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Average Surface Temperature Over the Circular Disk Heat Source/Sink Embedded in an Insulating Boundary Plane. Analytic Solution.

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Abstract. Reduction of a Bessel integral solution for the average transient temperature change at the surface of a constant flux thermal source/sink of circular disk aspect embedded in an otherwise insulating plane boundary of a homogeneous, isotropic, and conducting half-space is reported. The analytic solution comprises algebraic expressions of tabulated functions.

Keywords. Embedded disk, Average temperature, Analytic solution, Heat equation, Constant flux.

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1. Introduction

Determination of exact, analytic expressions, without integrals, sums, etc. for solutions to heat transfer problems concerned with finite dimension heat sources and/or sinks is frequently complicated by the apparent intractability of reducing infinite integrals containing multiple special functions. This is often the case for the resolution of parabolic differential equation systems wherein heat transport at the surface of an embedded, finite 2D source/sink of arbitrary aspect is the subject of interest; the circular disk structure remains important because of its common use as conveniently fabricated detectors or source devices found in various technical applications and studies of heat transfer.

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In a seminal work on the “constant flux” boundary problem on the disk, Selim [1] refers to applications of the general transient temperature solution to problems in heat transfer theory, as well as to studies in fluid flow, soil permeability, and percolation in porous materials. The authors used a double integral transform (Laplace–Hankel) procedure to find an exact solution for the spatial transient concentration profiles for the system comprising definite integrals. The integrands involved therein are relatively simple in structure with finite limits, are monotonic, and contain no special functions; these therefore are particularly amenable to numerical integration. Tables of concentration profiles of up to 4 significant figures as a function of normalized time and circular coordinates were provided therein.

Attention here is directed to the determination of a closed form analytic solution for the transient average surface temperature of a circular disk source/sink operating at constant thermal flux, which is embedded in the insulated plane $z = 0$ facing the half-space above. The measure of the average surface temperature of the source/sink, is often a transient indicator of the time to approach to steady state heating or cooling of the medium, and is used to determine other quantities of importance in heat transfer, e.g. surface contact resistance and conductance. The measure can be applied directly to temperature sensor data (e.g. from fast precision thermistors, thermocouples, modified circular foil gauges, infrared surface imaging techniques, etc.). Application of same is pertinent as well to describe thermal resistance between circular contact regions between conducting spheres of different temperatures; there has been work resulting in approximating series solutions [2] based on a previous technique[3]. Beck [4] and Cole [5] have applied series approximations for the values of a Green’s function-based approach for problems applied to partitioned (short, medium, and long-time (dimensionless variable)) domains, to determine numerical estimates of the average concentration measure over the disk. At long times (or small values of the disk radius), the nature of the mixed boundary condition at the interface of the disk and insulator affects the rate of convergence of approximating series, and significant errors in the average temperature estimate are known to occur if sufficient numbers of terms are not included. The regions of significant errors have been studied and characterized so that the approximations have therefore been useful for many present day applications.

In the present work, we report the reduction of an infinite Bessel integral from [1] to an analytic expression in algebraic, exponential, and Bessel functions for the transient average surface temperature. Comparison of several existing approximate solutions now in use to the analytic solution is included to demonstrate more accurately the magnitudes and ranges of existing errors in these applications. The results of an application from electrochemical mass transport which uses a numerical estimation for values of the subject Bessel integral is also included for comparison.

2. Analysis

Presented first is the statement of the problem and the replacement of the integral representation of the solution with the analytical result. Following this is a set of standard error comparisons of existing numerical and heuristic solutions in the literature to the analytic result.

2.1. *The problem and result*

The original exact analysis of [1] comprises the solution to the problem of the temperature $T = T(r, z, t)$ throughout a medium of thermal conductivity k and thermal diffusivity α, initially at temperature $T = 0$. A planar insulating boundary at $z = 0$ is assumed, which contains an embedded circular disk heat source (or sink) of radius a maintained at a constant thermal flux Q

beginning at time $z = 0$. Heat conduction into the medium was specified in circular cylindrical coordinates by the system of equations

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}, \quad T = 0, \quad r, z \geqslant 0 \;\; t = 0, \tag{1}$$

$$T(r, z, 0) = 0; \quad z > 0, \tag{2}$$

$$k\frac{\partial T}{\partial z}\bigg|_{z=0} = -Q, \quad 0 \leqslant r < a, \quad z = 0, \; t > 0, \tag{3}$$

$$= 0, \quad r > a, \quad z = 0, \quad t > 0, \tag{4}$$

$$\lim_{r\to\infty} T(r, z, t) = 0, \quad t > 0, \quad 0 < z < \infty, \tag{5}$$

which after integral transforming the time dependent temperature to its Laplace space equivalent, and the radial part of the result to its Hankel space equivalent in the usual way, exposes two tractable differential equations in the transformed variables; the first in terms of the Laplace space parameter temperature $[(T(r, z, t)] = \overline{T}(r, z, p)$

$$\overline{T}(r, z, p) = \int_0^\infty e^{-pt}\, T(r, z, t)\, dt$$

$$\frac{p\overline{T}}{\alpha} = \frac{\partial^2 \overline{T}}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{T}}{\partial r} + \frac{\partial^2 \overline{T}}{\partial z^2},$$

and the second in terms of the Hankel transform $\mathscr{H}_0 \overline{T}(r, z, p) = \overline{T_0}(\sigma, z, p)$,

$$\overline{T_0}(\sigma, z, p) = \int_0^\infty \sigma \mathrm{J}_0(\sigma r)\, \overline{T}(r, z, p)\, dr$$

$$\frac{\partial^2 \overline{T_0}}{\partial z^2} + \left(\sigma^2 - \frac{p}{\alpha}\right)\overline{T_0} = 0.$$

