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The scientific legacy of Roland Glowinski / *L'héritage scientifique de Roland Glowinski*

The Legacy of Roland Glowinski

L'héritage de Roland Glowinski

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Abstract. Roland Glowinski published 8 books and more than 300 articles. He was also an editor of many very well cited proceedings. Hence this attempt to summarize his scientific work is not likely to do justice to his work. Nevertheless, we will try to extract his major contributions, such as the augmented Lagrangian algorithm, various domain decomposition and fictitious methods and their performance on the Navier–Stokes equations in a moving domain. Roland has created a school of applied mathematicians remarkable by their rigor and efficiency for industrial applications. He marked his time and his books will be authorities as long as computer architectures are similar to their present structures.

Résumé. Roland Glowinski a publié 8 livres et plus de 300 articles. Il a également été rédacteur en chef de nombreux actes très bien cités. Il est donc peu probable que cette tentative de résumer son travail scientifique rende justice à son œuvre. Néanmoins, nous essaierons d'extraire ses principales contributions, telles que l'algorithme du Lagrangien augmenté, diverses méthodes de décomposition de domaines et méthodes de domaines fictifs et leurs performances sur les équations de Navier–Stokes dans un domaine mobile. Roland a créé une école de mathématiciens appliqués remarquables par leur rigueur et leur efficacité pour les applications industrielles. Il a marqué son temps et ses livres feront autorités tant que la structure des ordinateurs restera ce qu'elle est actuellement.

Keywords. Finite Element Method, Conjugate Gradient, Navier–Stokes equations, non-Newtonian fluid, fictitious domain, domain decomposition.

Mots-clés. Méthode des éléments finis, gradient conjugiué, équations de Navier–Stokes, fluide non-newtonien, domaine fictif, décomposition de domaine.

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1. Introduction

Roland Glowinski passed away suddenly on January 26th, 2022. He was 84 years old. Roland was the University of Houston's Cullen Professor of Mathematics and an internationally renowned computational and applied mathematician. He was a member of the French Académie des Sciences, the French Académie des Technologies, the Academia Europa. He received many honors such as the French Order of Mérite National, the French Palmes Académiques and the Légion d'Honneur.

He was also a member of the inaugural class of Fellows of both SIAM and the American Mathematical Society. During the course of his career, Roland was honored with numerous prestigious awards, including the Silver Medal of the City of Paris, the Seymour Cray Prize, the Grand Prix Marcel Dassault, and the Fluid Dynamics Award of the U.S. Association for Computational Mechanics Computational. He also received SIAM's Theodore Von Karman Prize in 2004 and the W.T. and Idalia Reid Prize in 2020.

Roland was born in Paris on March 9, 1937, to Polish and Bielorussian immigrants. During World War II, he and his brother Albert were among the legions of children of Jewish descent who were protected by French individuals. They lived with the Botineau family in Sargé-sur-Braye until the Allied forces liberated France in 1944. This period profoundly influenced Roland and is partly responsible for the sense of humanity and decency that he brought to the mathematics profession. Upon returning to Paris, Roland demonstrated an early talent in a variety of areas. He passed the entrance examination to the Lycée Charlemagne in 1948 and was admitted to the École Polytechnique in 1958, where he specialized in telecommunications. Following his graduation from École Polytechnique, Roland served as a French officer in the Algerian War and then joined the French National School of Telecommunication, as a member of the "Corps des Télécom".

His first position was with the Office de Radiodiffusion-Télévision Française from 1963 to 1968. However, he soon missed mathematics and enrolled in a correspondence course that led to a master's degree in pure and applied mathematics. Roland then registered for a doctoral program —the Diplôme d'Études Approfondies (DEA) d'Analyse Numérique— under the supervision of Jacques-Louis Lions and René de Possel. On the basis of his DEA work, Lions arranged for him to join the Institut de Recherche en Informatique et Automatique (now INRIA) as a research engineer. In 1970, Roland defended his thèse d'Etat at the Université Paris VI (later Université Pierre et Marie Curie and now Sorbonne Université). He was then appointed to a professorship in the same university and became a scientific director at INRIA.

In 1985, Roland accepted a chaired professorship at the University of Houston. He maintained his detached status at the Université Pierre et Marie Curie until 2000 and was a visiting professor, adjunct professor, or advisor at numerous institutions around the world.

From 1992 to 1994, Roland served as director of the Centre Européen de Recherche et Formation Avancée en Calcul Scientifique in France.

The legacy of Roland Glowinski is immense. As a scientist, as an applied mathematician, he will remain a reference for ever. Much before anyone he was attracted by the nonlinear problems of industry with a strategy grounded in optimal control through least squares leading to iterative solutions of linear problems, using functional analysis with a solid mathematical formulation, discretization, error analysis and convergence. For the same industrial reason Roland always worked with unstructured meshes and the Finite Element method.

He was also a leader. Not only he trained many students, who have become established scholars or talented engineers, but he was also a man of vision, a strategist. He was the best possible midway between science and applications. He established INRIA as a top worldwide center for computational mathematics and industrial applications. He was a pioneer in this domain. This explains not only his success with French industry in the first phase of his career, but also the interest he raised in the US, where multiple offers were made to him. The eminent position he was offered at the university of Houston speaks for itself.

In the second phase of his career, it was natural for him, not only to contribute beyond expectations to his new university, but to remain also faithful to his previous employers, the Université Pierre et Marie Curie and INRIA. As former President of INRIA, the first author wants to describe the role he has played and the huge help he has provided. This is a story, which is not recorded in published papers, but deserves to be told about, because it shows what an exceptional professional Roland was.

