INSTITUT DE FRANCE
Académie des sciences

# Comptes Rendus
# Mécanique

Sébastien Boyaval

**A viscoelastic flow model of Maxwell-type with a symmetric-hyperbolic formulation**

Volume 351, Special Issue S1 (2023), p. 251-259

Online since: 7 February 2023

**Part of Special Issue:** The scientific legacy of Roland Glowinski
**Guest editors:** Grégoire Allaire (CMAP, Ecole Polytechnique, Institut Polytechnique de Paris, Palaiseau, France),
Jean-Michel Coron (Laboratoire Jacques-Louis Lions, Sorbonne Université)
and Vivette Girault (Laboratoire Jacques-Louis Lions, Sorbonne Université)

https://doi.org/10.5802/crmeca.165



CENTRE MERSENNE

*The* Comptes Rendus. Mécanique *are a member of the Mersenne Center for open scientific publishing*
www.centre-mersenne.org — e-ISSN : 1873-7234

*Comptes Rendus*
*Mécanique*
**2023**, Vol. 351, Special Issue S1, p. 251-259
https://doi.org/10.5802/crmeca.165

The scientific legacy of Roland Glowinski / *L'héritage scientifique de Roland Glowinski*

# A viscoelastic flow model of Maxwell-type with a symmetric-hyperbolic formulation

**Sébastien Boyaval** [a, b]

[a] LHSV, Ecole des Ponts, EDF R&D, Chatou, France

[b] MATHERIALS, Inria, Paris, France

*E-mail:* sebastien.boyaval@enpc.fr

*Dedicated to the memory of Roland Glowinski, an inspiring pioneer*

**Abstract.** Maxwell models for viscoelastic flows are famous for their potential to unify elastic motions of solids with viscous motions of liquids in the continuum mechanics perspective. But the usual Maxwell models allow one to define well motions mostly for *one-dimensional* flows only. To define unequivocal *multi-dimensional* viscoelastic flows (as solutions to well-posed initial-value problems) we advocated in [*ESAIM:M2AN* **55** (2021), p. 807-831] an upper-convected Maxwell model for compressible flows with a symmetric-hyperbolic formulation. Here, that model is derived again, with new details.

**Keywords.** Viscoelastic flows, Maxwell fluids, Symmetric-hyperbolic systems of conservation laws, Elastodynamics of hyperelastic materials, Stress relaxation.

**2020 Mathematics Subject Classification.** 76A10,35L45,74D10.

*Manuscript received 29 November 2022, revised 16 December 2022, accepted 20 December 2022.*

## 1. Elastic and viscous motions in the continuum perspective

First, let us recall seminal systems of PDEs that unequivocally model the motions $\boldsymbol{\phi}_t : \mathscr{B} \to\subset \mathbb{R}^3$ of *continuum bodies* $\mathscr{B}$ on a time range $t \in [0,T)$. PDEs governing *elastic flows* are a starting point for all continuum bodies. PDEs governing *viscoelastic flows*, for liquid bodies in particular, shall come next in Section 2.

Let us denote $\{x^i, i = 1\dots3\}$ a Cartesian coordinate system for the Euclidean ambiant space $\mathbb{R}^3$. Let us assume, for $t \in [0,T)$, that $\mathscr{B}$ is a manifold equipped with a Cartesian coordinate system $\{a^\alpha, \alpha = 1\dots d\}$ ($d \in \{1,2,3\}$), and that $\boldsymbol{\phi}_t(\boldsymbol{a} \equiv a^\alpha \boldsymbol{e}_\alpha) = \phi^i_t(\boldsymbol{a})\boldsymbol{e}_i$ is a bi-Lipshitz function on $\mathscr{B} \ni \boldsymbol{a}$. Given a vector force field $\boldsymbol{f}$ in $\mathbb{R}^3$, Galilean physics requires the *deformation gradient* $F^i_\alpha := \partial_\alpha \phi^i_t \circ \boldsymbol{\phi}_t^{-1}$ and the *velocity* $u^i := \partial_t \phi^i_t \circ \boldsymbol{\phi}_t^{-1}$, to satisfy the conservation of linear momentum:

$$\hat\rho \partial_t(\boldsymbol{u}\circ\boldsymbol{\phi}_t) = \operatorname{div}_{\boldsymbol{a}} \boldsymbol{S} + \hat\rho(\boldsymbol{f}\circ\boldsymbol{\phi}_t) \quad \text{on } \mathscr{B} \tag{1}$$

given a mass-density $\hat\rho(\boldsymbol{a}) \geq 0$, see e.g. [1]. Neglecting heat transfers, the first Piola–Kirchoff stress tensor $\boldsymbol{S}(\boldsymbol{F})$ is defined by an internal energy functional $e(\boldsymbol{F})$:

$$S^{i\alpha} = \hat\rho \partial_{F^i_\alpha} e. \tag{2}$$

Then, when $\hat\rho \in \mathbb{R}^+_*$ is constant, motions can be unequivocally defined by solutions

$$(u^i \circ \boldsymbol{\phi}_t, F^i_\alpha \circ \boldsymbol{\phi}_t) \in C^0_t\left([0,T), H^s(\mathbb{R}^3)^3 \times H^s(\mathbb{R}^3)^{3\times3}\right) \quad \text{with } s > \frac{3}{2}$$

ISSN (electronic) : 1873-7234 

to (1)–(2) complemented by (3)–(5), if (1)–(5) defines a *symmetric-hyperbolic* system [2],

$$\partial_t(F^i_\alpha \circ \boldsymbol{\phi}_t) - \partial_\alpha(u^i \circ \boldsymbol{\phi}_t) = 0 \tag{3}$$

$$\partial_t(|F^i_\alpha| \circ \boldsymbol{\phi}_t) - \partial_\alpha(C^i_\alpha \circ \boldsymbol{\phi}_t\, u^i \circ \boldsymbol{\phi}_t) = 0 \tag{4}$$

$$\partial_t(C^i_\alpha \circ \boldsymbol{\phi}_t) + \sigma_{ijk}\sigma_{\alpha\beta\gamma}\partial_\beta(F^j_\gamma \circ \boldsymbol{\phi}_t\, u^k \circ \boldsymbol{\phi}_t) = 0 \tag{5}$$

denoting $\sigma_{ijk}$ Levi-Civita's symbol. But for physical applications, it is difficult to identify functionals $e(\boldsymbol{F})$ such that (1)–(5) defines a *symmetric-hyperbolic* system.