The latter differential equation is readily solved, and using the Laplace–Hankel transform of the system boundary condition at infinity, followed by inverse transformation of the the Hankel representation to the Laplace representation, and subsequent use of a known Bessel function identity, it is found that the infinite integral Laplace space representation

$$\overline{T}(r, z, p) = \frac{Qa}{pk} \cdot \int_0^\infty \frac{\mathrm{J}_0(\sigma r)\,\mathrm{J}_1(\sigma a)}{\sqrt{\sigma^2 + p/\alpha}} \exp\left(-\sqrt{\sigma^2 + \frac{p}{\alpha}}\, z\right) d\sigma$$

gives the transformed temperature when the radius of the disk is a. The result has also been confirmed for analogous mass transfer applications, e.g. for electrochemical analyses using the surface concentration over disk electrodes subject to constant flux conditions [6].

Attention to the temperature measure on the surface of the embedded disk source/sink $z = 0$:

$$\overline{T}(r, 0, p) = \frac{Qa}{pk}\int_0^\infty \frac{\mathrm{J}_0(\sigma r)\,\mathrm{J}_1(\sigma a)}{\sqrt{\sigma^2 + p/\alpha}}\, d\sigma \tag{6}$$

and considering the Bessel identities

$$\int_\delta^{\delta+2\pi} \mathrm{J}_0(\sigma R)\, d\psi' = 2\pi \mathrm{J}_0(\sigma\rho)\,\mathrm{J}_0(\sigma\rho')$$

$$\int_0^a \rho' \mathrm{J}_0(\sigma\rho')\, d\rho' = \frac{a}{\sigma}\mathrm{J}_1(\sigma a),$$

performing another integration of the Laplace temperature in Eq.(6) over the surface of the disk, and normalizing to the disk area πa^2 provides a representation of the average Laplace space temperature

$$\left\langle \overline{T}(p) \right\rangle = \frac{2Q}{pk}\int_0^\infty \frac{\mathrm{J}_1^2(\sigma a)}{\sigma}\frac{d\sigma}{\sqrt{\sigma^2 + \kappa^2}} = \frac{2Q}{p\sqrt{k}}\int_0^\infty \frac{\mathrm{J}_1^2(\sigma a)}{\sigma}\frac{d\sigma}{\sqrt{k\sigma^2 + p}}.$$

Following application of the inverse Laplace transformation

$$\frac{1}{p\sqrt{p+\beta}} \to \frac{1}{\sqrt{\beta}}\,\mathrm{erf}\left(\sqrt{\beta t}\right),$$

the result

$$\langle T(t)\rangle = \frac{2Q}{k}\int_0^\infty \frac{\mathrm{J}_1^2(\sigma a)}{\sigma^2}\,\mathrm{erf}\left(\sqrt{kt}\sigma\right)d\sigma \tag{7}$$

is an exact measure of the average temperature across the surface of the embedded disk source/sink. The reduction of the two parameter integral form therein,

$$\int_0^\infty \frac{\mathrm{J}_1^2(\gamma x)}{x^2}\,\mathrm{erf}(\beta x)\,dx, \tag{8}$$

is the main focus of this report, for which the details of reduction are presented in the Appendix. The use of the final analytic result Eq.(A.8) in Eq.(7) then gives the variable form

$$\langle T(t)\rangle = \frac{2Q}{k}\left[\frac{4\gamma}{3\pi} + \frac{\beta}{\sqrt{\pi}}\left\{1 - \left(1+\frac{2\gamma^2}{3\beta^2}\right)\exp\left(-\frac{\gamma^2}{2\beta^2}\right)\mathrm{I}_0\left(\frac{\gamma^2}{2\beta^2}\right) - \left(\frac{1}{3}+\frac{2\gamma^2}{3\beta^2}\right)\exp\left(-\frac{\gamma^2}{2\beta^2}\right)\mathrm{I}_1\left(\frac{\gamma^2}{2\beta^2}\right)\right\}\right],$$

and with the substitutions

$$\beta = \sqrt{kt}, \quad \gamma = a, \quad \tau = \beta^2/\gamma^2, \tag{9}$$

the average temperature at the active surface

$$\langle T(t)\rangle = \frac{2Qa}{k}\sqrt{\tau}\left[\frac{4}{3\pi\sqrt{\tau}} + \frac{1}{\sqrt{\pi}}\left\{1 - \left(1+\frac{2}{3\tau}\right)\exp\left(-\frac{1}{2\tau}\right)\mathrm{I}_0\left(\frac{1}{2\tau}\right) - \left(\frac{1}{3}+\frac{2}{3\tau}\right)\exp\left(-\frac{1}{2\tau}\right)\mathrm{I}_1\left(\frac{1}{2\tau}\right)\right\}\right] \tag{10}$$

is cast in terms of the dimensionless variable $\tau = kt/a^2$. At long times, and conditions under which the heat transfer characteristics of the medium above the disk remain under homogeneous heat conduction conditions, e.g. no phase changes, convection, etc., the average surface temperature $\langle T(r \leqslant a, 0, t)\rangle$ is seen to approach the known steady-state value $8Qa/3\pi k$, *e.g.* see [7].

2.2. *Comparison of the analytic result to literature approximations*

In general, applications in the literature which utilize the integral (8) typically comprise series approximations with dimensionless multipliers and variable changes to scale the integral to the particular infinite series functional form being used. For comparative analysis therefore, it is required to re-scale the series representatons to establish values equivalent to the analytical result containing the same dimensionless variable $\tau = kt/a^2$. Evidently, a convenient basis representation for this purpose is the form

$$\Phi_1\left(kt/a^2\right) = X \cdot \text{Integral Approximation}, \tag{11}$$

wherein X is the factor used to make the integral series approximation appropriately dimensionless and consistent.