As a person, Roland will be remembered for his kindness, empathy, compassion, and unique love of others. He had many hobbies and interests. He possessed a keen fascination for astronomy and science fiction, and was an avid fisherman as well as a voracious reader who typically devoured several books at once. How could he do that together with his prolific scientific activity remains a mystery.

Needless to say, Roland was dedicated to his family and his dear wife Angela of 58 years, who was his strongest supporter and cheerleader throughout his career and took an active part in all important decisions in his life. In fact, she pushed him to enter Polytechnique, and eventually to become a scientist. She understood remarkably well where Roland will blossom. Besides Angela, he is survived by his brother Albert; his daughters Anne Glowinski and Tania Glowinski Gonzalez; and five grandchildren. Roland will be dearly missed by his family and of course his former students, colleagues, and scientific friends. As authors of this article, we want to express our immense gratitude to Roland, for his support and his friendship, as well as our admiration for his legacy. [1]

Figure 1. Roland at 60. Middle: Proceedings of the very popular conference series at IRIA organized by Roland with J.L. Lions each year. Right: His most original and probably best book [1].

2. Thesis & First Article

Roland's thesis (Thèse d'état), with Prof. J.L. Lions as advisor, was defended in 1970 at the Université Paris VI (now Sorbonne-Université) with the title:

[1]The above text is adapted from an obituary written by Olivier Pironneau and William Fitzgibbon, professor of mathematics and Director of Graduate Studies in the Department of Mathematics of the University of Houston.

Analysis and Approximation of some integro-differential problems. [2].

The core of the content was published in an article which appeared in the proceedings of the *Symposium on Optimization* held in Nice, France, in 1969 [3].

The first page of his first article is shown on figure 2.

The aim is to solve numerically Problem (P_1) (see Figure 2). Although the problem is set on $\mathbb{R}^+$, one studies a regularized problem and shows that the solution has a compact support and that there is L finite such that the solution u^ϵ of

$$\min_{u\in H_0^2(0,L)\cap H_0^1(0,L)}\left\{\int_0^L\left(\frac{d^2u}{dx^2}\right)^2dx+\left(\int_0^L|u|^{1+\epsilon}dx\right)\,:\,\frac{du}{dx}(0)=1,\;\frac{du}{dx}(L)=0\right\},$$

converges to the solution of (P_1) when $\epsilon\to 0$. Then one writes the optimality conditions

$$\frac{d^4u}{dx^4}+(1+\epsilon)\mathrm{sgn}(u)\int_0^L|u|^\epsilon dx=0,\quad u(0)=u(L)=\frac{du}{dx}(L)=0,\;\frac{du}{dx}(0)=1.$$

The translation $v=u-x(x-L)^2/L^2$ renders the boundary conditions homogeneous. The Green function of the problem $\frac{d^4u}{dx^4}=f,\; u(0)=u(L)=\frac{du}{dx}(L)=0,\;\frac{du}{dx}(0)=0$ is known analytically. Thus the problem is converted into $v+(1+\epsilon)G(\mathrm{sgn}(u)\int_0^L|u|^\epsilon)=0$. More generally Roland showed that Hammerstein equations,

$$v+\lambda\int_0^L A(x,t)v|v|^{p-2}dt=f,$$

can be solved by the following algorithm,

$$v^{n+1}-v^n=\rho\left(v^n+\lambda\int_0^L A(x,t)v^n|v|^{p-2}dt-f\right)$$

which converges if ρ is small enough and $p>2$, $A\in L^\infty((0,L)\times(0,L))$ and $f\in C^0(0,L)$.

The problem was approximated by finite differences, convergence was shown and numerical solution provided by Roland is shown on Figure 2.

3. Variational Inequalities

In the mid-seventies, the variational inequalities came into fashion. Applications were many: Coulomb contacts in solid mechanics, visco-elasticity, mud dams etc. Roland proposed to discretize the formulations by using the finite element method. The technique was well known in solid mechanics but its generalization to abstract variational problem was quite new [4]. Then the discretized system is solved with Gauss–Seidel relaxation iterations. For example, for elasto-plasticity, one must solve

$$\min_u\frac{1}{2}a(u,u)-(f,u)\,:\,u\in K=\left\{v\in H_0^1(\Omega)\,:\,|\nabla u|\le 1\right\}$$

where Ω is a the material domain, a simply connected open set of $\mathbb{R}^2$ and where $a(\cdot,\cdot)$ is the bilinear form of elasticity. One may penalize the constraint and consider

$$\min_{u\in H_0^1(\Omega)}\frac{1}{2}a(u,u)-(f,u)+\frac{1}{4\epsilon}\left\||\nabla u|^2-1\right\|^2\}.$$

The optimality conditions are

$$a(u,w)+\frac{1}{\epsilon}\int_\Omega\left(|\nabla u|^2-1\right)\nabla u\cdot\nabla w\,\mathrm{d}x=(f,w)\quad\forall\, v\in H_0^1(\Omega),$$

The problem is discretized by a P^1 finite element method: Ω is triangulated and V_h denotes the space of continuous piecewise linear functions on the triangulation. To each of the N vertices

RESOLUTION NUMERIQUE D'UN PROBLEME
NON CLASSIQUE DE CALCUL DES VARIATIONS
PAR REDUCTION A UNE EQUATION INTEGRALE
NON LINEAIRE.

