalling that

$$[M_\perp]^S_R = N^S N_R, \quad [M_\parallel]^S_R + N^S N_R = \delta^S_R, \tag{63}$$

so that it is equal to

$$\int_\Sigma J f^{\text{ext}}_{\sigma n i}\,N^{\star R}\left([M_\parallel]^S_R + N^S N_R\right)\frac{\partial\delta\Pi^i}{\partial X^S}\,d\Sigma.$$

By recalling that $N^{\star R}N_R = \|\mathbf{F}^{-T}\mathbf{N}\|^2$, through integration by parts the work functional (61) is equal to

$$\int_\Sigma \left\|J\mathbf{F}^{-T}\mathbf{N}\right\|^2 f^{\text{ext}}_{\sigma n i}\,\frac{\partial\delta\Pi^i}{\partial N}\,d\Sigma + \int_\Sigma [M_\parallel]^S_{R'}\,\frac{\partial}{\partial X^S}\left\{J f^{\text{ext}}_{\sigma n i}\,N^{\star R}\,\delta\Pi^i\,[M_\parallel]^{R'}_R\right\}d\Sigma +$$
$$-\int_\Sigma [M_\parallel]^S_{R'}\,\frac{\partial}{\partial X^S}\left\{J f^{\text{ext}}_{\sigma n i}\,N^{\star R}\,[M_\parallel]^{R'}_R\right\}\delta\Pi^i\,d\Sigma.$$

Applying the divergence theorem to the faces constituting Σ (see e.g. [16,20,49]), we finally get

$$\int_\sigma f^{\text{ext}}_{\sigma n i}\,\frac{\partial\delta\Pi^i}{\partial n}\,d\sigma = \int_\Sigma \left\{\left\|J\mathbf{F}^{-T}\mathbf{N}\right\|^2 f^{\text{ext}}_{\sigma n i}\right\}\frac{\partial\delta\Pi^i}{\partial N}\,d\Sigma + \int_{\partial\Sigma}\left\{J f^{\text{ext}}_{\Sigma n i}\,N^{\star R}B_R\right\}\delta\Pi^i\,d\partial\Sigma$$
$$-\int_\Sigma \left\{[M_\parallel]^S_{R'}\,\frac{\partial}{\partial X^S}\left(J f^{\text{ext}}_{\Sigma n i}\,N^{\star R}\,[M_\parallel]^{R'}_R\right)\right\}\delta\Pi^i\,d\Sigma.$$

In the last three integral expressions, we have grouped in brackets the dual in work of virtual displacement $\delta\Pi^i$ (i.e. line and surface densities of force) and the dual in work of the normal derivative of virtual displacement $\frac{\partial\delta\Pi^i}{\partial N}$ ($i.e.$ surface density of double force).

We can conclude that the Eulerian double force surface density $f^{\text{ext}}_{\sigma n i}$ external work functional, when transformed into the Lagrangian description, is equivalent to the triple of work functionals corresponding to the external actions: i) surface density of double force density, ii) surface density of force and iii) line density of force.

The Lagrangian external forces corresponding to the Eulerian double force $f^{\text{ext}}_{\sigma n i}$ are given by the following three identities:

$$\begin{aligned} &\mathscr{F}^{\text{ext}}_{\Sigma N}\left(f^{\text{ext}}_{\sigma n}\right) = \left\|\mathbf{F}^{-T}\mathbf{N}\right\|^2 J f^{\text{ext}}_{\sigma n}; \\ &\mathscr{F}^{\text{ext}}_{\Sigma}\left(f^{\text{ext}}_{\sigma n}\right) = -\left[M_{\|}\right]^S_{R'} \frac{\partial}{\partial X^S}\left\{J f^{\text{ext}}_{\sigma n} N^{\star R}\left[M_{\|}\right]^{R'}_R\right\}; \\ &\mathscr{F}^{\text{ext}}_{L}\left(f^{\text{ext}}_{\sigma n}\right) = J f^{\text{ext}}_{\sigma n}\left(B_R N^{\star R}\right). \end{aligned} \tag{64}$$