For heat transfer applications, it is appropriate to consider the various Green's function based results of Cole and Beck. A number of their approximating series results corresponding to various specified time segments already expressed in terms of the dimensionless variable $\tau = kt/a^2$. One result for short diffusion times (small τ values), on the surface of the disk provides their approximation to $\Phi_1(kt/a^2)$, which results after multiplying the right-hand side of the series approximation ([5, Eq. (7.149) p. 275]) by the dimensionless factor $X = a/2\sqrt{kt}$ (which in their notation is $1/2\sqrt{t^+}$), i.e.

$$\Phi_{1,\,Cole\,short\,time}\left(\frac{kt}{a^2}\right) = \frac{a}{2\sqrt{kt}}\left\{2\sqrt{\frac{kt}{\pi a^2}} - \frac{kt}{\pi a^2}\left[2 - \frac{kt}{4a^2} - \frac{1}{4}\left(\frac{kt}{4a^2}\right)^2 - \frac{15}{4}\left(\frac{kt}{4a^2}\right)^3\right]\right\}$$

Likewise for long times, the estimating function is available by multiplication of the right-hand side ([5, Eq. (7.148) p. 275]) by the same dimensionless factor $X = a/2\sqrt{kt}$ $(= 1/2\sqrt{t^+})$:

$$\Phi_{1,\,Cole\,long\,time}\left(\frac{kt}{a^2}\right) = \frac{a}{2\sqrt{kt}}\left\{\frac{8}{3\pi} - \frac{a}{2\sqrt{\pi kt}}\left[1 - \frac{2a^2}{24kt} + \left(\frac{5a^4}{480k^2t^2}\right) - \left(\frac{19a^6}{10752k^3t^3}\right)\right]\right\}$$

Similarly, comparisons can be made to other reported short and long time approximations from [8,9] (see e.g. [9, Eqns. (3.158) and (3.159)]) after normalizing the series representations using the appropriate common multiplier $X = \pi a/8\sqrt{kt}$ on the right-hand sides of each:

$$\Phi_{1,\,Beck\,short\,time}\left(\frac{kt}{a^2}\right) = \frac{\pi a}{8\sqrt{kt}}\left\{\frac{8}{\pi}\sqrt{\frac{kt}{\pi a^2}} - \frac{kt}{\pi a^2} + \frac{1}{8\pi}\left(\frac{kt}{a^2}\right)^2 + \frac{1}{32\pi}\left(\frac{kt}{a^2}\right)^3 + \frac{15}{512\pi}\left(\frac{kt}{a^2}\right)^4\right\}$$

$$\Phi_{1,\,Beck\,long\,time}\left(\frac{kt}{a^2}\right) = \frac{\pi a}{8\sqrt{kt}}\left\{\frac{32}{3\pi^2} - \frac{2a}{\sqrt{\pi^3 kt}}\left[1 - \frac{a^2}{3(4kt)} + \frac{a^4}{6(4kt)^2} + \frac{a^6}{12(4kt)^3}\right]\right\}$$

An application to mass transfer experiments that comprises the integral in question has been reported [6, 7]. In that work, the average disk surface concentration measure $C(r,0,t)_{Av}$, is analogous to the temperature $\langle T(r \leqslant a,0,t)\rangle$ above, after replacement of the thermal conductivity k by the diffusion coefficient D.:

$$C(r,0,t)_{Av} = \frac{2Qa}{D}\cdot\sqrt{\tau}\int_0^\infty \mathrm{J}_1^2\left(\frac{\beta a}{l}\right)\mathrm{erf}(\beta)\frac{d\beta}{\beta^2}.$$

In contrast to the technique of developing approximating series for the values of the integral over long and short time regions, the work in [6, 7] uses numerical integration techniques to approximate the values of the Φ_1 integral over the dimensionless variable range. Efficiency in convergence of the integration was attained by recasting same to the sum of two integrals, with a precision of 4 significant figures over the reported range. The derived approximation to the integral for comparison to the others in this work becomes, after multiplication by the parameter $X = D/2Qa\cdot\sqrt{Dt/a^2} = D/2Qa\cdot\sqrt{\tau}$,

$$\Phi_{1,\,Fleisch}\left(\frac{kt}{a^2}\right) = \sqrt{\tau}\left\{\int_0^\infty \mathrm{J}_1^2\left(\frac{\beta}{\sqrt{\tau}}\right)\frac{d\beta}{\beta^2} + \int_0^\infty \mathrm{J}_1^2\left(\frac{\beta}{\sqrt{\tau}}\right)(\mathrm{erf}(\beta)-1)\frac{d\beta}{\beta^2}\right\}$$

Finally, the analytic result from (10) for determining the percentage error of the various approximations becomes

$$\Phi_{1,\,Anal.}\left(\frac{kt}{a^2}\right) = \sqrt{\tau}\left[\frac{4}{3\pi\sqrt{\tau}} + \frac{1}{\sqrt{\pi}}\left\{1 - \left(1 + \frac{2}{3\tau}\right)\exp\left(-\frac{1}{2\tau}\right)\mathrm{I}_0\left(\frac{1}{2\tau}\right) - \left(\frac{1}{3} + \frac{2}{3\tau}\right)\exp\left(-\frac{1}{2\tau}\right)\mathrm{I}_1\left(\frac{1}{2\tau}\right)\right\}\right]$$

and a plot of the analytic function is given in Figure 1.

Results for the percentage error for all the various estimates are given in Figures 2 and 3. In those figures, the dashed zero error (analytical reference) base line comprises the Fleischmann numerical error data, which is only precise to 4 significant figures.