R.GLOWINSKI (IRIA)

1 - INTRODUCTION :

Cet exposé a pour but d'indiquer une méthode de résolution numérique d'un problème de calcul des variations considéré par L.BERKOVITZ et H. POLLARD [1], L.BERKOVITZ [2]

Soit (P_1) :

$$(P_1)\begin{cases} \text{Minimiser } J_1(u) = \int_0^{+\infty} u''^2\,dx + \left(\int_0^{+\infty} |u|\,dx\right)^2 \\ \text{sur le convexe } K_1 \\ K_1 = [u/u \in L^1(R^+),\ u' \text{ absolument continue},\ u'' \in L^2(R^+)\ ] \\ \qquad u(0) = 0\ .\ u'(0) = 1 \end{cases}$$

On démontre dans [1] et [2] que (P_1) admet une solution optimale unique, à support compact et que

$$1.675828 < \inf J_1 < 1.745432.$$

Dans ce travail, on utilise la propriété de compacité du support de la solution optimale pour se ramener à un intervalle borné, soit $[0,L]$.

Les conditions d'optimalité sont données par une équation intégro-différentielle aux limites, d'ordre 4, non linéaire, réductible à une équation intégrale non linéaire par utilisation de la fonction de Green de l'opérateur

$$u \rightarrow u^{(4)}:\ H_0^2(0,L) \cap H^4(0,L) \rightarrow L^2(0,L).$$

C'est cette dernière équation intégrale qui est résolue numériquement par une méthode itérative dont la convergence est remarquablement rapide.

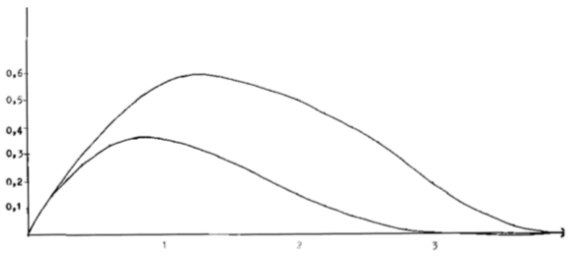
On donne également quelques résultats relatifs à un problème voisin en dimension deux.

2 - LE PROBLEME SUR UN INTERVALLE BORNE :

Soit (P_2) :

$$(P_2)\begin{cases} \text{Minimiser } J_2(u) = \int_0^{L} u''^2 dx + \left(\int_0^{L} |u|\,dx\right)^2 \\ \text{sur le convexe } K_2 \\ K_2 = [u/u \in H^2(0,L) \cap H_0^1(0,L),\ u'(0) = 1\ ,\ u'(L) = 0] \end{cases}$$

On remarquera que $\|u''\|_{L^2}$ est une norme sur $H^2(0,L) \cap H_0^1(0,L)$ équivalente



- La courbe inférieure représente la solution calculée.

Figure 2. Roland's first article and first FORTRAN result. But plotting hardwares didn't exist, so the curves are plotted by hand.

q^j is associated a "hat" functions $w^j \in V_h$ and $u \approx u_h = \sum_j u_j w(x)$. Then, with $A_{ij} = a(w^i, w^j)$, $B_i j(u) = (|\nabla u|^2 - 1)\nabla w^i, \nabla w^j)$, $(F, U \in \mathbb{R}^N$, one must solve

$$\mathbf{A}U + \mathbf{B}(u)U = F.$$

In [5] the following Gauss–Seidel iterations are shown to converge:

$$C_{ii}^n U_i^{n+1} + \sum_{j<i} C_{ij}^n U_j^{n+1} + \sum_{j>i} C_{ij}^n U_j^n = F_i,\ i = 1 \dots N,$$

where $\mathbf{C}^n = \mathbf{A} + \mathbf{B}(u^n)$.

When K is a cone, for instance $K = \{u \in H_0^1(\Omega)\ :\ u \geq 0\}$, a projection method is possible: find $U \in \mathbb{R}^N$ such that $\mathbf{A}U \geq F$, $U \geq 0$. A Gauss–Seidel i, n-step reads

$$A_{ii} U_i^{n+1} + \sum_{j<i} A_{ij} U_j^{n+1} + \sum_{j>i} A_{ij} U_j^n = F_i,\ i = 1 \dots N.$$

If U_i^{n+1} is negative then it is replaced by zero (projection) which produces an admissible iterate because then $\mathbf{A}U^{n+1} \geq F$.

The only step outdated above is the use of Gauss–Seidel iterations which one would replace by gradient iterations or a semi-smooth Newton method [6].

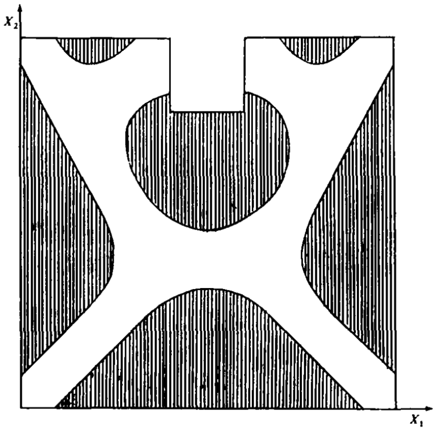


Fig. 3.5. Representation of the elastic and plastic regions for $C = 10$ (the plastic regions are shown hatched).

The computed solution is shown on Figure 3. In those days it was allowed to use a professional designer to beautify the figures, but still, 2D computed solutions were impressive, and several companies saw rightly the potential of this new technique for their industrial designs.

About 25 papers were published between 1970 and 1975 with collaborators from IRIA: J.F. Bourgat [7], M.-O. Bristeau [8], A. Marrocco, [9], D. Begis [10], to name just a few. All the papers use variational formulations, convergent algorithms and discretizations, a very attractive framework for industrial applications.