In the sequel, assuming $\hat{\rho} \in \mathbb{R}^+_*$ constant, we recall how one standardly defines $e(\boldsymbol{F})$ for solid and fluid dynamics, on considering the determinant $|F^i_\alpha|$ of the deformation gradient (also denoted $|\boldsymbol{F}|$ hereafter) and the cofactor matrix $C^i_\alpha$ of $F^i_\alpha$ ($\boldsymbol{C}$ in tensor notation) as variables independent of $\boldsymbol{F}$. Next, in Section 2, we recall with much details the function $e(\boldsymbol{F})$ that we proposed in [3] so as to properly define a viscoelastic dynamics of Maxwell type that *unifies* solids and fluids.

### 1.1. *Polyconvex elastodynamics*

If $e(\boldsymbol{F})$ in (2) is *polyconvex*, and if the initial conditions for $(\boldsymbol{u},\boldsymbol{F},|\boldsymbol{F}|,\boldsymbol{C}) \circ \boldsymbol{\phi}_t$ are given by $\big(\partial_t\boldsymbol{\phi}_t, \nabla_{\boldsymbol{a}}\boldsymbol{\phi}_t, |\nabla_{\boldsymbol{a}}\boldsymbol{\phi}_t|, \mathrm{Cof}(\nabla_{\boldsymbol{a}}\boldsymbol{\phi}_t)\big)\,(t=0) \in H^s(\mathbb{R}^3)$ with $s > 3/2$, such that $\nabla_{\boldsymbol{a}} \times \boldsymbol{F} = \boldsymbol{0} = \mathrm{div}_{\boldsymbol{a}}\boldsymbol{C}$ holds i.e.

$$\sigma_{\alpha\beta\gamma}\partial_\alpha F^i_\beta = 0 = \partial_\alpha C^i_\alpha \quad \forall i, \tag{6}$$

then (1)–(5) enters the framework of symmetric-hyperbolic systems. In particular, *a unique time-continuous solution* can be built in $H^s(\mathbb{R}^3)$ for $t \in [0,T)$, given initial conditions $F^i_\alpha(t=0) \in H^s(\mathbb{R}^3)^{3\times 3}$ and $u^i(t=0) \in H^s(\mathbb{R}^3)^3$ [2]. The latter solution, associated with a unique mapping $\boldsymbol{\phi}_t$, is equivalently defined by [4]

$$\partial_t(\rho u^i) + \partial_j(\rho u^i u^j - \sigma^{ij}) = \rho f^i \tag{7}$$

$$\partial_t(\rho F^i_\alpha) + \partial_j(\rho F^i_\alpha u^j - \rho u^i F^j_\alpha) = 0 \tag{8}$$

$$\partial_t \rho + \partial_j(\rho u^j) = 0 \tag{9}$$

$$\partial_t(\rho C^i_\alpha) + \partial_i(\rho C^j_\alpha u^j) = 0 \tag{10}$$

where $\sigma^{ij} := |\boldsymbol{F}|^{-1} S^{i\alpha} F^j_\alpha$ and $\rho := |\boldsymbol{F}|^{-1}\hat{\rho}$, provided the initial conditions satisfy

$$\partial_j(\rho F^j_\alpha) = 0 = \sigma_{ijk}\partial_j(\rho C^k_\alpha) \quad \forall \alpha. \tag{11}$$

Indeed, with the *Eulerian* description (7)–(10) of the body motions (i.e. in spatial coordinates, as opposed to the Lagrangian description (1)–(5) in material coordinates)

$$\partial_t(\rho\boldsymbol{u}) + \mathrm{div}\,(\rho\boldsymbol{u}\otimes\boldsymbol{u} - \boldsymbol{\sigma}) = \rho\boldsymbol{f} \tag{12}$$

$$\partial_t(\rho\boldsymbol{F}) - \nabla\times(\rho\boldsymbol{F}^T\times\boldsymbol{u}) = \boldsymbol{0} \tag{13}$$

$$\partial_t\rho + \mathrm{div}\,(\rho\boldsymbol{u}) = 0 \tag{14}$$

$$\partial_t(\rho\boldsymbol{C}) + \nabla\otimes(\rho\boldsymbol{C}^T\cdot\boldsymbol{u}) = \boldsymbol{0} \tag{15}$$

where $\boldsymbol{C}^T$ is the dual (matrix transpose) of $\boldsymbol{C}$, and with Piola's identity (11)

$$\mathrm{div}\,(\rho\boldsymbol{F}^T) = \boldsymbol{0} = \nabla\times(\rho\boldsymbol{C}^T), \tag{16}$$

one can show that, *when* $e(\boldsymbol{F})$ *is polyconvex*, the symmetric-hyperbolic framework applies to (12)–(16) insofar as smooth solutions also satisfy the conservation law

$$\partial_t\left(\frac{\rho}{2}|\boldsymbol{u}|^2 + \rho e\right) + \mathrm{div}\left(\left(\frac{\rho}{2}|\boldsymbol{u}|^2 + \rho e\right)\boldsymbol{u} - \boldsymbol{\sigma}\cdot\boldsymbol{u}\right) = \rho\boldsymbol{f}\cdot\boldsymbol{u} \tag{17}$$

for $(\rho/2)|\boldsymbol{u}|^2 + \rho e$, a functional *convex* in a set of independent conserved variables [2].