3. Summary

An analytical solution to a heretofore unreduced infinite integral, containing the square of a Bessel function, is described. The subject integral is expanded to a sum of two new Bessel integrals after a change of variables. It is then considered that a second order, two term scaled differentiation (with respect to the function parameter) of the Bessel component allows part of one Bessel integrand to be recast to a double integral comprising a finite one over a simpler infinite Bessel representation. The second integral of the expansion, after a similar differentiation with respect to the same parameter, leads to the required remaining and reducible integrals, after some further manipulation.

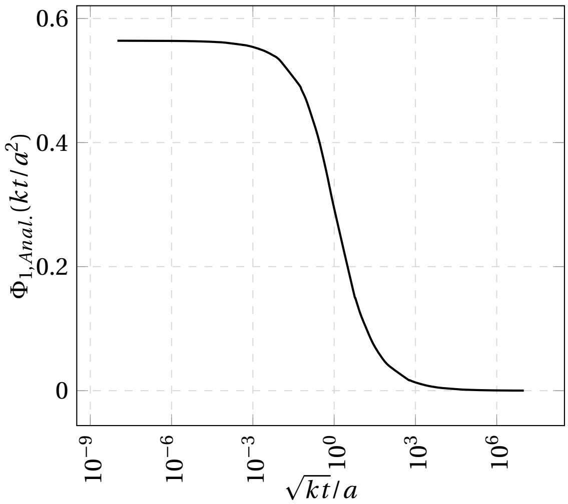

Figure 1. A plot of the values of the analytic Φ_1 function *vs.* the dimensionless variable $\sqrt{kt}/a$ (or $\sqrt{Dt}/a$).

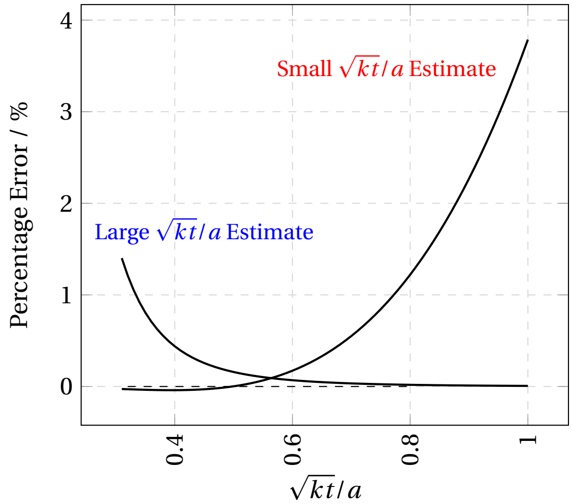

Figure 2. A plot of the percentage error of the Beck approximations $\Phi_{1,Beck}$ *vs.* the dimensionless variable $\sqrt{kt}/a$ with reference to the analytic value (dotted line).

The described solution can be used in error analysis of existing approximations to verify their usefulness as heuristic expressions for predicting physical quantities on active circular disk sources/sinks commonly used in diffusion analysis in heat and mass transfer measurements. Since an analytical solution which comprises well-known tabulated functions is in general more efficient to implement in test use, and inherently more accurate and precise than numerical integration techniques on modern laboratory computers, they are increasingly important in developing better models for improved high performance applications.

Since mathematically analogous differential equation systems exist for other areas of research (e.g. neutron density in reactors, velocity potential in incompressible fluids, (steady state) electric potential in electrostatics, displacement of vibrating stretched membranes, etc.) the solution is applicable to modeling in those applications as well, after appropriate unit parameter and

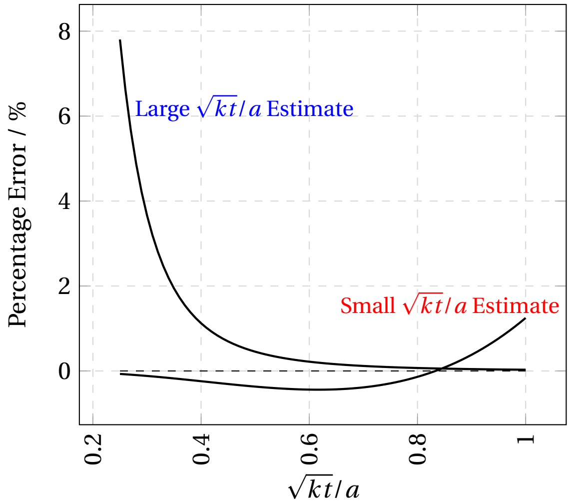

Figure 3. A plot of the percentage error of the Cole/Beck approximations $\Phi_{1,Cole}$ *vs.* the dimensionless variable $\sqrt{kt}/a$ with reference to the analytic value (dotted line).

variable substitutions changes to the the heat transfer differential equation system ((1)-(5)) introduced at the outset of this report.

4. Symbols used

a	Circular disk radius
C^{∞}	Bulk diffusant concentration
C_{Av}	Average transient surface concentration
$\langle T(t)\rangle$	Average transient surface temperature
D	Diffusion coefficient
erf	Error function
$\mathscr{H}$	Hankel transform operator
J_x	Bessel functions of the first kind
$\mathscr{L}$	Laplace transform
p	Laplace parameter operator
Q	Heat or mass Flux
r	Radial distance coordinate
t	Time
T	Temperature
z	$z-$distance coordinate
α	Thermal diffusivity
τ	Dimensionless variable kt/a^2 or Dt/a^2
$\Phi_{1,x}$	Scaled variants of $\int_0^{\infty}\left[\mathrm{J}_1^2(\gamma x)/x^2\right]\mathrm{erf}(\beta x)dx$
$\alpha-\omega$	In Appendix, various integration variables.