6.3. *Piola Transformation of surface external triple force*

In this subsection the Eulerian surface triple external force $f^{\text{ext}}_{\sigma n n i}$ density work functional is considered. It is peculiar of third or higher gradient continua and expends work on the second normal derivative of the virtual displacements:

$$\int_\sigma f^{\text{ext}}_{\sigma n n i} \frac{\partial^2 \delta \Pi^i}{\partial n^2}\, d\sigma. \tag{65}$$

6.3.1. *Eulerian surface triple force functional transformed into Lagrangian coordinates*

This work functional, by using the placement induced change of variables, becomes

$$\int_\Sigma \left(\left\|J\mathbf{F}^{-T}\mathbf{N}\right\| f^{\text{ext}}_{\sigma n n i}\, n^r n^s \frac{\partial^2 \delta\chi^i}{\partial x^r \partial x^s}\right) d\Sigma.$$

We then use the Piola transformation formula for the covariant normal n_r to get

$$\int_\Sigma \left(\left\|J\mathbf{F}^{-T}\mathbf{N}\right\| f^{\text{ext}}_{\sigma n n i}\, g^{rt} \frac{\left(\mathbf{F}^{-1}\right)^Q_t N_Q}{\left\|\mathbf{F}^{-T}\mathbf{N}\right\|} g^{sv} \frac{\left(\mathbf{F}^{-1}\right)^V_v N_V}{\left\|\mathbf{F}^{-T}\mathbf{N}\right\|} \left(\mathbf{F}^{-1}\right)^R_r \left(\mathbf{F}^{-1}\right)^S_s \frac{\partial^2 \delta\Pi^i}{\partial X^R \partial X^S}\right) d\Sigma.$$

which, after simplifications and using the definition of $N^{\star R}$, becomes:

$$\int_\Sigma \left\{\frac{J f^{\text{ext}}_{\sigma n n i}}{\left\|\mathbf{F}^{-T}\mathbf{N}\right\|} N^{\star A} N^{\star B}\right\} \frac{\partial^2 \delta\Pi^i}{\partial X^A \partial X^B} d\Sigma. \tag{66}$$

Since $N^{\star R} \neq N^R$, and exactly as it is true for double forces, we must remark that Eulerian surface triple force work functionals do NOT correspond to Lagrangian triple force work functionals only.

6.3.2. *Application of the general irreducible representation formula* (49)*: Lagrangian generalized forces associated to applied Eulerian external surface triple force density*

By equating

$$\mathbb{A}^{RS}_i = \frac{J f^{\text{ext}}_{\sigma n n i}}{\left\|\mathbf{F}^{-T}\mathbf{N}\right\|} N^{\star R} N^{\star S}$$

the general formula Eq. (49) derived above gives the following list of Lagrangian generalized forces corresponding to the Eulerian triple force surface density $f^{\text{ext}}_{\sigma n n i}$.

- Lagrangian surface triple force density

$$\mathscr{F}^{\text{ext}}_{\Sigma NN}\left(f^{\text{ext}}_{\sigma n n}\right) = \frac{J f^{\text{ext}}_{\sigma n n}}{\left\|\mathbf{F}^{-T}\mathbf{N}\right\|} \quad N^{\star R} N^{\star S} N_R N_S,$$

which becomes, by using the identity $N_R N^{\star R} = \|\mathbf{F}^{-T}\mathbf{N}\|^2$

$$\mathscr{F}^{\text{ext}}_{\Sigma NN}\left(f^{\text{ext}}_{\sigma n n}\right) = J f^{\text{ext}}_{\sigma n n} \left\|\mathbf{F}^{-T}\mathbf{N}\right\|^3; \tag{67}$$