A first example of a physically-meaningful internal energy is the *neo-Hookean*

$$e(F^k_\alpha F^k_\alpha) := \frac{c_1^2}{2}(F^k_\alpha F^k_\alpha - d) \tag{18}$$

with $c_1^2 > 0$. Then, the quasilinear system (1)–(3) is symmetric-hyperbolic insofar as smooth solutions additionally satisfy a conservation law for $|\boldsymbol{u}|^2/2 + e$ strictly convex in $(\boldsymbol{u}, \boldsymbol{F})$. Unequivocal motions can be defined.[1] The latter neo-Hookean model satisfyingly predicts the small motions of some solids.

However, Equation (18) is oversimplistic: it does not model the deformations that are often observed orthogonally to a stress applied unidirectionally, see e.g. [5] regarding rubber. Many observations are better fitted when the Cauchy stress $\boldsymbol{\sigma}$ contains an additional spheric term $-p\boldsymbol{I}$, with a pressure $p(\rho)$ function of volume changes.

Next, instead of (18), one can rather assume a *compressible neo-Hookean* energy

$$e(F^k_\alpha F^k_\alpha) := \frac{c_1^2}{2}(F^k_\alpha F^k_\alpha - d) - \frac{d_1^2}{1-\gamma}|\boldsymbol{F}|^{1-\gamma} =: \tilde{e}(|\boldsymbol{F}|, \boldsymbol{F}). \tag{19}$$

The functional (19) is polyconvex as soon as $\gamma > 1$ [2]. Thus, using either (1)–(5) or (7)–(10) one can define unequivocal smooth motions with $S^i_\alpha(\boldsymbol{F}) = \hat{\rho} c_1^2 F^i_\alpha - \hat{\rho} d_1^2 |\boldsymbol{F}|^{-\gamma} \mathrm{Cof}(\boldsymbol{F})^i_\alpha$ where an additional pressure term arises[2] in comparison with (18). Precisely, one can build unique solutions to a *symmetric* reformulation of a system of conservation laws for conserved variables $U(t, \boldsymbol{x}) : \mathbb{R}^+ \times \mathbb{R}^m \to \mathbb{R}^n$ i.e.

$$\partial_t U + \partial_\alpha G_\alpha(U) = 0 \tag{20}$$

with $k$ involutions $M_\alpha \partial_\alpha U = 0$, $M_\alpha \in \mathbb{R}^{k\times n}$ i.e. $M_\alpha G_\beta(U) = -G_\alpha(U) M_\beta$, $\alpha \neq \beta$.

An additional conservation law $\partial_t \eta(U) + \partial_\alpha Q_\alpha(U) = 0$ is satisfied by (20), for $\eta(U) = |\boldsymbol{u}|^2/2 + \tilde{e}(|\boldsymbol{F}|, \boldsymbol{F})$ a strictly convex functional of $U$. So a smooth function $\Xi(U) \in \mathbb{R}^k$ exists such that $DQ_\alpha(U) = D\eta(U) DG_\alpha(U) + \Xi(U)^T M_\alpha$ holds, $D^2\eta(U) DG_\alpha(U) + D\Xi(U)^T M_\alpha$ is a symmetric matrix, and (20) admits a symmetric-hyperbolic reformulation. The 2D Lagrangian case $\alpha \in \{a, b\}$, $c_1^2 \hat{\rho} \equiv 1$, reads

$$\partial_t u^x + \partial_a(F^y_b p - F^x_a) + \partial_b(-F^y_a p - F^x_b) = 0, \tag{21}$$

$$\partial_t u^y + \partial_a(-F^x_b p - F^y_a) + \partial_b(F^x_a p - F^y_b) = 0, \tag{22}$$

$$\partial_t |\boldsymbol{F}| = \partial_a(-F^x_b u^y + F^y_b u^x) + \partial_b(-F^y_a u^x + F^x_a u^y), \tag{23}$$

$$\partial_t F^x_a - \partial_a u^x = 0, \tag{24}$$

$$\partial_t F^x_b - \partial_b u^x = 0, \tag{25}$$

$$\partial_t F^y_a - \partial_a u^y = 0, \tag{26}$$

$$\partial_t F^y_b - \partial_b u^y = 0, \tag{27}$$

with $p(|\boldsymbol{F}|) := -\partial_{|\boldsymbol{F}|}\tilde{e} \equiv (d_1^2/c_1^2)|\boldsymbol{F}|^{-\gamma}$, abusively denoting $(\boldsymbol{u}, |\boldsymbol{F}|, \boldsymbol{F})$ the functions $(\boldsymbol{u}, |\boldsymbol{F}|, \boldsymbol{F}) \circ \boldsymbol{\phi}_t$ of material coordinates as usual. Involutions $M_\alpha \partial_\alpha U = 0$ hold with

$$M_a = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad M_b = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

---

[1] Not only with fields in $H^s(\mathbb{R}^3)$, $s > 3/2$, for $t \in [0, T)$, but in fact *whatever* $T > 0$ and $s \in \mathbb{R}$ here, insofar as $S^i_\alpha(\boldsymbol{F}) = \hat{\rho} c_1^2 F^i_\alpha$ so the Lagrangian description (1)–(3) reduces to *linear* PDEs.

[2] And thus the flux becomes nonlinear in the conservative variables, so $T > 0$ is definitely finite.