5. Acknowledgements

This paper is dedicated to the memory of Emeritus Professor Harold Levine, Department of Mathematics, Stanford University.

Appendix A. Reduction of the Integral in (8)

Denoting as I the integral in (8), a change of variables shows

$$
\begin{aligned}
I &= \int_0^\infty \frac{J_1^2(\gamma x)}{x^2}\,\mathrm{erf}(\beta x)\,dx = \int_0^\infty d\left(-\frac{1}{x}\right) J_1^2(\gamma x)\,\mathrm{erf}(\beta x) \\
&= \int_0^\infty \frac{1}{x}\left\{2\gamma J_1(\gamma x)\left[J_0(\gamma x) - \frac{J_1(\gamma x)}{\gamma x}\right]\mathrm{erf}(\beta x)\right\} dx + \frac{2}{\sqrt{\pi}}\beta \int_0^\infty \frac{1}{x} J_1^2(\gamma x)\,\mathrm{e}^{-\beta^2 x^2}\,dx
\end{aligned}
$$

$$
3I = 2\gamma \int_0^\infty J_0(\gamma x) J_1(\gamma x)\,\mathrm{erf}(\beta x)\,\frac{dx}{x} + \frac{2}{\sqrt{\pi}}\beta \int_0^\infty J_1^2(\gamma x)\,\mathrm{e}^{-\beta^2 x^2}\,\frac{dx}{x}. \tag{A.1}
$$

The first integral in (A.1)

$$
F_1 = \int_0^\infty J_0(\gamma x) J_1(\gamma x)\,\mathrm{erf}(\beta x)\,\frac{dx}{x},
$$

is rearranged according to a Bessel product identity

$$
\begin{aligned}
\text{Ĺ}J_0(\gamma x) J_1(\gamma x)_1 &= \frac{2}{\pi}\int_0^{\pi/2} J_1(2\gamma x\cos\vartheta)\cos\vartheta\,d\vartheta : \\
F_1 &= \frac{2}{\pi}\int_0^{\pi/2} \cos\vartheta \int_0^\infty J_1(2\gamma x\cos\vartheta)\,\mathrm{erf}(\beta x)\,\frac{dx}{x}.
\end{aligned} \tag{A.2}
$$

Now

$$
\int_0^\infty J_1(\gamma x)\,\mathrm{erf}(\beta x)\,\frac{dx}{x} = \mathrm{erfc}\left(\frac{\gamma}{2\beta}\right) + \frac{2\beta}{\sqrt{\pi}\gamma}\left[1 - \exp\left(-\frac{\gamma^2}{4\beta^2}\right)\right] \tag{A.3}
$$

as can be checked by differentiation with respect to β; thus

$$
\begin{aligned}
\frac{d}{d\beta}\int_0^\infty J_1(\gamma x)\,\mathrm{erf}(\beta x)\,\frac{dx}{x} &= \frac{2}{\sqrt{\pi}}\int_0^\infty J_1(\gamma x)\,\mathrm{e}^{-\beta^2 x^2}\,dx \\
1 &= \frac{2}{\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{2\beta}\exp\left(-\frac{\gamma^2}{8\beta^2}\right) I_{1/2}\left(\frac{\gamma^2}{8\beta^2}\right) = \frac{1}{\beta}\exp\left(-\frac{\gamma^2}{8\beta^2}\right)\cdot\sqrt{\frac{16\beta^2}{\pi\gamma^2}}\sinh\left(\frac{\gamma^2}{8\beta^2}\right) \\
1 &= \frac{2}{\sqrt{\pi}}\cdot\frac{1}{\gamma}\left[1 - \exp\left(-\frac{\gamma^2}{4\beta^2}\right)\right]
\end{aligned} \tag{A.4}
$$

and

$$
\begin{aligned}
&\frac{d}{d\beta}\left\{\mathrm{erfc}\left(\frac{\gamma}{2\beta}\right) + \frac{2\beta}{\sqrt{\pi}\gamma}\left[1 - \exp\left(-\frac{\gamma^2}{4\beta^2}\right)\right]\right\} \\
&\quad = \frac{2}{\sqrt{\pi}}\cdot\frac{\gamma^2}{2\beta^2}\exp\left(-\frac{\gamma^2}{4\beta^2}\right) + \frac{2}{\sqrt{\pi}}\cdot\frac{1}{\gamma}\left[1 - \exp\left(-\frac{\gamma^2}{4\beta^2}\right)\right] - \frac{2\beta}{\sqrt{\pi}\gamma}\exp\left(-\frac{\gamma^2}{4\beta^2}\right)\frac{2\gamma^2}{4\beta^3} \\
&\quad = \frac{2}{\sqrt{\pi}}\cdot\frac{1}{\gamma}\left[1 - \exp\left(-\frac{\gamma^2}{4\beta^2}\right)\right]
\end{aligned}
$$