- Lagrangian surface force density

By considering the dual in work of virtual displacement in the surface work functional we get:

$$\mathscr{F}_{\Sigma}^{\text{ext}}\left(f_{\sigma nn}^{\text{ext}}\right) = \left[M_{\|}\right]_{K'}^{R} \frac{\partial}{\partial X^{R}} \left\{ \left[M_{\|}\right]_{K}^{K'} \left(\left[M_{\|}\right]_{B}^{S} \frac{\partial \left(\frac{J f_{\sigma nn}^{\text{ext}}}{\|\mathbf{F}^{-T}\mathbf{N}\|} N^{\star K} N^{\star B} \right)}{\partial X^{S}} + \frac{J f_{\sigma nn}^{\text{ext}}}{\|\mathbf{F}^{-T}\mathbf{N}\|} N^{\star A} N^{\star B} N_{A} \frac{\partial N^{K}}{\partial X^{B}} - \frac{\partial N^{Q}}{\partial X^{Q}} \left(\frac{J f_{\sigma nn}^{\text{ext}}}{\|\mathbf{F}^{-T}\mathbf{N}\|} N^{\star K} N^{\star B} \right) N_{B} \right) \right\}. \quad (68)$$

By simplifying we get:

$$\mathscr{F}_{\Sigma}^{\text{ext}}\left(f_{\sigma nn}^{\text{ext}}\right) = \left[M_{\|}\right]_{K'}^{R} \frac{\partial}{\partial X^{R}} \left\{ \left[M_{\|}\right]_{K}^{K'} \left(\left[M_{\|}\right]_{B}^{S} \frac{\partial \left(\frac{J f_{\sigma nn}^{\text{ext}}}{\|\mathbf{F}^{-T}\mathbf{N}\|} N^{\star K} N^{\star B} \right)}{\partial X^{S}} + + \left\|\mathbf{F}^{-T}\mathbf{N}\right\| N^{\star B} \frac{\partial N^{K}}{\partial X^{B}} J f_{\sigma nn}^{\text{ext}} - \left\|\mathbf{F}^{-T}\mathbf{N}\right\| \frac{\partial N^{Q}}{\partial X^{Q}} \left(J f_{\sigma nn}^{\text{ext}} N^{\star K} \right) \right) \right\}. \quad (69)$$

- Lagrangian surface double force density

By considering the dual in work of normal derivative of virtual displacement in the surface work functional, we get:

$$\mathscr{F}_{\Sigma N}^{\text{ext}}\left(f_{\sigma nn}^{\text{ext}}\right) = - N_{K} \left[M_{\|}\right]_{Q}^{S} \left(\frac{\partial \left(\left(\frac{J f_{\sigma nn}^{\text{ext}}}{\|\mathbf{F}^{-T}\mathbf{N}\|} N^{\star A} N^{\star B} \right) N_{A} \left[M_{\|}\right]_{B}^{Q} N^{K} + \left(\frac{J f_{\sigma nn}^{\text{ext}}}{\|\mathbf{F}^{-T}\mathbf{N}\|} N^{\star K} N^{\star B} \right) \left[M_{\|}\right]_{B}^{Q} \right)}{\partial X^{S}} \right). \quad (70)$$

By simplifying we get:

$$\mathscr{F}_{\Sigma N}^{\text{ext}}\left(f_{\sigma nn}^{\text{ext}}\right) = - N_{K} \left[M_{\|}\right]_{Q}^{S} \left(\frac{\partial \left(J f_{\sigma nn}^{\text{ext}} \left\|\mathbf{F}^{-T}\mathbf{N}\right\| \left[M_{\|}\right]_{B}^{Q} N^{\star B} \left(N^{K} + \frac{N^{\star K}}{\|\mathbf{F}^{-T}\mathbf{N}\|^{2}} \right) \right)}{\partial X^{S}} \right). \quad (71)$$

- Lagrangian edge force density

By considering the dual in work of virtual displacement in the edge work functional we get:

$$\mathscr{F}_{\partial\Sigma}^{\text{ext}}\left(f_{\sigma nn}^{\text{ext}}\right) = - \left(\frac{J f_{\sigma nn}^{\text{ext}}}{\|\mathbf{F}^{-T}\mathbf{N}\|} N^{\star A} N^{\star B} \right) N_{A} \frac{\partial N^{S}}{\partial X^{B}} B_{S} - B_{K} \left[M_{\|}\right]_{Q}^{S} \frac{\partial \left(\left(\frac{J f_{\sigma nn}^{\text{ext}}}{\|\mathbf{F}^{-T}\mathbf{N}\|} N^{\star K} N^{\star B} \right) \left[M_{\|}\right]_{B}^{Q} \right)}{\partial X^{S}} + - T^{R} \frac{\partial}{\partial X^{R}} \left(T_{A} B_{B} \left(\frac{J f_{\sigma nn}^{\text{ext}}}{\|\mathbf{F}^{-T}\mathbf{N}\|} N^{\star A} N^{\star B} \right) \right). \quad (72)$$