4. Finite Elements for Fluids

It so happened that Jacques Periaux had also graduated from the DEA of the University Paris VI a few years after Roland. At Dassault Aviation. The head of the simulation team, Pierre Perrier, made a bold move at the time by betting that only unstructured body fitted meshes have a future for the aerodynamics of airplanes. But how to move away from linear potential flow to nonlinear viscous and/or compressible flow? The problem was put to Roland's team at INRIA. The solutions came in 3 steps.

4.1. *A biharmonic solver for the $\psi - \omega$ formulation of the Navier–Stokes equations*

Incompressible two-dimensional flows with viscosity ν can be modeled by the system,

$$-\Delta\psi + \omega = 0, \quad \partial_t\omega - \nu\Delta\omega + (\bar{\nabla}\times\psi)\cdot\nabla\omega = 0. \tag{1}$$

The fluid velocity is then $u = \bar{\nabla}\times\psi$, i.e. $(u_1, u_2, 0)^T = \nabla\times(0,0,\psi)^T$.

An iterative solution which applies $-\Delta$ to the second equation is faced with the problem of finding an algorithm to solve

$$\Delta^2\psi^{n+1} = f^n \text{ in } \Omega, \quad \psi^{n+1}|_{\partial\Omega} \quad \text{and} \quad \frac{\partial\phi^{n+1}}{\partial n}|_{\partial\Omega} \text{ given.}$$

The problem is easy when $\omega = \Delta\phi|_{\partial\Omega}$ is given, so the idea is to study the linear map $\omega|_{\partial\Omega} \mapsto \frac{\partial\phi}{\partial n}|_{\partial\Omega}$ and propose an iterative solution to the underlying linear system [11]. This would only require to be able to compute $\frac{\partial\phi}{\partial n}|_{\partial\Omega}$ for a given $\omega|_{\partial\Omega}$, an easy task.

4.2. *An abstract Least Square Method for the Compressible Potential Flow equation*

The velocity of a compressible inviscid flow is given by $u = \nabla\phi$ where the potential ϕ solves

$$\nabla\cdot\left(\left(1-|\nabla\phi|^2\right)^{\frac{1}{\gamma-1}}\nabla\phi\right) = 0, \tag{2}$$

where $\gamma > 1$ is the adiabatic constant of air, with ϕ or $\frac{\partial\phi}{\partial n}$ given on subsonic boundaries. The solution is not unique unless an entropy condition condition is added. For a flow around an airplane from left to right without recirculation, shocks through which the velocity increases are not allowed. Roland converted this condition into $\Delta\phi < M$, M large enough but finite. Furthermore, it is not easy to find a convergent iteration scheme so the following was suggested to the Dassault company:

$$\min_{\phi-\phi_d\in H^1_0(\Omega)}\left\{\int_\Omega\left[|\nabla\epsilon|^2 + \beta\left|(\Delta\phi - M)^+\right|^2\right]\mathrm{d}x \,:\, -\Delta\epsilon = \nabla\cdot\left(\left(1-|\nabla\phi|^2\right)^{\frac{\gamma}{\gamma-1}}\nabla\phi\right), \quad \epsilon|_{\partial\Omega} = 0\right\}.$$

Notice that this formulation drives (2) to zero in H^{-1}, as expected. It is a least square method, but unlike most formulation which minimize a L^2 criteria, this one minimizes a criteria in H^{-1}; for this reason it was called an *abstract least-square method.*

4.3. *An abstract Least Square Method for the Stationary Navier–Stokes equations*

The same idea can be applied to the Navier–Stokes equations:

$$\min_{u-u_d\in H^1_0(\Omega,div)}\int_\Omega|\nabla\epsilon|^2\mathrm{d}x \,:\, \int_\Omega\nabla\epsilon\cdot\nabla v\mathrm{d}x = \int_\Omega\left[\nu\nabla u\cdot\nabla v + (u\cdot\nabla u)\cdot v\right]\mathrm{d}x,\ \forall\ v\in H^1(\Omega,div),$$

where

$$H^1(\Omega,div) = \left\{v\in\left(H^1_0(\Omega)\right)^d \,:\, \nabla\cdot v = 0\right\}.$$

To solve the problem on uses a conjugate gradient method. The gradient is computed by calculus of variations:

The method was only recently shown to converge [12].

Applications to aerospace by the Dassault Aviation Lab became a success story in 1979 because it was the first time a complete aircraft was computed on an unstructured body-fitted mesh (Figure 3). Roland received the Seymour Cray prize for his participation [13].

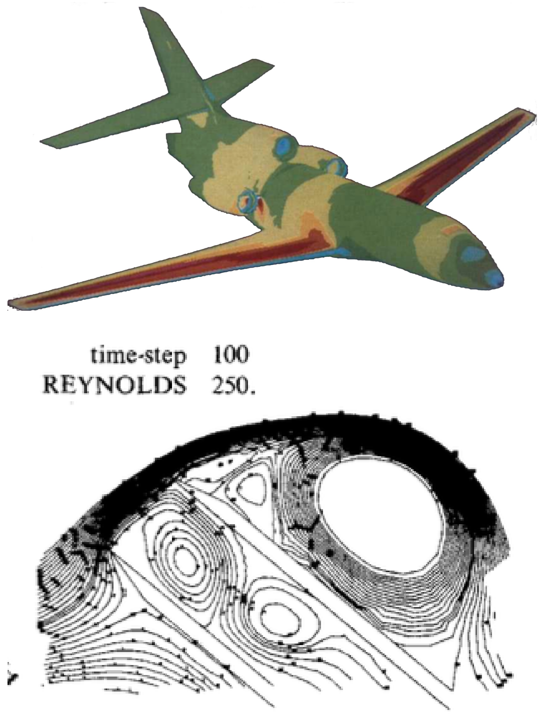


Figure 3. Left Potential flow around a complete aircraft at transonic speed. Right: Navier–Stokes flow near the intake of an aircraft engine. Both are computed using an abstract least square formulation. (Courtesy of Dassault Aviation)