in agreement with (A.4). Combining (A.2) and (A.3),

$$
\begin{aligned}
F_1 &= \frac{2}{\pi}\int_0^{\pi/2}\cos\vartheta\left\{\mathrm{erfc}\left(\frac{\gamma\cos\vartheta}{\beta}\right) + \frac{\beta}{\sqrt{\pi}\gamma\cos\vartheta}\left[1 - \exp\left(-\frac{\gamma^2\cos^2\vartheta}{4\beta}\right)\right]\right\} d\vartheta \\
&= \frac{2}{\pi}\int_0^{\pi/2}\cos\vartheta\,\mathrm{erfc}\left(\frac{\gamma\cos\vartheta}{\beta}\right) d\sin\vartheta + 2\frac{\beta}{\gamma}\frac{1}{\pi^{3/2}}\int_0^{\pi/2}\left[1 - \exp\left(-\frac{\gamma^2\cos^2\vartheta}{4\beta}\right)\right] d\vartheta \\
&= \frac{2}{\pi} - \frac{2}{\pi}\frac{2}{\sqrt{\pi}}\frac{\gamma}{\beta}\int_0^{\pi/2}\sin^2\vartheta\exp\left(-\frac{\gamma^2\cos^2\vartheta}{4\beta}\right) d\sin\vartheta \\
&\qquad + \frac{\beta}{\gamma}\frac{1}{\sqrt{\pi}} - 2\frac{\beta}{\gamma}\frac{1}{\pi^{3/2}}\int_0^{\pi/2}\exp\left(-\frac{\gamma^2\cos^2\vartheta}{\beta^2}\right) d\vartheta.
\end{aligned} \tag{A.5}
$$

Before continuing with the integrals in (A.5), consider

$$
\begin{aligned}
F_2 &= \int_0^\infty \mathrm{J}_1^2\left(\gamma x\right)\mathrm{e}^{-\beta^2 x^2}\frac{dx}{x}\\
D_2 &= \left(\frac{d^2}{d\gamma^2}+\frac{3}{\gamma}\frac{d}{d\gamma}\right)F_2 = \frac{4}{\pi}\int_0^\pi \cos^2\vartheta \int_0^\infty \mathrm{J}_0\left(2\gamma x\cos\vartheta\right)x\mathrm{e}^{-\beta^2 x^2}\,dx\,d\vartheta\\
&= \frac{4}{\pi}\int_0^\pi \cos^2\vartheta \int_0^\pi \mathrm{J}_0\left(2\gamma x\cos\vartheta\right)d\left(\frac{\mathrm{e}^{-\beta^2x^2}}{-2\beta^2}\right)d\vartheta\\
&= \frac{4}{\pi}\int_0^\pi \cos^2\vartheta\, d\vartheta\left\{\frac{1}{2\beta^2}-\frac{\gamma}{\beta^2}\cos\vartheta\int_0^\infty \mathrm{J}_1\left(2\gamma x\cos\vartheta\right)\mathrm{e}^{-\beta^2x^2}\,dx\right\}\\
&= \frac{2}{\pi\beta^2}\int_0^\pi \cos^2\vartheta\, d\vartheta - \frac{4\gamma}{\pi\beta^2}\int_0^\pi \cos^3\frac{1}{2\gamma\cos\vartheta}\left[1-\exp\left(-\frac{\gamma^2\cos^2\vartheta}{\beta^2}\right)\right]d\vartheta\\
&= \frac{2}{\pi\beta^2}\int_0^\pi \cos^2\vartheta \exp\left(-\frac{\gamma^2\cos^2\vartheta}{\beta^2}\right)d\vartheta.
\end{aligned}
$$

Thus

$$
\frac{d}{d\gamma}\left(\gamma^3\frac{dF_2}{d\gamma}\right) = \frac{2}{\pi\beta^2}\int_0^\pi \cos^2\vartheta\gamma^3\exp\left(-\frac{\gamma^2\cos^2\vartheta}{\beta^2}\right)d\vartheta,
$$

and since

$$
\begin{aligned}
\int_0^\gamma \exp\left(-\lambda x^2\right)dx &= \int_0^\gamma x^2 d\frac{\exp\left(-\lambda x^2\right)}{-2\lambda}dx\\
&= -\frac{\gamma^2}{2\lambda}\exp\left(-\lambda x^2\right)\exp\left(-\lambda\gamma^2\right)+\frac{1}{\lambda}\int_0^\gamma x\exp\left(-\lambda x^2\right)dx\\
&= -\frac{\gamma^2}{2\lambda}\exp\left(-\lambda\gamma^2\right)+\frac{1}{2\lambda^2}\left[1-\exp\left(-\lambda\gamma^2\right)\right],
\end{aligned}
$$

it follows that

$$
\begin{aligned}
\frac{dF_2}{d\gamma} &= \frac{2}{\pi\gamma^3\beta^2}\int \cos^2\vartheta\left\{\frac{\beta^4}{2\cos^4\vartheta}\left[1-\exp\left(-\frac{\gamma^2\cos^2\vartheta}{\beta^2}\right)\right]-\frac{\gamma^2\beta^2}{2\cos^2\vartheta}\exp\left(-\frac{\gamma^2\cos^2\vartheta}{\beta^2}\right)\right\}d\vartheta\\
&= \frac{\beta^2}{\pi\gamma^3}\int_0^\pi \frac{1}{\cos^2\vartheta}\left\{\left[1-\exp\left(-\frac{\gamma^2\cos^2\vartheta}{\beta^2}\right)\right]\frac{\gamma^2}{\beta^2}\cos^2\vartheta\exp\left(-\frac{\gamma^2\cos^2\vartheta}{\beta^2}\right)\right\}d\vartheta.
\end{aligned}
$$