They can be combined together with (20) by using $\Xi(U)^T = \begin{pmatrix} pu^y & -pu^x \end{pmatrix}$ to yield a symmetric system after premultiplication by $D^2\eta(U)$: note $\nu_\alpha(D^2\eta(U)DG_\alpha(U)+D\Xi(U)^T M_\alpha)$ reads

$$\begin{pmatrix} 0 & 0 & (\boldsymbol{e}_x\boldsymbol{C}\nu)\partial_{|\boldsymbol{F}|}p & -\nu_a & -\nu_b & 0 & 0 \\ 0 & 0 & (\boldsymbol{e}_y\boldsymbol{C}\nu)\partial_{|\boldsymbol{F}|}p & 0 & 0 & -\nu_a & -\nu_b \\ (\boldsymbol{e}_x\boldsymbol{C}\nu)\partial_{|\boldsymbol{F}|}p & (\boldsymbol{e}_y\boldsymbol{C}\nu)\partial_{|\boldsymbol{F}|}p & 0 & 0 & 0 & 0 & 0 \\ -\nu_a & 0 & 0 & 0 & 0 & 0 & 0 \\ -\nu_b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\nu_a & 0 & 0 & 0 & 0 & 0 \\ 0 & -\nu_b & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{28}$$

denoting $\boldsymbol{e}_x\boldsymbol{C}\nu \equiv F_b^y\nu_a - F_a^y\nu_b$, $\boldsymbol{e}_y\boldsymbol{C}\nu \equiv -F_b^x\nu_a + F_a^x\nu_b$ and $\nu^T = (\nu_a\ \nu_b) \in \mathbb{R}^m$ a unit vector. The symmetric formulation allows one to establish the key energy estimates in the existence proof of smooth solutions [2], as well as self-similar weak solutions to the 1D Riemann problem using generalized eigenvectors $R$ solutions to

$$\nu_\alpha\left(D^2\eta(U)DG_\alpha(U)+D\Xi(U)^T M_\alpha\right)R = \sigma D^2\eta(U)R \tag{29}$$

with eigenvalues $\sigma \in \left\{0, \pm 1, \pm\sqrt{1+(|\boldsymbol{e}_x\boldsymbol{C}\nu|^2+|\boldsymbol{e}_y\boldsymbol{C}\nu|^2)\partial_{|\boldsymbol{F}|}p}\right\}$. For application to real materials,[3] one important question remains: how to choose $c_1^2$ and $d_1^2$.

In most real applications of elastrodynamics, the *material parameters* $c_1^2$ and $d_1^2$ should vary, as functions of $\boldsymbol{F}$ e.g., but also as functions of an additional *temperature* variable so as to take into account microscopic processes not described by the macroscopic elastodynamics system. For instance, the deformations endured by stressed elastic solids increase with temperature, until the materials become viscous liquids. Then, one natural question arises: could (19) remain useful for liquids which are mostly incompressible (i.e. $\operatorname{div}\boldsymbol{u} \approx 0$ holds) and much less elastic than solids?

In Section 1.2, we recall the limit case when the volumic term dominates the internal energy, and $p = C_0\rho^\gamma$ dominates $\boldsymbol{\sigma}$, which coincides with seminal PDEs for *perfect fluids* (fluids without viscosity). In Section 2, we next consider how to rigorously connect fluids like liquids to solids using an enriched elastodynamics system.

### 1.2. *Fluid dynamics*

Consider the general Eulerian description (12)–(15) for continuum body motions. It is noteworthy that given $\boldsymbol{u}$, each *kinematic* equation (10), (8) and (9) is autonomous. As a consequence, *in spatial coordinates*, motions can be defined by *reduced versions* of the full Eulerian description (7)–(10), with an internal energy $e$ strictly convex in $\rho$ but not in $\boldsymbol{F}$! One famous case is the *polytropic law*

$$e(\rho) := \frac{C_0}{\gamma-1}\rho^{\gamma-1} \tag{30}$$

with $C_0 > 0$. Then, one obtains Euler's system for perfect (inviscid) fluids

$$\begin{aligned} \partial_t\rho + \partial_i(u^i\rho) &= 0 \\ \rho(\partial_t u^i + u^j\partial_j u^i) + \partial_i p &= \rho f^i \end{aligned} \tag{31}$$

with a pressure $p := -\partial_{\rho^{-1}}e = C_0\rho^\gamma$ characterizing *spheric* stresses:

$$\sigma^{ij} = -p\delta_{ij}. \tag{32}$$

The system (31) is symmetric-hyperbolic. It is useful to define unequivocal time-evolutions of Eulerian fields (on finite time ranges) [2], although multi-dimensional solutions are then not equivalently described by one well-posed Lagrangian description [6]. In fact, for applications

[3] So far, the only parameters to be specified for real application are $\hat{\rho}$, $c_1^2$ and $d_1^2$.

to real fluids, the system (31) is better understood as the limit of a kinetic model based on Boltzmann's statistical description of molecules [7], and the model indeed describes gaseous fluids better than condensed fluids (liquids). In any case, the fluid model (31) still lacks viscosity.

One classical approach adds viscous stresses as an *extra-stress* term $\boldsymbol{\tau}$ in (32) i.e.

$$\boldsymbol{\sigma} = -p\boldsymbol{\delta} + \boldsymbol{\tau}. \tag{33}$$

The extra-stress is required symmetric (to preserve angular momentum), objective (for the sake of Galilean invariance), and "dissipative" (to satisfy thermodynamics principles) [8]. Precisely, introducing the entropy $\eta$ as an additional state variable for heat exchanges at temperature $\theta = \partial_s e > 0$, thermodynamics requires

$$\partial_t \eta + (u^j \partial_j)\eta = \mathscr{D}/\theta$$

with a *dissipation* term $\mathscr{D} \geq 0$. Usually, denoting $D(u)^{ij} := (1/2)(\partial_i u^j + \partial_j u^i)$, one then postulates a *Newtonian* extra-stress with two constant parameters $\ell, \dot{\mu} > 0$

$$\tau^{ij} = 2\dot{\mu} D(u)^{ij} + \ell\, D(u)^{kk} \delta_{ij} \tag{34}$$

which satisfies $\mathscr{D} \equiv \tau^{ij}\partial_j u^i \geq 0$ [8]. The Newtonian model allows for the definition of causal motions through the resulting Navier–Stokes equations. But it is not obviously unified with elastodynamics; and letting alone that (34) is far from some real "non-Newtonian" materials, it implies that shear waves propagate infinitely-fast, an idealization that is also a difficulty for the unification with elastodynamics.