5. Augmented Lagrangian Methods

The incompressiblity constraint in the Navier–Stokes equations was a serious difficulty at that time. Roland generalized his earlier approach to solve variational problems with constraints and came with a very successful idea which is the subject of a book with Michel Fortin [14]. Let A, B be linear operators. To solve

$$Au + B^T q = b, \quad Bu = 0,$$

when A is symmetric, one applies Uzawa iterations on p and a direct method on u to the problem:

$$\min_{u\in U}\max_{q\in Q} \mathscr{L}_r(v,q) := \frac{1}{2}\langle Au, u\rangle - \langle f, u\rangle + \langle q, Bu\rangle + \frac{r}{2}\|Bu\|^2. \tag{3}$$

For the Stokes problem forced by b, $Au = -\Delta u$, $Bu = \nabla\cdot u$ and the Uzawa algorithm to solve (3) was shown to converge [14]:

Theorem 1. *Let Ω be a bounded open set in $\mathbb{R}^d$. Let $f \in L^2(\Omega)^d$. Given $p^0 \in L^2(\Omega)$ and $\rho \in (0, 2(r+\frac{1}{d}))$. Let $u^{n+1} \in H^1_0(\Omega)^d$, $p^n \in L^2(\Omega)$ be given by*

$$-\Delta u^{n+1} - r\nabla\left(\nabla\cdot u^{n+1}\right) = f - \nabla p^n, \quad p^{n+1} = p^n - \rho\nabla\cdot u^{n+1},$$

then the solution of the Stokes problem is reached at the limit:

$$\lim_{n\to\infty}\left\{u^n, p^n\right\} = \{u, p\} \text{ in } H^1(\Omega)^d \times L^2(\Omega) \text{ strongly.}$$

6. Books

Roland liked to write books. He wrote 9 books. The first one, written with J.L. Lions and R. Tremolieres, is the reference for variational inequalities [15]. The table of content is

- Methods of approximation of steady-state inequality problems
- Optimisation algorithms
- The elasto-plastic torsion of a cylindrical bar, penalty and duality.
- Boundary unalitary problems and elliptic inequalities of order 4.
- Steady Bingham fluid in a cylindrical duct
- General approximation methods for time-dependent variational inequlities
- Further discussion on the obstacle problem, optimisation algorithms etc.

We have mentioned above his second book with Michel Fortin which deals mostly on applications to fluids [14]. Its table of content is

- Augmented Lagrangian methods and quadratic programing
- Application to the stokes and Navier–Stokes equations
- On decomposition-coordination methods using an augmented Lagrangian.
- Application to mildly nonlinear problems (with T.F. Chan)
- Application to strongly nonlinear problems of order 2 (with A. Marrocco)
- Application to incompressible visco-plastic fluids (with D. Begis)
- Further discussion on the obstacle problem, optimisation algorithms etc.

Although applications are in very different fields, the books offer a unified framework for all of them with robust discretization schemes for which the discretization errors are known and established in the book [9]. Usually, however, Roland does not discuss the computer implementation, numerical libraries, programming techniques etc.

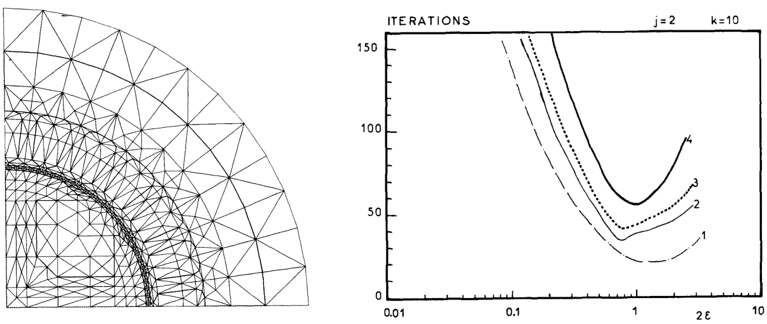


Figure 4. Left: 1/4 of the triangulation of an electric motor (Maxwell stationary equations). Right: convergence study for different values of the penalisation parameter ϵ for 4 different stopping tests (from [9]).

His third book entitled *Numerical Methods for Nonlinear Variational Problems* [1] teaches the reader to formulate the problems in a sound variational framework, then choose an adapted discretization method and finally a programmable algorithm.

This third book deals with the following topics:

- Variational Inequalities
- Discretization with Finite Elements
- Duality, decomposition
- Applications to heat transfer, electromagnetism, solid and fluid mechanics.

In 2003, Roland is invited to write an article in the Handbook of numerical analysis. He returns to the editor an 1176 pages long article. The editor decides to make a separate book [16]. It is the second major book written by Roland.