By simplifying we get

$$\mathscr{F}_{\partial\Sigma}^{\text{ext}}\left(f_{\sigma nn}^{\text{ext}}\right) = - J f_{\sigma nn}^{\text{ext}} \left\|\mathbf{F}^{-T}\mathbf{N}\right\| N^{\star B} \frac{\partial N^{S}}{\partial X^{B}} B_{S} - B_{K} \left[M_{\|}\right]_{Q}^{S} \frac{\partial \left(\frac{J f_{\sigma nn}^{\text{ext}}}{\|\mathbf{F}^{-T}\mathbf{N}\|} N^{\star K} N^{\star B} \left[M_{\|}\right]_{B}^{Q} \right)}{\partial X^{S}} + - T^{R} \frac{\partial}{\partial X^{R}} \left(\frac{J f_{\sigma nn}^{\text{ext}}}{\|\mathbf{F}^{-T}\mathbf{N}\|} N^{\star A} T_{A} N^{\star B} B_{B} \right). \quad (73)$$

- Lagrangian edge double force densities

By considering the dual in work of the normal derivatives of virtual displacement along the directions B and N in the edge work functional we get:

$$\mathscr{F}^{\text{ext}}_{\partial\Sigma B}\left(f^{\text{ext}}_{\sigma nn}\right) = \frac{J f^{\text{ext}}_{\sigma nn}}{\left\|\mathbf{F}^{-T}\mathbf{N}\right\|}\left(N^{\star K}B_K\right)^2;$$
$$\mathscr{F}^{\text{ext}}_{\partial\Sigma N}\left(f^{\text{ext}}_{\sigma nn}\right) = 2J f^{\text{ext}}_{\sigma nn}\left\|\mathbf{F}^{-T}\mathbf{N}\right\| N^{\star B}B_B.$$

- Lagrangian wedge concentrated forces

By considering the dual in work of the virtual displacement in the wedge work functional we get:

$$\mathscr{F}^{\text{ext}}_{P}\left(f^{\text{ext}}_{\sigma nn}\right) = \sum_{\partial\partial\Sigma} T_A B_B \left(\frac{J f^{\text{ext}}_{\sigma nn}}{\left\|\mathbf{F}^{-T}\mathbf{N}\right\|} N^{\star A}N^{\star B}\right).$$

6.4. *Piola transformation of edge contact forces*

The Eulerian edge external work functional, relative to the line force density f^{ext}_{li}, is easily transformed directly into the Lagrangian edge work functional. In fact one finds

$$\int_l f^{\text{ext}}_{li}\,\delta\Pi^i\,dl = \int_L \|\mathbf{FT}\|_g f^{\text{ext}}_{li}\,\delta\Pi^i\,dL, \tag{74}$$

from which we get

$$\mathscr{F}^{\text{ext}}_{L}\left(f^{\text{ext}}_{li}\right) = \|\mathbf{FT}\|_g f^{\text{ext}}_{l}.$$

6.5. *Piola Transformation of edge double force f^{ext}_{lni}*

The Eulerian external work functional relative to the edge double force f^{ext}_{lni} can be transformed as follows:

$$\int_l f^{\text{ext}}_{lni}\,g^{rs}n_s\frac{\partial\delta\Pi^i}{\partial x^r}\,dl = \int_L \|\mathbf{FT}\|_g f^{\text{ext}}_{lni}\,g^{rs}\frac{\left(\mathbf{F}^{-1}\right)^Q_s N_Q}{\left\|\mathbf{F}^{-T}\mathbf{N}\right\|_g}\left(\mathbf{F}^{-1}\right)^S_r\frac{\partial\delta\Pi^i}{\partial X^S}\,dL. \tag{75}$$