Making use of the reduction

$$
\begin{aligned}
\int_0^\gamma \frac{1}{x^3}\left\{1-\exp\left(-\lambda x^2\right)-\lambda x^2\exp\left(-\lambda x^2\right)\right\}dx &= \int_0^\gamma d\left(-\frac{1}{2x^2}\right)\left\{1-\exp\left(-\lambda x^2\right)-\lambda x^2\exp\left(-\lambda x^2\right)\right\}\\
&= \frac{1}{2\gamma^2}\left\{1-\exp\left(-\lambda x^2\right)-\lambda x^2\exp\left(-\lambda x^2\right)\right\}+\frac{1}{2}\int_0^\gamma\frac{1}{x^2}\Big\{2\lambda x\exp\left(-\lambda x^2\right)-2\lambda x\exp\left(-\lambda x^2\right)\\
&\quad +2\lambda^2x^3\exp\left(-\lambda x^2\right)\Big\}dx\\
&= -\frac{1}{2\gamma^2}\left\{1-\exp\left(-\lambda\gamma^2\right)-\lambda\gamma^2\exp\left(-\lambda x^2\right)\right\}+\frac{\lambda}{2}\Big(1-\exp\left(-\lambda\gamma x^2\right)\Big),
\end{aligned}
$$

it follows next that

$$\begin{aligned}
F_2 =& \frac{1}{2\pi}\int_0^{\pi}\left[1-\exp\left(-\frac{\gamma^2\cos^2\vartheta}{\beta^2}\right)\right]d\vartheta \\
&\quad -\frac{1}{2\pi}\frac{\beta^2}{\gamma^2}\cdot\int_0^{\pi}\frac{d\vartheta}{\cos^2\vartheta}\left\{1-\exp\left(-\frac{\gamma^2\cos^2\vartheta}{\beta^2}\right)-\frac{\gamma^2\cos^2\vartheta}{\beta^2}\cdot\exp\left(-\frac{\gamma^2\cos^2\vartheta}{\beta^2}\right)\right\} \\
=& \frac{1}{2\pi}\int_0^{\pi}\left[1-\exp\left(-\frac{\gamma^2\cos^2\vartheta}{\beta^2}\right)\right]d\vartheta \\
&\quad +\frac{1}{2\pi}\int_0^{\pi}\exp\left(-\frac{\gamma^2\cos^2\vartheta}{\beta^2}\right)d\vartheta-\frac{1}{2\pi}\frac{\beta^2}{\gamma^2}\cdot\int_0^{\pi}\frac{d\vartheta}{\cos^2\vartheta}\left\{1-\exp\left(-\frac{\gamma^2\cos^2\vartheta}{\beta^2}\right)\right\} \\
=& \frac{1}{2}-\frac{1}{2\pi}\frac{\beta^2}{\gamma^2}\int_0^{\pi}d\left(\frac{\sin\vartheta}{\cos\vartheta}\right)\left\{1-\exp\left(-\frac{\gamma^2\cos^2\vartheta}{\beta^2}\right)\right\} \\
=& \frac{1}{2}-\frac{1}{2\pi}\frac{\beta^2}{\gamma^2}\int_0^{\pi}\frac{\sin\vartheta}{\cos\vartheta}\cdot\frac{\gamma^2}{\beta^2}\cos\vartheta\sin\vartheta\cdot\exp\left(-\frac{\gamma^2\cos^2\vartheta}{\beta^2}\right)d\vartheta \\
=& \frac{1}{2}-\frac{1}{\pi}\int_0^{\pi}\sin^2\vartheta\exp\left(-\frac{\gamma^2\cos^2\vartheta}{\beta^2}\right)d\vartheta \\
=& \frac{1}{2}-\frac{2}{\pi}\int_0^{\pi/2}\sin^2\vartheta\exp\left(-\frac{\gamma^2}{\beta^2}\cos^2\vartheta\right)d\vartheta.
\end{aligned} \tag{A.6}$$

A simpler determination of F_2 which also serves as a check is the following

$$\begin{aligned}
\frac{dF_2}{d\beta} &= -2\beta\int_0^{\infty}\mathrm{J}_1^2(\gamma x)\,x\mathrm{e}^{-\beta^2x^2}dx = -2\beta\cdot\frac{1}{2\beta^2}\exp\left(-\frac{\gamma^2}{2\beta^2}\right)\mathrm{I}_1\left(\frac{\gamma^2}{2\beta^2}\right) \\
\therefore F_2 &= \int_{\beta}^{\infty}\frac{1}{x}\exp\left(-\frac{\gamma^2}{2\beta^2}\right)\mathrm{I}_1\left(\frac{\gamma^2}{2\beta^2}\right)dx = \frac{1}{2}\int_{\beta}^{\gamma^2/2\beta^2}\frac{1}{x}\mathrm{e}^{-y}\mathrm{I}_1(y)\frac{dy}{y}.
\end{aligned}$$

Now

$$\begin{aligned}
\int_0^{\sigma}\exp(-y)\,\mathrm{I}_1(y)\frac{dy}{y} &= \int_0^{\sigma}d\left(-\frac{1}{y}\right)y\exp(-y)\,\mathrm{I}_1(y) \\
&= -\mathrm{I}_1(\sigma)\exp(-\sigma)+\int_0^{\sigma}\frac{1}{y}\exp(-y)\left[y\mathrm{I}_0(y)-y\mathrm{I}_1(y)\right]dy \\
&= -\mathrm{I}_1(\sigma)\exp(-\sigma)+\int_0^{\sigma}\exp(-y)\,\mathrm{I}_0(y)\,dy-\int_0^{\sigma}\exp(-y)\,\mathrm{I}_1(y)\,dy \\
&= -\mathrm{I}_1(\sigma)\exp(-\sigma)+\int_0^{\sigma}\exp(-y)\,\mathrm{I}_0(y)\,dy-\int_0^{\sigma}\exp(-y)\,d\mathrm{I}_0 \\
&= -\mathrm{I}_1(\sigma)\exp(-\sigma)+\int_0^{\sigma}\exp(-y)\,\mathrm{I}_0(y)\,dy-\exp(-\sigma)\,\mathrm{I}_0(\sigma)+1-\int_0^{\sigma}\exp(-y)\,\mathrm{I}_0(y)\,dy \\
&= 1-\exp(-\sigma)\left[\mathrm{I}_0(\sigma)+\mathrm{I}_1(\sigma)\right].
\end{aligned}$$