(1) The Navier–Stokes equations. Introduction to the variational formulations of the equations for incompressible viscous fluids.
(2) A family of operator splitting methods

$$\partial_t\phi + (A+B)(\phi)(t) = 0,\ t \in \left]t^n, t^{n+1}\right]$$
$$\partial_t\phi + A(\phi)\left(t^n + \alpha(t-t^n)\right) = 0$$
$$\partial_t\phi + B(\phi)\left(t^n + \alpha(t^{n+1}-t^n) + \beta(t-t^n)\right) = 0\,.$$

(3) Iterative solution of the Advection equation
(4) The Stokes problem. Various methods to handle incompressibility, error analysis.
(5) The Finite Element Method.
(6) The wave equation for advection

$$u(x,0) = u_0(x),\ u(0,t) = g(t),\ x \in (0,L), t>0,$$
$$\partial_t u + a\partial_x u = 0, \Leftrightarrow \partial_{tt}u - a^2\partial_{xx}u = 0,$$
$$\text{and}\quad \partial_t u(x,0) = -a\partial_x u_0(x),\ a(\partial_t u + a\partial_x u)|_{L,t} = 0$$

(7) The fictitious domain method

6.1. *A Book in Press*

In July 2021 we received a manuscript from De Gruyter for evaluation. It was the preliminary version of a book with T.W. Pan [17]; its content is:

(1) Operator splitting: error estimates
(2) Advection by the wave equation
(3) Finite Element Method for spatial discretization
(4) Solution by preconditioned Conjugate Gradient
(5) Lagrange multipliers and fictitious domains
(6) Application to Oldroyd-B fluid

7. Second Scientific Life at the University of Texas in Houston

In 1985, at 46, at the top of his scientific career in France, Roland moved to the US. Yet he kept his French connections. In some sense one can say that he pursued two scientific careers, one in France and one in the USA.

In the USA he needed new partners such T.W. Pan and Dan Joseph, and new industrial collaborations such as NASA-Houston, Schlumberger and the NSF.

Roland used a sabbatical leave to become the director of CERFACS[2] for 2 years. He invited his former close collaborators, Vivette Girault [18], Patrick LeTallec [19], Jacques Periaux [20], the authors and others. He took French trained post-docs such as Jiwen He and Bertrand Maury [21].

[2] Centre Européen de Recherche et de Formation Avancée en Calcul Scientifique.

Finally he organized from Houston an international series of conference, the popular DDM (i.e.*Domain Decomposition Methods*.

Roland's research included now 3 major new areas:

7.1. *Operator Splitting Algorithms*

The second order splitting of Strang [22] was adapted to fluids; error estimates and convergence results were shown:

$$\frac{dy}{dt} = L_1(y) + L_2(y) \quad \Rightarrow \quad y(t+\delta t) = e^{L_1\delta t} e^{L_2\delta t} y(t),$$

is replaced by

$$\tilde{y}_1 = e^{L_1\delta t} y(t), \qquad y_1 = e^{L_2\delta t}\tilde{y}_1, \qquad \tilde{y}_2 = e^{L_1\delta t} y_1, \qquad y(t+\delta t) = e^{L_2\delta t}\tilde{y}_2.$$

For the Navier–Stokes equations, L_1 is the Laplacian and L_2 is the projection on divergence free function.

7.2. *Domain Decomposition Methods*

Parallel computing became a necessity for large problems around the nineties. Solutions to distribute tasks were given at the level of the linear systems and parallelization of loops for vector computing. Roland was one who had the brilliant idea to tackle the problem at the level of its formulation. Following the Russian school (Y. Kuznetsov [23], [24]) and some early work [25], Roland proposed to solve the Laplace equation $-\Delta u = f$ in $\Omega = \Omega_1 \cup \Omega_2$ with Dirichlet conditions u_Γ on $\Gamma = \partial\Omega$, iteratively as an optimal control problem

$$\min_v \left\{ \int_{\bar{\Omega}_1 \cap \bar{\Omega}_2} |u_1 - u_2|^2 \;:\; -\Delta u_i = f \text{ in } \Omega_i, \quad u_i|_{\partial\Omega} = u_\Gamma,\; u_i|_{\partial\Omega_i \dot{-} \Gamma} = v,\; i = 1,2 \right\}.$$

An example is given on Figure 5.

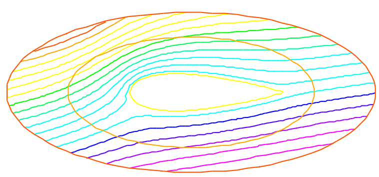


Figure 5. Potential flow around a NACA0012 airfoil with lift, solved by non-overlapping DDM with $\bar{\Omega}_1 \cap \bar{\Omega}_2$ reduced to a curve.

Soon many more similar formulations were proposed, Dirichlet–Neumann accelerators, multigrid skeletons to handle large number of subdomains, etc, and this community became connected via an annual conference co-founded by Roland: The DDM conferences and the Neumann–Neumann preconditioner was proposed by his research team which is still today one of the best method for non-overlapping subdomains [26].

7.3. *Fictitious Domain Methods (FDM)*

A PDE discretized by a finite difference method on a rectangular domain generates a linear system which is much easier to solve than the one generated by a finite element methods. Again, following the Russian school, Roland proposed a variational approach to the so called Fictitious Domain Method.

When the domain of the PDE is embedded into a rectangle Ω, the following optimal control problem is solved (see Figure 6):

$$\min_v \left\{ \int_\gamma |u - u_\Gamma|^2 \ : \ -\Delta u = f \text{ in } \Omega,\ u|_\Gamma = v \right\}.$$

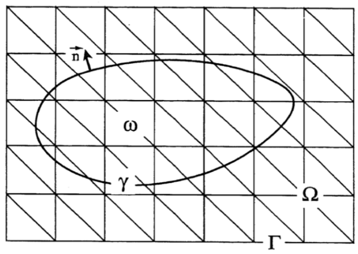


Figure 6. The solution of the PDE is seek in ω which is embedded into the rectangle Ω and the Dirichlet condition on γ is imposed in a weak sense.