By contrast, Maxwell's viscoelastic fluid models for $\boldsymbol{\tau}$ possess well-defined shear waves of finite-speed, and they can be connected with elastodynamics with a view to unifying solids and fluids (liquids) in a single continuum description.

## 2. Viscoelastic flows with Maxwell fluids

Maxwell's models [9] with viscosity $\dot{\mu} > 0$, relaxation time $\lambda > 0$, time-rate $\overset{\diamondsuit}{\boldsymbol{\tau}}$

$$\lambda \overset{\diamondsuit}{\boldsymbol{\tau}} + \boldsymbol{\tau} = 2\dot{\mu}\boldsymbol{D}(\boldsymbol{u}) \tag{35}$$

are widely recognized as physically useful to link fluids where $\boldsymbol{\tau} \xrightarrow{\lambda \to 0} 2\dot{\mu}\boldsymbol{D}(\boldsymbol{u})$ in the Newtonian limit, with solids governed by elastodynamics when $\lambda \sim \dot{\mu} \to \infty$. In particular, one often considers the Upper-Convected Maxwell (UCM) model, with objective time-rate $\overset{\diamondsuit}{\boldsymbol{\tau}}$ in (35) defined by the Upper-Convected (UC) derivative:[4]

$$\overset{\triangledown}{\boldsymbol{\tau}} := \partial_t \boldsymbol{\tau} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{\tau} - (\nabla \boldsymbol{u})\boldsymbol{\tau} - \boldsymbol{\tau}(\nabla \boldsymbol{u})^T \tag{36}$$

because $\overset{\triangledown}{\boldsymbol{\tau}} = 2(\dot{\mu}/\lambda)\boldsymbol{D}(\boldsymbol{u})$ is compatible with elastodynamics when $\boldsymbol{\tau} = (\dot{\mu}/\lambda)(\boldsymbol{F}\boldsymbol{F}^T - \boldsymbol{I})$.

However, a difficulty arises with the quasilinear system (12)–(14)–(33)–(35)–(36) to define general *multi-dimensional* motions for any $\lambda \in (0,\infty)$ from solutions to Cauchy problems: the system may not be hyperbolic and numerical simulations may become unstable [10]. As a cure, we proposed in [3] a symmetric-hyperbolic reformulation of (12)–(14)–(33)–(35)–(36) using a new variable $\boldsymbol{A}$ in $\boldsymbol{\tau} = \rho c_1^2(\boldsymbol{F}\boldsymbol{A}\boldsymbol{F}^T - \boldsymbol{I})$.

We review the reformulation in Section 2.2, after recalling in Section 2.1 well-known 1D solutions to (12)–(14)–(33)–(35)–(36) which show the interest for Maxwell's models.

---

[4] Other objective derivatives than UC can be used, which also allow symmetric-hyperbolic reformulations. They will not be considered here for the sake of simplicity.

### 2.1. *Viscoelastic 1D shear waves for solids and fluids*

Some particular solutions to (12)–(14)–(33)–(35)–(36) unequivocally model viscoelastic flows, and rigorously link solids to fluids. Shear waves e.g. for a 2D body moving along $\boldsymbol{e}_x \equiv \boldsymbol{e}_{x^1}$ following $b = y \equiv x^2$, $a = x - X(t,y)$, $X(0,y) = 0$ are well-defined by (7) i.e.

$$\partial_t u = \partial_y \tau^{xy} \tag{37}$$

where we recall $u := \partial_t X$, and Maxwell's constitutive relation (35) i.e.

$$\lambda \partial_t \tau^{xy} + \tau^{xy} = \dot{\mu} \partial_y u, \tag{38}$$

given enough initial and boundary conditions. Denoting $G := \dot{\mu}/\lambda > 0$ the shear elasticity, (37)–(38) indeed coincides with the famous hyperbolic system for 1D damped waves, which implies $\lambda \partial^2_{tt} u(t,y) + \partial_t u(t,y) = \dot{\mu} \partial^2_{yy} u(t,y)$ and $\lambda \partial^2_{tt} \tau^{xy}(t,y) + \partial_t \tau^{xy}(t,y) = \dot{\mu} \partial^2_{yy} \tau^{xy}(t,y)$. Time-continuous solutions to (37)–(38) are well defined given initial conditions plus possibly boundary conditions when the body has finite dimension along $\boldsymbol{e}_y \equiv \boldsymbol{e}_{x^2}$, such as $y \equiv x^2 > 0$ in Stokes first problem see e.g. [11]. Moreover, the latter 1D shear waves rigorously unify solids and fluids insofar as they are *structurally stable* [12, 13]: when $\lambda \equiv (1/G)\,\dot{\mu} \to \infty$, they satisfy

$$\partial^2_{tt} \tau^{xy} = G \partial^2_{yy} \tau^{xy} \quad \partial^2_{tt} u = G \partial^2_{yy} u$$

like elastic solids, and when $\lambda \to 0$, they satisfy

$$\tau^{xy} = \dot{\mu} \partial_y u \quad \partial_t u = \dot{\mu} \partial^2_{yy} u$$

like viscous liquids. So the 1D shear waves illustrate well the structural capability of Maxwell's model to unify solid and Newtonian fluid motions.