By decomposing the identity as done in (9), applying the divergence theorem for the curved edge, the above functional becomes:

$$\begin{aligned}\int_L \|\mathbf{FT}\|_g\left\|\mathbf{F}^{-T}\mathbf{N}\right\|_g f^{\text{ext}}_{lni}\frac{\partial\delta\Pi^i}{\partial N}\,dL + \int_L \frac{\|\mathbf{FT}\|_g}{\left\|\mathbf{F}^{-T}\mathbf{N}\right\|_g} f^{\text{ext}}_{lni}\left(N^{\star S}B_S\right)\frac{\partial\delta\Pi^i}{\partial B}\,dL&\\ + \sum_{\alpha=1}^{l}\sum_{\beta=1}^{k_\alpha}\lim\left(\frac{\|\mathbf{FT}\|}{\left\|\mathbf{F}^{-T}\mathbf{N}\right\|} f^{\text{ext}}_{lni}\left(T_S N^{\star S}\right)\right)\left(e_{\beta,\alpha},P_\alpha\right)\delta\Pi^i&\\ - \int_L T^C\frac{\partial}{\partial X^C}\left\{\frac{\|\mathbf{FT}\|}{\left\|\mathbf{F}^{-T}\mathbf{N}\right\|} f^{\text{ext}}_{lni}N^{\star S}T_S\right\}\delta\Pi^i\,dL.&\end{aligned} \tag{76}$$

Therefore we can conclude:

$$\mathscr{F}^{\text{ext}}_{LN}\left(f^{\text{ext}}_{ln}\right) = \|\mathbf{FT}\|_g\left\|\mathbf{F}^{-T}\mathbf{N}\right\|_g f^{\text{ext}}_{lni};$$
$$\mathscr{F}^{\text{ext}}_{LB}\left(f^{\text{ext}}_{ln}\right) = \frac{\|\mathbf{FT}\|_g}{\left\|\mathbf{F}^{-T}\mathbf{N}\right\|_g} f^{\text{ext}}_{Ln}\left(B_S N^{\star S}\right);$$
$$\mathscr{F}^{\text{ext}}_{L_\star}\left(f^{\text{ext}}_{ln}\right) = -T^C T_{S'}\frac{\partial}{\partial X^C}\left\{\frac{\|\mathbf{FT}\|_g}{\left\|\mathbf{F}^{-T}\mathbf{N}\right\|_g} f^{\text{ext}}_{Lni}\left(N^{\star S}T_S\right)T^{S'}\right\};$$
$$\mathscr{F}^{\text{ext}}_{P_\alpha} = \sum_{\beta=1}^{k_\alpha}\lim\left(\frac{\|\mathbf{FT}\|}{\left\|\mathbf{F}^{-T}\mathbf{N}\right\|} f^{\text{ext}}_{lni}\left(T_S N^{\star S}\right)\right)\left(e_{\beta,\alpha},P_\alpha\right),$$

where the concentrated force at a single wedge was considered.

6.6. *Piola Transformation of edge double force* f^{ext}_{lbi}

The Eulerian external work functional relative to the edge double force f^{ext}_{lbi} can be transformed as follows, by recalling (55) and by decomposing the identity using the orthonormal basis (T_S, N_S, B_S) as follows:

$$\int_l f^{\text{ext}}_{lbi} b^r \frac{\partial \delta \chi^i}{\partial x^r}\, dl = \int_L \frac{\|\mathbf{FT}\|^2}{\|J\mathbf{F}^{-T}\mathbf{N}\|} f^{\text{ext}}_{lbi} \left\{ B^S - \frac{G^\star_{LM} B^L T^M}{G^\star_{PN} T^P T^N} T^S \right\} \left(T^Q T_S + N^Q N_S + B^Q B_S \right) \frac{\partial \delta \Pi^i}{\partial X^Q}\, dL. \tag{77}$$