The problem can be stated as a saddle point: find $u \in H^1_0(\Omega)$, $p \in H^{-\frac{1}{2}}(\Gamma)$ such that

$$\int_\Omega \nabla u \cdot \nabla v + \int_\gamma p\, v = \int_\Omega f\, v,\ \forall\, v \in H^1_0(\Omega), \qquad \int_\gamma q\, u = \int_\Gamma u_\gamma v,\ \forall\, q \in H^{-\frac{1}{2}}(\gamma).$$

In a beautiful article with V. Girault [18] an error estimate was given for a finite element approximation of the problem: find u_h, p_h:

$$\int_\Omega \nabla u_h \nabla v_h + \int_\gamma \left(u_h q_h + v_h p_h\right) = \int_\Omega f v_h \quad \forall\, v \in V_h(\Omega),\ q \in Q_h(\gamma)$$

7.4. *Fluidized Suspensions*

Roland's interest in FDM was also stirred by a problem given to him by D. Joseph [27]: can one reduced the drag of a thick fluid flowing in a pipe (like petrol) by injecting small balls in the fluid? One must solve the Navier–Stokes equations in a moving domain $\Omega(t)$ made of the fluid container minus the balls

$$\rho\,(\partial_t u + u \cdot \nabla u) - \nabla \cdot \sigma = 0, \quad \nabla \cdot u = 0, \quad u|_{\Gamma_i(t)} = V_i + \omega_i \times (x - G_i), \tag{4}$$

$$\sigma := \mu\left(\nabla u + \nabla u^T\right) - pI, \quad m\frac{dV_i}{dt} = -\int_{\Gamma_i(t)} \sigma \cdot n + \tilde{m} g, \quad J\frac{d\omega_i}{dt} = \int_{\Gamma_i(t)} (\sigma \times (x - G_i)) \cdot n \tag{5}$$

where m is the particle mass, $\tilde{m}$ the apparent mass in the fluid, $G_i(t)$ is the position of the center of ball i, ρ is the density, J the inertial moment of the particle and g is gravity. To avoid a remeshing procedure at every time step one uses FDM for the boundary conditions on the balls.

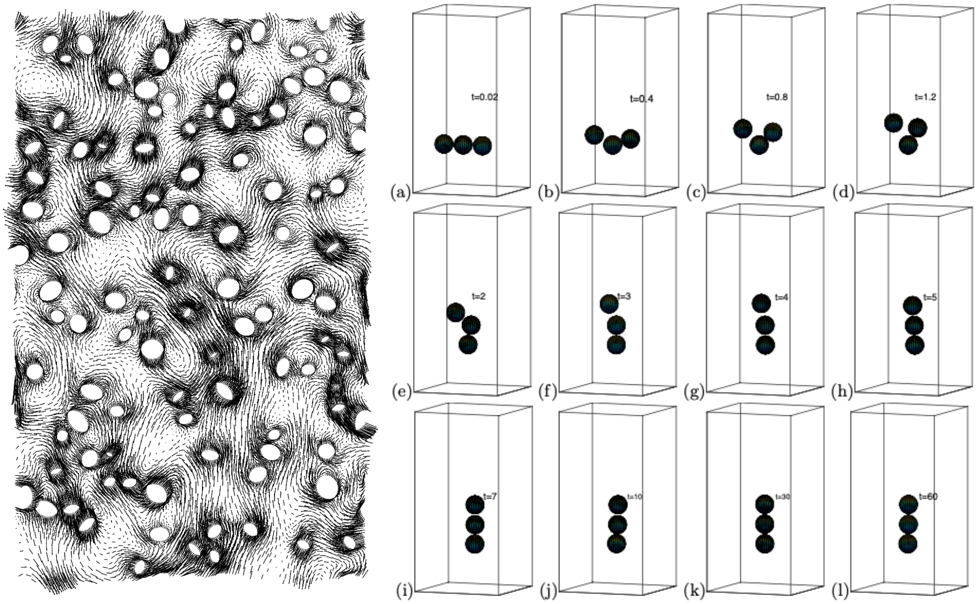


Figure 7. Particles falling freely in a fluid at rest under gravity. Left: the fluid is Newtonian. Right: [28]. Particles falling freely in an Oldroyd-B fluid [29] (Courtesy of T.W. Pan).

7.5. *Complex Fluids*

With T.W. Pan, Roland's latest research was on Oldroyd and other complex fluids. The equations are (4) but with a viscous tensor σ given by a non linear convection equation:

$$\partial_t \sigma^p + u \cdot \nabla \sigma^p - \left(\nabla u + \nabla u^T\right)\left(\sigma^p - \xi I\right) + \lambda \sigma = 0, \quad \sigma^p \text{ given at} t = 0 \text{and on } \Gamma \text{ when} u \cdot n < 0.$$

with ξ and λ constant coefficients.. In (4), the stress tensor is now $\mu(\nabla u + \nabla u^T) + \sigma^p$. This makes the analysis and the simulations very hard (see Figure 7).

8. Additional Remarks

The above description of Roland's contributions could lead to the conclusion that he was not interested by applications in Economics and Management Science, nor by stochastic equations. The first author can indicate that Roland was very well aware of potential connections. For instance, coordination and decentralization in relation with decomposition methods were clearly inspired by economic concepts. In addition, A. Bensoussan coauthored with Roland Glowinski and Robert Eliott in 1989 two articles applying the splitting up method to stochastic equations, in particular Zakai equation.