But a problem arises with *multi-dimensional* motions: solutions to (12)–(14)–(33)–(35)–(36) are not well-defined in general.

### 2.2. *Maxwell flows with a symmetric-hyperbolic formulation*

To establish multi-dimensional motions satisfying (35), we introduced in [3] a 2-tensor $\boldsymbol{A}$:

$$\lambda(\partial_t + \boldsymbol{u} \cdot \nabla)\boldsymbol{A} + \boldsymbol{A} = \boldsymbol{F}^{-1} \boldsymbol{F}^{-T} \tag{39}$$

which can be understood as a material property that relaxes in fluid flows.

**Proposition 1.** *Set $\dot{\mu} = \lambda c_1^2$. Then $\boldsymbol{\tau} := \rho c_1^2 (\boldsymbol{F}\boldsymbol{A}\boldsymbol{F}^T - \boldsymbol{I})$ satisfies (35) with*

$$\overset{\diamondsuit}{\boldsymbol{\tau}} := \partial_t \boldsymbol{\tau} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{\tau} - (\nabla \boldsymbol{u})\,\boldsymbol{\tau} - \boldsymbol{\tau}\,(\nabla \boldsymbol{u})^T + (\operatorname{div} \boldsymbol{u})\,\boldsymbol{\tau}. \tag{40}$$

**Proof.** Recall that $(\partial_t + \boldsymbol{u} \cdot \nabla)\boldsymbol{F}^T = \boldsymbol{F}^T \cdot (\nabla \boldsymbol{u})^T$ holds, using (8) and (11). Then compute $(\partial_t + \boldsymbol{u} \cdot \nabla)\boldsymbol{\tau}$ straightforwardly using $\boldsymbol{\tau} := \rho c_1^2 (\boldsymbol{F}\boldsymbol{A}\boldsymbol{F}^T - \boldsymbol{I})$. □

Noteworthily (35)–(40) coincides with a version of Maxwell's models for compressible fluids [14]. Moreover, it is contained in a larger symmetric-hyperbolic system, which allows one to rigorously define viscoelastic motions unequivocally.

**Proposition 2.** *With* (33) *such that $\boldsymbol{\tau} := \rho c_1^2 (\boldsymbol{F}\boldsymbol{A}\boldsymbol{F}^T - \boldsymbol{I})$ and $p(\rho) + c_1^2 \rho = -\partial_{\rho^{-1}} e_0$ for $e_0$ strictly convex in $\rho^{-1}$, (12)–(13)–(14)–(39) becomes symmetric-hyperbolic provided* $\operatorname{div}(\rho \boldsymbol{F}^T) = 0$ *and $\boldsymbol{A}$ is symmetric positive-definite ($\boldsymbol{A} \in \boldsymbol{S}^3_{+,*}$).*

**Proof.** Using $\operatorname{div}(\rho \boldsymbol{F}^T) = \boldsymbol{0}$, (12)–(13)–(14)–(39) rewrites in material coordinates as the Lagrangian system (1)–(3)–(4) plus $\lambda \partial_t \boldsymbol{A} + \boldsymbol{A} = \boldsymbol{F}^{-1} \boldsymbol{F}^{-T}$ where

$$\boldsymbol{S} = (p(|\boldsymbol{F}|)\,\boldsymbol{C} + c_1^2 \boldsymbol{F}^{-T}) + \hat{\rho} c_1^2 \boldsymbol{F}\boldsymbol{A} = \hat{\rho} \partial_{\boldsymbol{F}} \left( e_0 + \frac{c_1^2}{2} \boldsymbol{F}\boldsymbol{A} : \boldsymbol{F} \right).$$

Then, $\boldsymbol{A} \in \boldsymbol{S}^3_{+,*}$ allows the variable change $\boldsymbol{Y} = \boldsymbol{A}^{-2}$. The resulting Lagrangian system for $(\boldsymbol{u}, \boldsymbol{F}, |\boldsymbol{F}|, \boldsymbol{Y})$ with involution $\nabla_{\boldsymbol{a}} \times \boldsymbol{F} = \boldsymbol{0}$ admits a "mathematical entropy" [15] so it is therefore symmetric-hyperbolic. For details we refer to [3]. □

A unique smooth solution can be constructed for (12)–(13)–(14)–(39) using an initial condition satisfying $\rho|\boldsymbol{F}| =: \hat{\rho} > 0$, $\operatorname{div}(\rho \boldsymbol{F}^T) = 0$, $\boldsymbol{A} \in \boldsymbol{S}_{+,*}$ [2]. On small time intervals, it unequivocally defines viscoelastic *multi-dimensional* motions governed by the compressible UCM law (35)–(40) as long as hyperbolicity holds and the solution remains bounded. Those motions satisfy thermodynamics with

$$e = e_0 + \frac{c_1^2}{2}(\boldsymbol{FA} : \boldsymbol{F} - \log\det \boldsymbol{FA} : \boldsymbol{F}). \tag{41}$$

**Proposition 3.** *With* (33), $\boldsymbol{\tau} := \rho c_1^2(\boldsymbol{FAF}^T - \boldsymbol{I})$ *and* $p(\rho) + c_1^2\rho = -\partial_{\rho^{-1}} e_0$, *smooth solutions to (12)–(13)–(14)–(39) additionally satisfy*

$$\partial_t\left(\frac{\rho}{2}|\boldsymbol{u}|^2 + \rho e\right) + \operatorname{div}\left(\left(\frac{\rho}{2}|\boldsymbol{u}|^2 + \rho e\right)\boldsymbol{u} - \boldsymbol{\sigma}\cdot\boldsymbol{u}\right) = \rho \boldsymbol{f}\cdot\boldsymbol{u} + \frac{\rho c_1^2}{2\lambda}(\boldsymbol{I} - \boldsymbol{c}^{-1}) : (\boldsymbol{c} - \boldsymbol{I})$$

*provided* $\operatorname{div}(\rho \boldsymbol{F}^T) = 0$ *and* $\boldsymbol{A} \in \boldsymbol{S}_{+,*}$, *on denoting* $\boldsymbol{c} = \boldsymbol{FAF}^T \in \boldsymbol{S}_{+,*}$.