The chain of equalities continues by using the orthonormality of (T_S, N_S, B_S), integrating by parts and applying the divergence theorem on the edge:

$$\begin{aligned}
&= \int_L \frac{\|\mathbf{FT}\|^2}{\|J\mathbf{F}^{-T}\mathbf{N}\|} f^{\text{ext}}_{lbi} B^Q \frac{\partial \delta \Pi^i}{\partial X^Q}\, dL \\
&\qquad + \int_L T_{S'} T^Q \frac{\partial}{\partial X^Q} \left(\frac{\|\mathbf{FT}\|^2}{\|J\mathbf{F}^{-T}\mathbf{N}\|} f^{\text{ext}}_{lbi} \left\{ B^S - \frac{G^\star_{LM} B^L T^M}{G^\star_{PN} T^P T^N} T^S \right\} \delta \Pi^i\, T^{S'} T_S \right) dL \\
&\qquad - \int_L T_{S'} T^Q \frac{\partial}{\partial X^Q} \left(\frac{\|\mathbf{FT}\|^2}{\|J\mathbf{F}^{-T}\mathbf{N}\|} f^{\text{ext}}_{Lbi} \left\{ B^S - \frac{G^\star_{LM} B^L T^M}{G^\star_{PN} T^P T^N} T^S \right\} T^{S'} T_S \right) \delta \Pi^i dL \\
&\qquad = \int_L \frac{\|\mathbf{FT}\|^2}{\|J\mathbf{F}^{-T}\mathbf{N}\|} f^{\text{ext}}_{lbi} \frac{\partial \delta \Pi^i}{\partial B}\, dL + \\
&\qquad - \sum_{\alpha=1}^{l} \sum_{\beta=1}^{k_\alpha} \lim \left(\frac{\|\mathbf{FT}\|^2}{\|J\mathbf{F}^{-T}\mathbf{N}\|} f^{\text{ext}}_{lbi} \frac{G^\star_{LM} B^L T^M}{G^\star_{PN} T^P T^N} \right) \left(e_{\beta,\alpha}, P_\alpha \right) \delta \Pi^i \\
&\qquad + \int_L T^Q T_{S'} \frac{\partial}{\partial X^Q} \left(\frac{\|\mathbf{FT}\|^2}{\|J\mathbf{F}^{-T}\mathbf{N}\|} f^{\text{ext}}_{Lbi} \left\{ \frac{G^\star_{LM} B^L T^M}{G^\star_{PN} T^P T^N} T^S \right\} T_S T^{S'} \right) \delta \Pi^i dL.
\end{aligned} \tag{78}$$

where we have used the notation introduced in Eq. (44).

Therefore we have:

$$\begin{aligned}
\mathscr{F}^{\text{ext}}_{LB}\left(f^{\text{ext}}_{Lb}\right) &= + \frac{\|\mathbf{FT}\|^2}{\|J\mathbf{F}^{-T}\mathbf{N}\|} f^{\text{ext}}_{Lb}\,; \\
\mathscr{F}^{\text{ext}}_{L}\left(f^{\text{ext}}_{Lb}\right) &= + T^L T_{S'} \frac{\partial}{\partial X^L} \left\{ \frac{\langle \mathbf{F}, \mathbf{FT} \rangle}{\|J\mathbf{F}^{-T}\mathbf{N}\|} f^{\text{ext}}_{Lb} T^{S'} \right\}; \\
\mathscr{F}^{\text{ext}}_{P_\alpha}\left(f^{\text{ext}}_{Lb}\right) &= - \sum_{\beta=1}^{k_\alpha} \left(\frac{\|\mathbf{FT}\|^2}{\|J\mathbf{F}^{-T}\mathbf{N}\|} f^{\text{ext}}_{lb} \left\{ \frac{G^\star_{LM} B^L T^M}{G^\star_{PN} T^P T^N} \right\} \right),
\end{aligned} \tag{79}$$

where the concentrated force at a single wedge was considered.

6.7. *Piola transformation of wedge forces*

The transformation of concentrated wedge forces from the Eulerian to the Lagrangian description trivially leads to the following relationship

$$\mathscr{F}^{\text{ext}}_{P_{\text{w}}} = f^{\text{ext}}_{p_{\text{w}}}, \tag{80}$$

being $\mathbf{P}_{\text{w}}$ and $\mathbf{p}_{\text{w}} = \mathbf{\Pi}(\mathbf{P}_{\text{w}})$ corresponding points through the placement, and for the virtual displacements one has $\delta\Pi^i = \delta x^i(\mathbf{\Pi}^{-1})$.