The most striking connection concerns "variational inequalities". This theory introduced by J. L. Lions and G. Stampacchia was motivated by nonlinear elasticity and other problems related to friction. Roland was instrumental in making this theory applicable and extensively used in concrete applications. It turns out that the same theory is exactly the right one to solve completely different problems, namely optimal stopping problems. These stopping time problems are motivated by economics and management science, to such an extent that the theory of Q.V.I. (Quasi Variational Inequalities), which goes beyond variational inequalities had originally no physical or mechanical motivation. Later O. Pironneau worked on numerical algorithms for V.I. in finance for the pricing of American options.

So, in this case, publications reflect partly the reality of scientific connections. In fact, the reason is that we were all students of J.L. Lions. Under his leadership, there was a natural sharing of tasks. This is why, it would be very reducing to describe the legacy of Roland Glowinski, not to mention his role in Lions huge group, called “Les Lionceaux”. This is materialized by his role at IRIA and INRIA.

Many Lions’ students have worked in numerical analysis, following the brilliant use by Lions of Functional Analysis in setting a rigorous approach to numerical analysis. But Roland had a special role. For Lions, Roland was a magician of algorithms. He used to say “If you want to know whether an algorithm works well or not, ask Glowinski”. It is true that even if an algorithm converges, it might not be efficient in practice. For Lions, Roland was able to foresee whether an algorithm was efficient or not.

The first author has very often asked the advice of Roland, on his own research. In the months preceding his death, we were discussing a lot on telephone, between Dallas and Houston, on Machine Learning. Roland knew very well the important papers, and what can be expected from ML methodologies.

9. Leadership at IRIA and INRIA

As former President of INRIA, the first author considers that the role played by Roland at IRIA (renamed INRIA in 1980) is an essential part of his legacy. I will use the first person, for convenience. His scientific publications cannot describe alone accurately his lasting influence. One needs to understand the role of J.L. Lions, in this process. INRIA is not a university. The Institute has a more specific mission and we had a responsibility in that respect.

A little bit of history, to begin with! IRIA was created in 1967 as a part of “Plan Calcul” an ambitious action decided by the Government of General de Gaulle, to provide the country with a computer industry and first tier expertise. IRIA was the research part of it, and essential for innovation and preparation for the future. The first Director, Professor Michel Laudet had 4 scientific directors, and J. L. Lions was one of them. I (Informatique) and A (Automatique) were wisely supposed to be developed on equal footing and interact. For J.L. Lions, “Automatique” has always meant Numerical Analysis (soon called Scientific Computing) and Control (may be what was generally considered as “Automatique”). I was recruited in 1967 to work on Control, and Roland was recruited in 1968 to work on numerical analysis. Roland and I had some professional experience before turning to research, which was not common at that time. For J. L. Lions, it was considered as an asset. Although we had a different professional experience, the reason to switch to research was the same for Roland and me. We had been captivated by lectures of J. L. Lions and definitely were eager to become his students. We did not know each other at all. A third person was recruited by J. L. Lions, to work in control: an exceptionally brillant scholar, Pierre Faurre. He was my school mate at Ecole Polytechnique and ranked first in entrance and graduation.

In 1972, an important event occurred. IRIA missions were extended, to become also a research agency in information technology. A. Danzin was appointed Director. The research activities were regrouped in an entity, called LABORIA. J. L. Lions became Director of Laboria, and thus responsible for all the research, I and A. At the same time P. Faurre went to industry, as Secretary General of Sagem, on track to become CEO of the company. Roland and I were Scientific Directors on a part time basis (we were both university professors). Clearly J.L. Lions gave us the mission to develop the “A”, scientific computing and control, with an encouragement towards economics and management science. We understood that this was not a standard academic position. He wanted top research, with strong industrial impact and first tier international recognition.

This mission was further emphasized after a new transformation, which occurred in 1980. The research and agency missions of IRIA were separated. INRIA was created as an autonomous government research organisation (N means national) J. L. Lions became President of INRIA. There was a big challenge. The momentum was with Information Technology, which means I and much less A (at least at that time). To do solid mathematical work, even motivated by applications was not the issue. We had to acquire credibility with industry. If this credibility is well established now, it is thanks to Roland Glowinski. This is the legacy I want to emphasize. It is indeed unique. In those days the gap was huge. Roland has pushed very ambitious projects, in particular computations for real airplanes using his algorithms (cooperation with Dassault mentioned above). Beyond that, the development of Modulef, a software library for finite elements, the first project of this kind. This allowed to initiate Simulog, the first start up of INRIA, launched in 1984. If the first start up was in the A, not in the I, it is thanks to Roland. Needless to say, the success followed. Roland was asked to help many companies, from different industrial sectors. Credibility was attained. Credibility was even beyond borders, since indeed, nobody had done so much, worldwide. This explains why he got these outstanding offers in the US, and finally decided to go to Houston.

This is not the end of the story. J. L. Lions was an influential man. He was the scientific adviser of Laurent Fabius, the Minister of research, who became Prime Minister in 1984. J. L. Lions was asked to become President of CNES, and thus left INRIA. He proposed my name to succeed to him as President of INRIA. Roland continued to devote his vision and energy to maintain the momentum of scientific computing at INRIA. I could rely on him. That was so important for my own mission.

As we can say that the legacy of J. L. Lions goes beyond his scientific works, the same can be said for Roland Glowinski. Aside his talent for algorithms and for applied mathematics, probably what characterizes him the most as a human being is his willingness to help. I am a major beneficiary of it. This explains 54 years of friendship. I want also to come back to Angela. She has always encouraged our friendship, by pushing family relations. It is true that she has also done that inside the group of "Lionceaux". Friendship is part of her genes.

Conflicts of interest

The authors have no conflict of interest to declare.

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