**Proof.** We will show (3) in material coordinates (the Lagrangian description). On one hand, computing $\partial_t|\boldsymbol{u}|^2 = 2\boldsymbol{u}\cdot\partial_t\boldsymbol{u}$ is straightforward. One the other hand, using (1) and $\partial_t \boldsymbol{F} = \nabla_{\boldsymbol{a}}\boldsymbol{u}$ one computes

$$\partial_t e = \partial_t e_0 + \frac{c_1^2}{2}(\boldsymbol{I} - \boldsymbol{c}^{-1}) : \partial_t \boldsymbol{c} = -\frac{\rho c_1^2}{2\lambda}(\boldsymbol{I} - \boldsymbol{c}^{-1}) : (\boldsymbol{c} - \boldsymbol{I}) + \nabla_{\boldsymbol{a}}\boldsymbol{u} : \boldsymbol{S}/\hat{\rho} \tag{42}$$

where $(\boldsymbol{I} - \boldsymbol{c}^{-1}) : (\boldsymbol{c} - \boldsymbol{I}) \geq 0$ is a dissipation. □

Interestingly, notice that our free energy (41) is not useful for well-posedness: it is not strictly convex in conserved variables. Morover, our formulation (12)–(13)–(14)–(39) for a sound Maxwell model admits the 1D shear waves examined in Section 2.1 as solutions, so it preserves some well-established interesting properties of the standard (incompressible) formulation of Maxwell model.

Let us finally present the symmetric structure of our hyperbolic formulation for (compressible) viscoelastic flows of Maxwell-type, with Lagrangian description

$$\partial_t \boldsymbol{u} = \operatorname{div}_{\boldsymbol{a}} \boldsymbol{S} + \boldsymbol{f} \tag{43}$$
$$\partial_t |\boldsymbol{F}| = \operatorname{div}_{\boldsymbol{a}}(\boldsymbol{C}^T \boldsymbol{u}) \tag{44}$$
$$\partial_t \boldsymbol{C}^T = \nabla_{\boldsymbol{a}} \times (\boldsymbol{u} \times \boldsymbol{F}) \tag{45}$$
$$\partial_t \boldsymbol{F}^T = \nabla_{\boldsymbol{a}} \otimes \boldsymbol{u} \tag{46}$$
$$\partial_t \boldsymbol{A} = (\boldsymbol{F}^{-1}\boldsymbol{F}^{-T} - \boldsymbol{A})/\lambda \tag{47}$$

where $\boldsymbol{S} = -p\,\boldsymbol{C} + \boldsymbol{FA}$, $p(|\boldsymbol{F}|) = |\boldsymbol{F}|^{-1} + (d_1^2/c_1^2)|\boldsymbol{F}|^{-\gamma}$, assuming $c_1^2\hat{\rho} \equiv 1$ in (41)

$$e(\boldsymbol{F}) = \frac{c_1^2}{2}(F^k_\alpha A^{\alpha\beta} F^k_\beta - 2\log|F^k_\beta|) - \frac{d_1^2}{1-\gamma}|\boldsymbol{F}|^{1-\gamma}.$$

To that aim, we consider a 2D system when $\lambda \to \infty$:

$$\partial_t u^x + \partial_a(F_b^y p - (A^{aa}F_a^x + A^{ab}F_b^x)) + \partial_b(-F_a^y p - (A^{ab}F_a^x + A^{bb}F_b^x)) = 0, \tag{48}$$

$$\partial_t u^y + \partial_a(-F_b^x p - (A^{aa}F_a^y + A^{ab}F_b^y)) + \partial_b(F_a^x p - (A^{ab}F_a^y + A^{bb}F_b^y)) = 0, \tag{49}$$

$$\partial_t |\boldsymbol{F}| = \partial_a(-F_b^x u^y + F_b^y u^x) + \partial_b(-F_a^y u^x + F_a^x u^y), \tag{50}$$

$$\partial_t F_a^x - \partial_a u^x = 0, \tag{51}$$

$$\partial_t F_b^x - \partial_b u^x = 0, \tag{52}$$

$$\partial_t F_a^y - \partial_a u^y = 0, \tag{53}$$

$$\partial_t F_b^y - \partial_b u^y = 0, \tag{54}$$

$$\partial_t Y^{aa} = \partial_t Y^{ab} = \partial_t Y^{bb} = 0 \tag{55}$$

where, denoting $\Delta = Y^{aa}Y^{bb} - Y^{ab}Y^{ab}$, $\delta = \sqrt{Y^{aa} + Y^{bb} + 2\sqrt{\Delta}}$, we have

$$A^{aa} = \frac{Y^{bb} + \sqrt{\Delta}}{\delta}, \quad A^{ab} = \frac{-Y^{ab}}{\delta}, \quad A^{bb} = \frac{Y^{bb} + \sqrt{\Delta}}{\delta}.$$

Rewriting $\partial_t U + \partial_\alpha G_\alpha(U) = 0$ the system above, involutions $M_\alpha \partial_\alpha U = 0$ hold with

$$M_a = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad M_b = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $\partial_t \eta(U) + \partial_\alpha Q_\alpha(U) = 0$ is satisfied for $\eta(U) = |\boldsymbol{u}|^2/2 + e$, using $\Xi(U)^T = (pu^y - pu^x)$ in $DQ_\alpha(U) = D\eta(U)DG_\alpha(U) + \Xi(U)^T M_\alpha$.