Hence

$$F_2 = \frac{1}{2}-\frac{1}{2}\exp\left(-\frac{\gamma^2}{2\beta^2}\right)\left[\mathrm{I}_0\left(\frac{\gamma^2}{2\beta^2}\right)+\mathrm{I}_1\left(\frac{\gamma^2}{2\beta^2}\right)\right] \tag{A.7}$$

in full agreement with (A.6), since

$$\begin{aligned}
\int_0^{\pi/2}\sin^2\vartheta\exp\left(-\lambda\cos^2\vartheta\right)d\vartheta &= \int_0^{\pi/2}\frac{1-\cos2\vartheta}{2}\exp\left(-\frac{\lambda}{2}\right)\exp\left(-\frac{\lambda}{2}\cos2\vartheta\right)d\vartheta \\
&= \frac{1}{4}\int_0^{\pi}(1-\cos2\varphi)\exp\left(-\frac{\lambda}{2}\right)\exp\left(-\frac{\lambda}{2}\cos2\varphi\right)d\varphi \\
&= \frac{\pi}{4}\exp\left(-\frac{\lambda}{2}\right)\mathrm{I}_0\left(\frac{\lambda}{2}\right)+\frac{\pi}{4}\exp\left(-\frac{\lambda}{2}\right)\mathrm{I}_1\left(\frac{\lambda}{2}\right).
\end{aligned}$$

Note that

$$F_2 \to \int_0^\infty \frac{J_1^2(\gamma x)\,dx}{x} = \frac{1}{2}, \quad \text{when}\, \beta \to 0$$

as implied by (A.7).

Upon collecting the results obtained

$$\begin{aligned}
3I &= 2\gamma F_1 + \frac{2}{\sqrt{\pi}}\beta F_2 \\
F_1 &= \frac{2}{\pi} + \frac{\beta}{\gamma\sqrt{\pi}} - \frac{\beta}{\gamma\sqrt{\pi}}\exp\left(-\frac{\gamma^2}{2\beta^2}\right)\mathrm{I}_0\left(\frac{\gamma^2}{2\beta^2}\right) - \frac{\gamma}{\beta\sqrt{\pi}}\exp\left(-\frac{\gamma^2}{2\beta^2}\right)\left[\mathrm{I}_0\left(\frac{\gamma^2}{2\beta^2}\right) + \mathrm{I}_1\left(\frac{\gamma^2}{2\beta^2}\right)\right] \\
F_2 &= \frac{1}{2} - \frac{1}{2}\exp\left(-\frac{\gamma^2}{2\beta^2}\right)\left[\mathrm{I}_0\left(\frac{\gamma^2}{2\beta^2}\right) + \mathrm{I}_1\left(\frac{\gamma^2}{2\beta^2}\right)\right],
\end{aligned}$$

so that

$$\begin{aligned}
3I = &\frac{4\gamma}{\pi} + \frac{2\beta}{\sqrt{\pi}} - \frac{2\beta}{\sqrt{\pi}}\exp\left(-\frac{\gamma^2}{2\beta^2}\right)\mathrm{I}_0\left(\frac{\gamma^2}{2\beta^2}\right) - \frac{2\gamma^2}{\beta\sqrt{\pi}}\exp\left(-\frac{\gamma^2}{2\beta^2}\right)\left[\mathrm{I}_0\left(\frac{\gamma^2}{2\beta^2}\right) + \mathrm{I}_1\left(\frac{\gamma^2}{2\beta^2}\right)\right] \\
&+ \frac{\beta}{\sqrt{\pi}} - \frac{\beta}{\sqrt{\pi}}\exp\left(-\frac{\gamma^2}{2\beta^2}\right)\left[\mathrm{I}_0\left(\frac{\gamma^2}{2\beta^2}\right) + \mathrm{I}_1\left(\frac{\gamma^2}{2\beta^2}\right)\right] \\
= &\frac{4\gamma}{\pi} + \frac{3\beta}{\sqrt{\pi}} - \frac{3\beta}{\sqrt{\pi}}\exp\left(-\frac{\gamma^2}{2\beta^2}\right)\mathrm{I}_0\left(\frac{\gamma^2}{2\beta^2}\right) - \frac{2\gamma^2}{\beta\sqrt{\pi}}\exp\left(-\frac{\gamma^2}{2\beta^2}\right)\mathrm{I}_0\left(\frac{\gamma^2}{2\beta^2}\right) \\
&- \frac{\beta}{\sqrt{\pi}}\exp\left(-\frac{\gamma^2}{2\beta^2}\right)\mathrm{I}_1\left(\frac{\gamma^2}{2\beta^2}\right) - \frac{2\gamma^2}{\beta\sqrt{\pi}}\exp\left(-\frac{\gamma^2}{2\beta^2}\right)\mathrm{I}_1\left(\frac{\gamma^2}{2\beta^2}\right),
\end{aligned}$$

and, finally

$$I = \frac{4\gamma}{3\pi} + \frac{\beta}{\sqrt{\pi}}\left\{1 - \left(1 + \frac{2\gamma^2}{3\beta^2}\right)\exp\left(-\frac{\gamma^2}{2\beta^2}\right) - \mathrm{I}_0\left(\frac{\gamma^2}{2\beta^2}\right)\left(\frac{1}{3} + \frac{2\gamma^2}{3\beta^2}\right)\exp\left(-\frac{\gamma^2}{2\beta^2}\right)\mathrm{I}_1\left(\frac{\gamma^2}{2\beta^2}\right)\right\} \quad \text{(A.8)}$$

represents the full reduction of the integral (8).

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