7. Conclusions and research perspectives

The approach to continuum mechanics "imposed" by Cauchy postulation represents a true straightjacket for the development of the discipline. The scheme that is imposed by the format used by the entire Truesdellian school assumes that externally applied loads can be forces and torques, only. As remarked by Paul Germain: i) torques are a kind of double forces, for instance the tangent part of surface double force are surface contact torques, ii) instead normal surface double forces, expending work on the normal component of the normal derivative of virtual displacement, are loads that "tend" to elongate the continuum and do not influence the resultant force and the resultant torque applied to any of its sub-bodies.

The inability of describing in their scheme this kind of external load led Truesdellians to deny, for a long time, even the "existence" of higher gradient continua. This statement has some delicate epistemological implications: in fact they assume an inductivist viewpoint, in which the physical entities and the mathematical models used for describing them are confused. This implies that they deny the existence of phenomena which cannot be described by their models (a complete discussion about this point can be found in [5]). In other words their *non sequitur* reasoning is the following: i) higher gradient continua cannot be described in the framework of Cauchy postulation, ii) every phenomenon must be described by Cauchy postulation, as it has been induced « experimentally » iii) higher gradient continua, a mathematical model that they confuse with some really existing physical objects, do not exist.

The French school, as represented by Paul Germain, assumes a more realistic epistemological viewpoint and bases its analyses on the principle of virtual work, which is recognized to be a very powerful tool in mathematical models formulation.

Following the indications put forward by Germain (see [7, 8, 50–52]) the development of continuum mechanics must be based on the theory of distributions, as formulated by Laurent Schwartz (see [15, 47]).

Therefore, when one decides to develop the theory of those continua, whose deformation energy depends on the third gradient of placement, the wisest choice is to formulate it based on D'Alembert–Lagrange postulation of mechanics. In fact we prove in this paper that the Piola Transformation of Eulerian external loads that can be sustained by third gradient continua contradicts all beliefs of those scholars using Cauchy postulation scheme.

In fact, in third gradient continua such loads include: i) *double* and *triple* forces surface density to be prescribed over the boundary face, that expend work on the first and second normal derivatives of the virtual displacements, ii) *force* and *double force* line density prescribed over the boundary edge, expending work on the virtual displacement and its first derivatives, iii) *forces* concentrated on wedges. Clearly balance of force and balance of torque are not affected by double completely normal and triple forces, and, as shown in [53] and [40], the presence of edge forces implies the existence of double forces in considered continua: as a consequence Cauchy postulation scheme does not allow for the construction of second and third gradient continuum models. It has to be remarked that, as discussed in the introduction and the references there cited, higher gradient continuum models are proving to be able to supply the needed conceptual and theoretical basis for the development of novel metamaterials, and therefore the aforementioned theoretical problems do have an immediate impact in applications.

In order to formulate well-posed equilibrium and dynamic problems in the theory of third gradient solid continua it is necessary to formulate these problems in the Lagrangian description: therefore Piola transformations are necessarily to be looked for in this generalized framework.

This was the aim of the present paper.

The results are surprising: Eulerian triple forces produce, once transformed into the Lagrangian description, all sustainable types of loads: hence, the type of externally applied loads does depend on the description used. This fact has many consequences to be investigated: in particular, we can question whether the concept of dead load, as used so far, needs to be modified to have a more general validity.

It is therefore proven, once more, that the most fundamental concept in continuum mechanics is not that of applied load but that of work functional related to a load: work expended by an externally applied load is invariant under transformation from Lagrangian to Eulerian description.

The mathematical reason, which is at the basis of the fact that the type of applied loads are not invariant under Piola transformation, is purely related to a differential geometric property: the normals to a differential boundary are not transported from Lagrangian to Eulerian description by the gradient of placement, as already established by Piola. This fact has all the mechanical consequences described in this paper.

Many are the research perspectives: we plan to generalize the presented results to the more general case of n^{th} gradient continua. Also of interest is the application of the results presented here to solve, with numerical methods, problems relevant in modern metamaterials theory.

Conflicts of interest

The authors declare no competing financial interest.

Dedication

The manuscript was written through contributions of both the authors. The authors have given approval to the final version of the manuscript.

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