A symmetric formulation is obtained for our quasilinear formulation of Maxwell (compressible) viscoelastic flows similarly to the standard compressible elastodynamics case: on premultiplying the system (48)–(55) by $D^2\eta(U)$, insofar as the matrix $(D^2\eta(U)DG_\alpha(U) + D\Xi(U)^T M_\alpha)\nu_\alpha$ is symmetric given a unit vector $\nu = (\nu_a, \nu_b) \in \mathbb{R}^2$. We do not detail the symmetric matrix $(D^2\eta(U)DG_\alpha(U) + D\Xi(U)^T M_\alpha)\nu_\alpha$ here: its upper-left block coincides with (28), but the other blocks are complicate and depend on the choice of the variable $\boldsymbol{Y} = \boldsymbol{A}^{-1/2}$ (key to exhibit the symmetric-hyperbolic structure using a fundamental convexity result from [16]—Theorem 2 p. 276 with $r = 1/2$ and $p = 0$) a choice which is not unique (ours may not be optimal). In any case, the symmetric structure yields a key energy estimate for the construction of unique smooth solutions, and it also allows one to construct 1D waves similarly from (29) when $\lambda \to \infty$ (otherwise one has to take into account the source term of relaxation-type).

## 3. Conclusion and perpsectives

Our symmetric-hyperbolic formulation of viscoelastic flows of Maxwell type [3] allows one to rigorously describe *multidimensional* motions, within the same continuum perspective as elastodynamics and Newtonian fluid models. It remains to exploit that mathematically sound framework, e.g. to establish the structural stability of the model and rigorously unify (liquid) fluid and solid motions through parameter variations in our model: see [13] regarding the nonsingular limit toward elastodynamics. Another step in that direction is to drive the transition between (liquid) fluid and solid motions more physically, e.g. on taking into account heat transfers: see [17] for a model of Cattaneo-type for the heat flux, which preserves the symmetric-hyperbolic structure. Last, one may want to add physical effects for particular applications: the purely Hookean internal energy in (41) can be modified to include finite-extensibility effects as in FENE-P or Gent models, or to use another measure of strain, with lower-convected time-rate for instance, see [17].

## Conflicts of interest

The author has no conflict of interest to declare.

## Acknowledgment

The author acknowledges the support of ANR (grant 15-CE01-0013 for SEDIFLO project).

## References


[1] J. E. Marsden, T. J. R. Hughes, *Mathematical Foundations of Elasticity*, Dover Civil and Mechanical Engineering, Dover Publications, New York, 2012.

[2] C. M. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics*, 4th ed., Springer-Verlag, Berlin, 2016.

[3] S. Boyaval, "Viscoelastic flows of Maxwell fluids with conservation laws", *ESAIM Math. Model. Numer. Anal.* **55** (2021), no. 3, p. 807-831.

[4] D. H. Wagner, "Symmetric-hyperbolic equations of motion for a hyperelastic material", *J. Hyperbolic Differ. Equ.* **6** (2009), no. 3, p. 615-630.

[5] L. R. G. Treloar, *The Physics of Rubber Elasticity*, Oxford University Press, USA, 2005.

[6] B. Després, C. Mazeran, "Lagrangian gas dynamics in two dimensions and lagrangian systems", *Arch. Rat. Mech. Anal.* **178** (2005), no. 3, p. 327-372.

[7] F. Golse, "The Boltzmann equation and its hydrodynamic limits", in *Evolutionary Equations*, Handbook of Differential Equations, vol. II, Elsevier/North-Holland, Amsterdam, 2005.

[8] B. D. Coleman, W. Noll, "The thermodynamics of elastic materials with heat conduction and viscosity", *Arch. Rat. Mech. Anal.* **13** (1963), no. 1, p. 167-178.

[9] J. C. Maxwell, "IV. On double refraction in a viscous fluid in motion", *Proc. R. Soc. Lond.* **22** (1874), no. 148-155, p. 46-47.

[10] R. G. Owens, T. N. Philips, *Computational Rheology*, Imperial College Press/World Scientific, Singapore, 2002.

[11] L. Preziosi, D. D. Joseph, "Stokes' first problem for viscoelastic fluids", *J. Non-Newton. Fluid Mech.* **25** (1987), no. 3, p. 239-259.

[12] L. E. Payne, B. Straughan, "Convergence of the equations for a Maxwell fluid", *Stud. Appl. Math.* **103** (1999), no. 3, p. 267-278.

[13] S. Boyaval, "About the structural stability of Maxwell fluids: convergence toward elastodynamics", Tech. report, Laboratoire d'Hydraulique Saint-Venant (LHSV), Ecole des Ponts, 2022, hal-03899611.

[14] P. C. Bollada, T. N. Phillips, "On the mathematical modelling of a compressible viscoelastic fluid", *Arch. Rat. Mech. Anal.* **205** (2012), no. 1, p. 1-26.

[15] C. M. Dafermos, "Non-convex entropies for conservation laws with involutions", *Philos. Trans. R. Soc. Lond. Ser. A* **371** (2013), no. 2005, article no. 20120344.

[16] E. H. Lieb, "Convex trace functions and the Wigner–Yanase–Dyson conjecture", *Adv. Math.* **11** (1973), p. 267-288.

[17] S. Boyaval, M. Dostalík, "Non-isothermal viscoelastic flows with conservation laws and relaxation", *J. Hyperbolic Differ. Equ.* **19** (2022), no. 2, p. 337-364.